Newman-Penrose formalism and gravitational wave polarizations

Manuel Hohmann

December 11, 2020

1 Newman-Penrose basis

We know from general relativity that the observable effect of gravitational waves on a set of test masses, whose trajectories are timelike geodesics, is given by the geodesic deviation. We have seen that a suitable coordinate system is given by Fermi coordinates around the trajectory of a given observer. Here the time coordinate t is given by the proper time along the observer geodesic and the spatial coordinates x^i are chosen so that on the observer trajectory $x^i = 0$, the metric is given by the Minkowski metric $\eta_{\mu\nu}$ and the Christoffel symbols $\Gamma^{\mu}{}_{\nu\rho}$ vanish. In these coordinates the acceleration of a test mass is given by

$$\frac{\mathrm{d}^2x^i}{\mathrm{d}t^2} = -R_{0i0j}x^j\,. \tag{1.1}$$

The 6 components R_{0i0j} of the Riemann tensor are denoted electric components.
It is convenient to use a complex double null basis of the tangent spaces introduced by Newman and Penrose which is spanned by the vectors $l^\mu, n^\mu, m^\mu, \bar{m}^\mu$ given by

$$l = \partial_0 + \partial_3\,, \quad n = \frac{1}{2}(\partial_0 - \partial_3)\,, \quad m = \frac{1}{\sqrt{2}}(\partial_1 + i\partial_2)\,, \quad \bar{m} = \frac{1}{\sqrt{2}}(\partial_1 - i\partial_2)\,, \tag{1.2}$$

and to express tensors in terms of this basis [NP62]. For example, the Minkowski metric in this basis takes the form

$$\eta_{\mu\nu} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{1.3}$$

The diagonal elements vanish since $l^\mu, n^\mu, m^\mu, \bar{m}^\mu$ are null vectors, and the only non-vanishing scalar products are $n_\mu l^\mu = -1$ and $m_\mu \bar{m}^\mu = 1$.
We now consider a plane wave propagating in the positive x^3 direction, which is given by

$$h_{\mu\nu} = \hat{h}_{\mu\nu}e^{-i\omega l_\mu x^\mu} = \hat{h}_{\mu\nu}e^{i\omega(t-x^3)} = \hat{h}_{\mu\nu}e^{i\omega u}\,, \quad \chi = \hat{\chi}e^{i\omega u}\,, \tag{1.4}$$

where we have introduced the retarded time $u = t - x^3$. This means that all quantities which are constructed from the metric perturbation $h_{\mu\nu}$ and the perturbations χ of other gravitational fields depend only on u. In particular we can calculate the Riemann tensor for this plane wave and find that it is fully determined by the components in the Newman-Penrose basis given by

$$\begin{gathered}\Psi_2 = -\frac{1}{6}R_{nlnl} = \frac{1}{12}\ddot{h}_{ll}\,, \quad \Psi_3 = -\frac{1}{2}R_{nln\bar{m}} = -\frac{1}{2}\overline{R_{nlnm}} = \frac{1}{4}\ddot{h}_{l\bar{m}} = \frac{1}{4}\overline{\ddot{h}_{lm}}\,,\\ \Psi_4 = -R_{n\bar{m}n\bar{m}} = -\overline{R_{nmnm}} = \frac{1}{2}\ddot{h}_{\bar{m}\bar{m}} = \frac{1}{2}\overline{\ddot{h}_{mm}}\,, \quad \Phi_{22} = -R_{nmn\bar{m}} = \frac{1}{2}\ddot{h}_{m\bar{m}}\,,\end{gathered} \tag{1.5}$$

where dots denote derivatives with respect to u. Note that Ψ_3 and Ψ_4 are complex. From these one can easily calculate the electric components of the Riemann tensor and see its influence on a set of test masses. These are shown in figure 1.

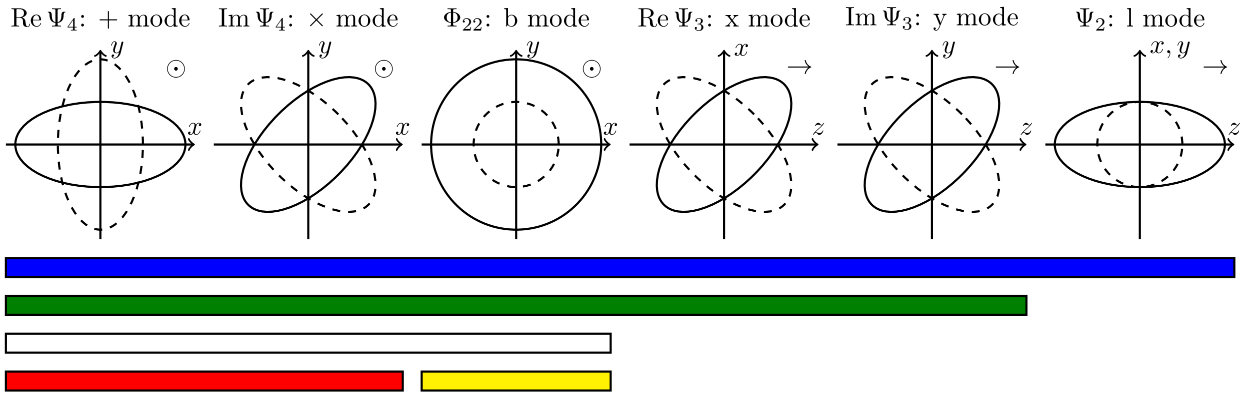

Figure 1: Effects of gravitational waves on a set of test masses, from left to right, for the wave components $\Psi_4, \Phi_{22}, \Psi_3, \Psi_2$ [Wil93, Wil14, Wil18]. For the two complex components Ψ_4, Ψ_3 the real part and the imaginary part are shown separately. The colors indicate in which $\mathrm{E}(2)$ class which mode is present.

2 Lorentz transformation and $\mathrm{E}(2)$ classes

We now consider two observers located at the same point $x^\mu = x'^\mu = 0$ whose coordinate systems are related by a Lorentz transform, such that both agree on the observed frequency ω and direction of the wave. This means that the Lorentz transform must leave the retarded time $u = u'$ and thus the wave vector $l^\mu = l'^\mu$ unchanged. In the Newman-Penrose basis this Lorentz transform can be parametrized in the form

$$\begin{pmatrix} l'^\mu \\ n'^\mu \\ m'^\mu \\ \bar{m}'^\mu \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \alpha\bar{\alpha} & 1 & \bar{\alpha}e^{i\phi} & \alpha e^{-i\phi} \\ \alpha & 0 & e^{i\phi} & 0 \\ \bar{\alpha} & 0 & 0 & e^{-i\phi} \end{pmatrix} \begin{pmatrix} l^\mu \\ n^\mu \\ m^\mu \\ \bar{m}^\mu \end{pmatrix} = \Lambda(\phi, \alpha) \begin{pmatrix} l^\mu \\ n^\mu \\ m^\mu \\ \bar{m}^\mu \end{pmatrix} \tag{2.1}$$

using parameters $\phi \in [0, 2\pi)$ and $\alpha \in \mathbb{C}$. The product of two such matrices is given by

$$\Lambda(\phi', \alpha')\Lambda(\phi, \alpha) = \Lambda(\phi + \phi', \alpha' + \alpha e^{i\phi'}), \tag{2.2}$$

which shows that these Lorentz transforms form a group isomorphic to $\mathrm{U}(1) \ltimes \mathbb{C} \cong \mathrm{SO}(2) \ltimes \mathbb{R}^2 = \mathrm{E}(2)$, the two-dimensional Euclidean group spanned by rotations with angle ϕ and translations by $(\Re\alpha, \Im\alpha)$ [ELL+73b, ELL73a].

If we apply the Lorentz transform shown above to the components of the Riemann tensor we see that they transform as

$$\begin{pmatrix} \Psi'_2 \\ \Psi'_3 \\ \bar{\Psi}'_3 \\ \Psi'_4 \\ \bar{\Psi}'_4 \\ \Phi'_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 3\bar{\alpha} & e^{-i\phi} & 0 & 0 & 0 & 0 \\ 3\alpha & 0 & e^{i\phi} & 0 & 0 & 0 \\ 6\bar{\alpha}^2 & 4\bar{\alpha}e^{i\phi} & 0 & e^{-2i\phi} & 0 & 0 \\ 6\alpha^2 & 0 & 4\alpha e^{-i\phi} & 0 & e^{2i\phi} & 0 \\ 6\alpha\bar{\alpha} & 2\alpha e^{-i\phi} & 2\bar{\alpha}e^{i\phi} & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \Psi_2 \\ \Psi_3 \\ \bar{\Psi}_3 \\ \Psi_4 \\ \bar{\Psi}_4 \\ \Phi_{22} \end{pmatrix} \tag{2.3}$$

From these transformations we can read off some properties of the corresponding waves. Under pure rotations $\alpha = 0$ the matrix becomes diagonal and we see that Ψ_2 and Φ_{22} have helicity 0, so they are scalar modes; Ψ_3 and $\bar{\Psi}_3$ have helicity ± 1, so they are vector modes; Ψ_4 and $\bar{\Psi}_4$ have helicity ± 2, so they are tensor modes; the latter are the two polarizations found in general relativity.

Decomposing the matrix above into its real and imaginary parts we see that we have constructed a six-dimensional, real representation of the Euclidean group. This representation is not irreducible, since it contains subspaces which are invariant under the group action. These subspaces allow a decomposition of the full six-dimensional representation space into subsets which is invariant under Lorentz transformations. This means that if a two observers measure a gravitational wave, they may not agree on the individual components $\Psi_2, \Psi_3, \Psi_4, \Phi_{22}$, but they agree on the subset in which this wave is located. The subsets are as follows, and are labeled by the Petrov type of the non-vanishing Weyl tensor and by the dimension of the corresponding representation of $\mathrm{E}(2)$:

- II_6: 6 polarizations, all modes are allowed.
- III_5: 5 polarizations, $\Psi_2 = 0$, all other modes are allowed.
- N_3: 3 polarizations, $\Psi_2 = \Psi_3 = 0$, tensor and breathing modes are allowed.
- N_2: 2 polarizations, $\Psi_2 = \Psi_3 = \Phi_{22} = 0$, only tensor modes are allowed.
- O_1: 1 polarization, $\Psi_2 = \Psi_3 = \Psi_4 = 0$, only breathing mode is allowed.
- O_0: no gravitational waves.

3 The E(2) class of gravity theories

In any theory of gravity some of these wave types may be allowed, while other may be prohibited by the gravitational field equations. The largest set of allowed waves determines the E(2) class of the theory.

3.1 Scalar-tensor gravity

As an example, consider the linearized vacuum field equations of scalar tensor gravity given by

$$\square\psi = 0\,, \quad R_{\mu\nu} = \psi_{,\mu\nu}\,, \tag{3.1}$$

together with the wave ansatz

$$h_{\mu\nu} = \hat{h}_{\mu\nu}e^{i\omega u}\,, \quad \psi = \hat{\psi}e^{i\omega u}\,. \tag{3.2}$$

This ansatz already solves the field equation for the scalar field. The field equation for the metric yields

$$R_{\mu\nu} = \psi_{,\mu\nu} = -\omega^2\hat{\psi}e^{i\omega u}l_\mu l_\nu\,, \tag{3.3}$$

so that the only non-vanishing component of the Ricci tensor is R_{nn}. From the wave ansatz we find the Ricci tensor

$$R_{nn} = -\ddot{h}_{m\bar{m}}\,, \quad R_{nm} = -\frac{1}{2}\ddot{h}_{lm}\,, \quad R_{n\bar{m}} = -\frac{1}{2}\ddot{h}_{l\bar{m}}\,, \quad R_{m\bar{m}} = -\frac{1}{2}\ddot{h}_{ll} \tag{3.4}$$

and all other components vanish identically. The field equations thus yield

$$\hat{h}_{lm} = \hat{h}_{l\bar{m}} = \hat{h}_{ll} = 0\,, \tag{3.5}$$

so that $\Psi_2 = \Psi_3 = 0$. Gravitational waves with $\Psi_4 \neq 0$ and $\Phi_{22} \neq 0$ solve the field equations, so that the E(2) class of scalar tensor gravity is N_3.

3.2 General relativity

From the equations above we can easily get to general relativity by setting $\psi \equiv 0$, so that we obtain the vacuum Einstein equations

$$R_{\mu\nu} = 0\,. \tag{3.6}$$

Now also $R_{nn} = -\ddot{h}_{m\bar{m}}$ must vanish, so that we get $\Phi_{22} = 0$. Hence, there is no scalar (breathing) mode and the E(2) class is N_2, containing only two tensor modes. Conversely, if any of the other (scalar or vector) modes is found in nature, this would be a clear hint beyond general relativity.

References

[ELL73a] D. M. Eardley, D. L. Lee, and A. P. Lightman. Gravitational-wave observations as a tool for testing relativistic gravity. *Phys. Rev.*, D8:3308–3321, 1973. doi:10.1103/PhysRevD.8.3308.

[ELL+73b] D. M. Eardley, D. L. Lee, A. P. Lightman, R. V. Wagoner, and C. M. Will. Gravitational-wave observations as a tool for testing relativistic gravity. *Phys. Rev. Lett.*, 30:884–886, 1973. doi:10.1103/PhysRevLett.30.884.

[NP62] Ezra Newman and Roger Penrose. An Approach to gravitational radiation by a method of spin coefficients. *J. Math. Phys.*, 3:566–578, 1962. doi:10.1063/1.1724257.

[Wil93] Clifford M. Will. *Theory and experiment in gravitational physics*. Cambridge University Press, 1993. doi:10.1017/CBO9780511564246.

[Wil14] Clifford M. Will. The Confrontation between General Relativity and Experiment. *Living Rev. Rel.*, 17:4, 2014. arXiv:1403.7377, doi:10.12942/lrr-2014-4.

[Wil18] Clifford M. Will. *Theory and Experiment in Gravitational Physics*. Cambridge University Press, 2018. URL: https://www.cambridge.org/academic/subjects/physics/cosmology-relativity-and-gravitation/theory-and-experiment-gravitational-physics-2nd-edition?format=AR&isbn=9781108679824.