Cosmological perturbations and gauge transformations

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1 Energy-momentum tensor

For the background value of the energy-momentum tensor the requirement that it is homogeneous and isotropic leads to the perfect fluid form

$$\bar{T}_{\mu\nu} = (\bar{\rho} + \bar{p})\bar{n}_\mu\bar{n}_\nu + \bar{p}\bar{g}_{\mu\nu} = \bar{\rho}\bar{n}_\mu\bar{n}_\nu + \bar{p}\bar{h}_{\mu\nu}\,, \tag{1.1}$$

where $\bar{\rho}$ and $\bar{p}$ are the background values of the mass-energy density and pressure. Both are functions of time. For constructing perturbations of the energy-momentum tensor, it turns out to be useful to interpret these variables in a geometric way, which relates them to invariant properties on the energy-momentum tensor. For this purpose it is useful to first raise one index of the background energy-momentum tensor,

$$\bar{T}^\mu{}_\nu = (\bar{\rho} + \bar{p})\bar{n}^\mu\bar{n}_\nu + \bar{p}\delta^\mu_\nu = \bar{\rho}\bar{n}^\mu\bar{n}_\nu + \bar{p}\bar{h}^\mu_\nu\,, \tag{1.2}$$

so that it takes the form of an endomorphism on the tangent bundle. One then realizes that $\bar{n}^\mu$ is an eigenvector of this endomorphism, since

$$\begin{aligned}\bar{T}^\mu{}_\nu\bar{n}^\nu &= (\bar{\rho} + \bar{p})\bar{n}^\mu\bar{n}_\nu\bar{n}^\nu + \bar{p}\delta^\mu_\nu\bar{n}^\nu \\ &= -(\bar{\rho} + \bar{p})\bar{n}^\mu + \bar{p}\bar{n}^\mu \\ &= -\bar{\rho}\bar{n}^\mu\,,\end{aligned} \tag{1.3}$$

with eigenvalue $-\bar{\rho}$. We wish to retain a similar property for the perturbed energy-momentum tensor

$$T^\mu{}_\nu = \bar{T}^\mu{}_\nu + \delta T^\mu{}_\nu\,. \tag{1.4}$$

First, note that n^μ is just the normalized four-velocity of the background matter (which is at rest with respect to the co-moving observers). For the perturbed matter, we conclude the existence of an eigenvector u^μ with eigenvalue ρ,

$$T^\mu{}_\nu u^\nu = -\rho u^\mu\,, \tag{1.5}$$

which are described by linear perturbations of the unperturbed quantities,

$$\rho = \bar{\rho} + \delta\rho\,, \quad u^\mu = \bar{n}^\mu + \delta u^\mu\,. \tag{1.6}$$

Here the perturbed four-velocity is normalized with the perturbed four-metric, so that we have the relation

$$\begin{aligned}-1 &= g_{\mu\nu}u^\mu u^\nu \\ &= \bar{g}_{\mu\nu}n^\mu n^\nu + \delta g_{\mu\nu}n^\mu n^\nu + 2\bar{g}_{\mu\nu}n^\mu\delta u^\nu \\ &= -1 - 2\phi + 2n_\mu\delta u^\mu\,,\end{aligned} \tag{1.7}$$

which fixes the time component of the perturbation δu^μ. Hence, one needs to treat only the spatial components as independent. For these one defines the three-velocity v^i such that the four-velocity becomes

$$u^\mu\partial_\mu = a^{-1}[(1 - \phi)\partial_\eta + v^i\partial_i]\,. \tag{1.8}$$

Lowering the indices of the energy-momentum tensor, we can finally write the full, perturbed tensor in the form

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + pg_{\mu\nu} + \pi_{\mu\nu}\,, \tag{1.9}$$

where we have included the pressure, which we expand as

$$p = \bar{p} + \delta p \tag{1.10}$$

into its background value and a perturbation, as well as the anisotropic stress $\pi_{\mu\nu}$, which is subject to the constraints

$$\pi_{\mu\nu}u^{\nu} = 0\,, \quad \pi^{\mu}{}_{\mu} = 0\,. \tag{1.11}$$

Since we have defined the perturbation of the energy-momentum tensor with mixed indices, we find that it is given by

$$\delta T^{\mu}{}_{\nu}\partial_{\mu} \otimes \mathrm{d}x^{\nu} = -\delta\rho\partial_{\eta} \otimes \mathrm{d}\eta - (\bar{\rho}+\bar{p})v^i\partial_i \otimes \mathrm{d}\eta + (\bar{\rho}+\bar{p})(v_i + \tilde{B}_i)\partial_{\eta} \otimes \mathrm{d}x^i + (\delta p\delta^i_j + \pi^i_j)\partial_i \otimes \mathrm{d}x^j\,. \tag{1.12}$$

2 Gauge transformations of the metric

When we studied linear perturbations of the metric around the Minkowski metric, we have seen that the possibility to consider the metric as such a perturbation is retained under coordinate transformations of the form

$$x^{\mu} \mapsto \tilde{x}^{\mu} = x^{\mu} + \xi^{\mu}(x)\,, \tag{2.1}$$

provided that the components of the vector field ξ^{μ} are sufficiently small. Further, we have seen that by considering both the original metric $g_{\mu\nu}$ and the transformed metric $\tilde{g}_{\mu\nu}$ as perturbations around the same background metric $\bar{g}_{\mu\nu}$,

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu}\,, \quad \tilde{g}_{\mu\nu} = \bar{g}_{\mu\nu} + \delta\tilde{g}_{\mu\nu}\,, \tag{2.2}$$

we could relate the perturbations by

$$\delta g_{\mu\nu} = \delta\tilde{g}_{\mu\nu} + (\mathcal{L}_{\xi}\bar{g})_{\mu\nu}\,, \tag{2.3}$$

up to linear order in both the metric perturbations and the vector field components, and the Lie derivative is given by the well-known formula

$$(\mathcal{L}_{\xi}\bar{g})_{\mu\nu} = \xi^{\rho}\partial_{\rho}\bar{g}_{\mu\nu} + \partial_{\mu}\xi^{\rho}\bar{g}_{\rho\nu} + \partial_{\nu}\xi^{\rho}\bar{g}_{\mu\rho} = 2\bar{\nabla}_{(\mu}\xi_{\nu)}\,, \tag{2.4}$$

where in the last expression the indices have been lowered with the background metric. In the following we will assume that the background metric $\bar{g}_{\mu\nu}$ is given by the Friedmann-Lemaître-Robertson-Walker metric, for which we introduced the form

$$\bar{g}_{\mu\nu}\mathrm{d}x^{\mu} \otimes \mathrm{d}x^{\nu} = a^2(-\mathrm{d}\eta \otimes \mathrm{d}\eta + \gamma_{ij}\mathrm{d}x^i \otimes \mathrm{d}x^j) \tag{2.5}$$

in the last lecture, working with the conformal time η. To study its Lie derivative and thereby derive the gauge transformation of the metric perturbation components, it is helpful to decompose ξ_{μ} in the form

$$\xi_{\mu}\mathrm{d}x^{\mu} = a^2[-\alpha\,\mathrm{d}\eta + (\mathrm{d}_i\beta + \zeta_i)\mathrm{d}x^i]\,, \quad \mathrm{d}^i\zeta_i = 0\,, \tag{2.6}$$

where we introduced a factor a^2 for convenience, which is related to lowering the indices with the metric. Hence, we have decomposed the gauge transforming vector field in a similar way as the metric perturbation, into two scalars and a divergence-free vector. The covariant derivative then decomposes as

$$\begin{aligned}\bar{\nabla}_{\mu}\xi_{\nu}\mathrm{d}x^{\mu} \otimes \mathrm{d}x^{\nu} &= (\partial_{\mu}\xi_{\nu} - \bar{\Gamma}^{\rho}{}_{\nu\mu}\xi_{\rho})\mathrm{d}x^{\mu} \otimes \mathrm{d}x^{\nu}\\ &= a^2\big[-(\alpha' + \mathcal{H}\alpha)\mathrm{d}\eta \otimes \mathrm{d}\eta + (\mathrm{d}_i\beta' + \zeta_i' + \mathcal{H}\mathrm{d}_i\beta + \mathcal{H}\zeta_i)\mathrm{d}\eta \otimes \mathrm{d}x^i\\ &\quad + (-\mathrm{d}_i\alpha - \mathcal{H}\mathrm{d}_i\beta - \mathcal{H}\zeta_i)\mathrm{d}x^i \otimes \mathrm{d}\eta + (\mathrm{d}_i\mathrm{d}_j\beta + \mathrm{d}_i\zeta_j + \mathcal{H}\alpha\gamma_{ij})\mathrm{d}x^i \otimes \mathrm{d}x^j\big]\,.\end{aligned} \tag{2.7}$$

Its symmetric part yields the change of the metric perturbation

$$\begin{aligned}&a^{-2}(\mathcal{L}_{\xi}\bar{g})_{\mu\nu}\mathrm{d}x^{\mu} \otimes \mathrm{d}x^{\nu}\\ &= -2(\alpha' + \mathcal{H}\alpha)\mathrm{d}\eta \otimes \mathrm{d}\eta + (-\mathrm{d}_i\alpha + \mathrm{d}_i\beta' + \zeta_i')(\mathrm{d}\eta \otimes \mathrm{d}x^i + \mathrm{d}x^i \otimes \mathrm{d}\eta) + 2(\mathrm{d}_i\mathrm{d}_j\beta + \mathrm{d}_{(i}\zeta_{j)} + \mathcal{H}\alpha\gamma_{ij})\mathrm{d}x^i \otimes \mathrm{d}x^j\,.\end{aligned} \tag{2.8}$$

Hence, by comparing this transformation with the irreducible decomposition of the metric perturbation $\delta g_{\mu\nu}$, and decomposing the transformed perturbation $\delta\tilde{g}_{\mu\nu}$ in the same fashion, we find that the components are related by

$$\phi = \tilde{\phi} + \mathcal{H}\alpha + \alpha' \,, \tag{2.9a}$$
$$\psi = \tilde{\psi} - \mathcal{H}\alpha \,, \tag{2.9b}$$
$$B = \tilde{B} - \alpha + \beta' \,, \tag{2.9c}$$
$$E = \tilde{E} + \beta \,, \tag{2.9d}$$
$$B_i = \tilde{B}_i + \zeta_i' \,, \tag{2.9e}$$
$$E_i = \tilde{E}_i + \zeta_i \,, \tag{2.9f}$$
$$E_{ij} = \tilde{E}_{ij} \,. \tag{2.9g}$$

We thus see that the tensor perturbations are invariant under the gauge transformation, while this is not the case for the other components. However, we can find a number of combinations of these components which are gauge-invariant. Defining

$$\Phi = \phi + \mathcal{H}(B - E') + B' - E'' \,, \tag{2.10a}$$
$$\Psi = \psi - \mathcal{H}(B - E') \,, \tag{2.10b}$$
$$I_i = B_i - E_i' \,, \tag{2.10c}$$

we find that these are also gauge-invariant,

$$\Phi = \tilde{\Phi} \,, \quad \Psi = \tilde{\Psi} \,, \quad I_i = \tilde{I}_i \,. \tag{2.11}$$

Since the field equations are independent of the choice of coordinates, one may therefore expect that they can be expressed in terms of these gauge-invariant variables. This is indeed the case and straightforward to derive, even though not immediate.

3 Gauge transformation of the energy-momentum tensor

Similarly to the metric, the energy-momentum tensor transforms under a gauge transformation as

$$T^\mu{}_\nu = \tilde{T}^\mu{}_\nu + (\mathcal{L}_\xi \bar{T})^\mu{}_\nu \,. \tag{3.1}$$

Assuming that both $T^\mu{}_\nu$ and $\tilde{T}^\mu{}_\nu$ are expanded around the same background, we see that the perturbation transforms by the Lie derivative of this background. Using the expressions (1.2) for the background energy-momentum tensor and (2.6) for the gauge transformation, we can write the Lie derivative as

$$(\mathcal{L}_\xi \bar{T})^\mu{}_\nu = (\bar{\rho} + \bar{p})\left[(\mathcal{L}_\xi \bar{n})^\mu \bar{n}_\nu + \bar{n}^\mu (\mathcal{L}_\xi \bar{n})_\nu\right] + \mathcal{L}_\xi \bar{\rho} \bar{n}^\mu \bar{n}_\nu + \mathcal{L}_\xi \bar{p}(\bar{n}^\mu \bar{n}_\nu + \delta^\mu_\nu) \,, \tag{3.2}$$

where

$$(\mathcal{L}_\xi \bar{n})_\mu \mathrm{d}x^\mu = a(\alpha' + \mathcal{H}\alpha)\mathrm{d}\eta + a\mathrm{d}_i\alpha\, \mathrm{d}x^i \,, \tag{3.3a}$$
$$(\mathcal{L}_\xi \bar{n})^\mu \partial_\mu = a^{-1}(\alpha' + \mathcal{H}\alpha)\partial_\eta + a^{-1}(\mathrm{d}^i\beta' + \zeta'^i)\partial_i \,, \tag{3.3b}$$
$$\mathcal{L}_\xi \bar{\rho} = \alpha\bar{\rho}' \,, \tag{3.3c}$$
$$\mathcal{L}_\xi \bar{p} = \alpha\bar{p}' \,. \tag{3.3d}$$

In summary, we thus find the Lie derivative

$$(\mathcal{L}_\xi \bar{T})^\mu{}_\nu \partial_\mu \otimes \mathrm{d}x^\nu = -\alpha\bar{\rho}'\partial_\eta \otimes \mathrm{d}\eta + (\bar{\rho} + \bar{p})(\mathrm{d}^i\beta' + \zeta'^i)\partial_i \otimes \mathrm{d}\eta - (\bar{\rho} + \bar{p})\mathrm{d}_i\alpha\partial_\eta \otimes \mathrm{d}x^i + \alpha\bar{p}'\partial_i \otimes \mathrm{d}x^i \,. \tag{3.4}$$

Comparing with the perturbation (1.12) of the energy-momentum tensor, we thus find that its constituents transform as

$$\delta\rho = \widetilde{\delta\rho} + \alpha\bar{\rho}' \,, \tag{3.5a}$$
$$\delta\rho = \widetilde{\delta p} + \alpha\bar{p}' \,, \tag{3.5b}$$
$$v_i = \tilde{v}_i - \mathrm{d}_i\beta' - \zeta_i' \,, \tag{3.5c}$$
$$\pi_{ij} = \tilde{\pi}_{ij} \,. \tag{3.5d}$$

In order to form gauge-invariant irreducible components, we decompose these relations into their irreducible components and compare them with the gauge transformation (2.9) of the irreducible components of the metric. For the density and the pressure we can then define the gauge-invariant quantities

$$\mathcal{E} = \delta\rho + \bar{\rho}'(B - E')\,, \quad \mathcal{P} = \delta p + \bar{p}'(B - E')\,. \tag{3.6}$$

The velocity can be decomposed into a scalar and a vector part, and we define

$$\mathrm{d}_i\mathcal{L} + \mathcal{X}_i = v_i + \tilde{E}'_i = v_i + \mathrm{d}_iE' + E'_i\,. \tag{3.7}$$

Finally, the anisotropic stress is already gauge-invariant, and we simply decompose it into the components

$$\pi_{ij} = \mathrm{d}_i\mathrm{d}_j\mathcal{S} - \frac{1}{3}\triangle\mathcal{S}\gamma_{ij} + \mathrm{d}_{(i}\mathcal{V}_{j)} + \mathcal{T}_{ij}\,. \tag{3.8}$$

Note that $\mathcal{X}_i$ and $\mathcal{V}_i$ are divergence-free vectors,

$$\mathrm{d}_i\mathcal{X}^i = \mathrm{d}_i\mathcal{V}^i = 0\,, \tag{3.9}$$

while $\mathcal{T}_{ij}$ is symmetric, trace-free and divergence-free,

$$\mathcal{T}_{[ij]} = 0\,, \quad \mathrm{d}^i\mathcal{T}_{ij} = 0\,, \quad \mathcal{T}^i{}_i = 0\,. \tag{3.10}$$

In the following, we will work with these quantities.

4 Energy-momentum conservation

As a first, simple example, we will study the energy-momentum conservation

$$\nabla_\mu T^\mu{}_\nu = 0\,. \tag{4.1}$$

We start by calculating the background, which reads

$$\bar{\nabla}_\mu\bar{T}^\mu{}_\nu\mathrm{d}x^\nu = [\bar{\rho}' + 3\mathcal{H}(\bar{\rho} + \bar{p})]\mathrm{d}\eta\,. \tag{4.2}$$

This is the well-known conservation law for a cosmological perfect fluid. In the following, we will make use of this result in order to replace $\bar{\rho}'$ in places where it appears useful. We then continue by calculating the perturbation, which we can write in the form

$$\delta[\nabla_\mu T^\mu{}_\nu] = \bar{\nabla}_\mu\delta T^\mu{}_\nu + \delta\Gamma^\mu{}_{\rho\mu}\bar{T}^\rho{}_\nu - \delta\Gamma^\rho{}_{\nu\mu}\bar{T}^\mu{}_\rho\,. \tag{4.3}$$

The calculation is rather lengthy and thus omitted here. Decomposing the result by using the irreducible decomposition, we find that the time component yields the scalar equation

$$\mathcal{E}' + 3\mathcal{H}(\mathcal{E} + \mathcal{P}) + (\bar{\rho} + \bar{p})(\triangle\mathcal{L} - 3\Psi') = 0\,. \tag{4.4}$$

The spatial part splits into two parts. The pure divergence part reads

$$[(\bar{\rho} + \bar{p})\mathcal{L}]' + 4\mathcal{H}(\bar{\rho} + \bar{p})\mathcal{L} + \mathcal{P} + \frac{2}{3}\triangle\mathcal{S} + 2K\mathcal{S} + (\bar{\rho} + \bar{p})\Phi = 0\,, \tag{4.5}$$

and so depends only on the scalar perturbations. Similarly, we have a divergence-free vector part

$$[(\bar{\rho} + \bar{p})(\mathcal{X}_i + I_i)]' + 4\mathcal{H}(\bar{\rho} + \bar{p})(\mathcal{X}_i + I_i) + \frac{1}{2}\triangle\mathcal{V}_i + K\mathcal{V}_i = 0\,, \tag{4.6}$$

which contains only vector perturbations. These relations will turn out to be useful later.

5 Gauge invariance of the field equations

In the following we will not assume a priori that the field equations are given by Einstein's equations of general relativity, but we will assume a generic metric gravity theory, whose metric field equations can be expressed in the form

$$E^{\mu}{}_{\nu} = T^{\mu}{}_{\nu}\,. \tag{5.1}$$

We now perform a perturbative expansion of this equation, from which we obtain a background equation

$$\bar{E}^{\mu}{}_{\nu} = \bar{T}^{\mu}{}_{\nu}\,, \tag{5.2}$$

as well as a linear perturbation

$$\delta E^{\mu}{}_{\nu} = \delta T^{\mu}{}_{\nu}\,. \tag{5.3}$$

The key ingredient in perturbation theory is to first impose the background equation (5.2) and to consider a perturbation around a solution of this equation, for which then the perturbed equation (5.3) is imposed. For the background, we have chosen a homogeneous and isotropic solution, given by the Friedmann-Lemaître-Robertson-Walker metric and perfect fluid energy-momentum tensor, and we have fixed the coordinates by demanding that the metric takes a particular form. Of course, the field equations are tensor equations, and so we could have used any other coordinates as well - we have simply made a convenient choice. The same holds true for the perturbations. As we have seen above, fixing the coordinates for the background still allows infinitesimal coordinate transformations, under which the background retains its form, but the components of the perturbations change in the form of the gauge transformations we have found. Also here we can make a convenient choice, by demanding a particular form of the perturbations, and this is often done in different ways, depending on which aspects of cosmological perturbations are being studied. Equivalently, one can work with the gauge-invariant variables we have introduced (or a different linear combination thereof). It follows from general covariance of the field equations that the perturbed equations are gauge-invariant, i.e., they do not depend on the chosen gauge, and so they can be expressed in terms of gauge-invariant quantities. However, as we have seen above, the components of the energy-momentum tensor and the metric, and thus presumable also of the tensor $\delta E^{\mu}{}_{\nu}$, do depend on the choice of the gauge. To resolve this apparent contradiction, recall that under an infinitesimal coordinate transformation (of the same order as the perturbations) the two sides of the perturbed gravitational field equations change as

$$\delta E^{\mu}{}_{\nu} = \delta\tilde{E}^{\mu}{}_{\nu} + (\mathcal{L}_{\xi}\bar{E})^{\mu}{}_{\nu}\,, \quad \delta T^{\mu}{}_{\nu} = \delta\tilde{T}^{\mu}{}_{\nu} + (\mathcal{L}_{\xi}\bar{T})^{\mu}{}_{\nu}\,. \tag{5.4}$$

For the background, however, we assume that the field equations (5.2) hold. In this case the two Lie derivative terms agree, and so both sides of the field equations change by the same amount. It follows that even though each side of the perturbed field equations undergoes a non-trivial transformation, the equation as a whole is gauge-invariant.

6 Gauge-invariant decomposition of the Einstein equations

We now apply the gauge-invariant cosmological perturbation theory to Einstein's equations of general relativity,

$$G^{\mu}{}_{\nu} = 8\pi G T^{\mu}{}_{\nu}\,, \tag{6.1}$$

which we write here with mixed indices, as this is how we defined the perturbations. It is well known that for a homogeneous and isotropic background they yield the Friedmann equations

$$\mathcal{H}^2 + K = \frac{8\pi G a^2}{3}\bar{\rho}\,, \quad 2\mathcal{H}' + \mathcal{H}^2 + K = -8\pi G a^2\bar{p}\,. \tag{6.2}$$

In the following, we will assume that these background equations are solved, and so we can use them to replace $\bar{\rho}$ and $\bar{p}$ with the corresponding left hand sides and vice versa, at our convenience.
We now come to the perturbed field equations. We will not derive these equations, since their derivation is rather lengthy, and only show them and their relations. We start with the tensor equations, which are the easiest, and whose vacuum equations we derived in a previous lecture. Adding also the matter contribution, we find the equation

$$E''_{ij} + 2\mathcal{H}E'_{ij} - \triangle E_{ij} + 2KE_{ij} = 8\pi G a^2\mathcal{T}_{ij}\,. \tag{6.3}$$

As we have seen before, this equation is expressed in terms of gauge-invariant variables, since the tensor part of the equations is in any case gauge-invariant.
We then come to the vector equations. From the field equations we obtain two independent equations, which are given by

$$I_i' + 2\mathcal{H}I_i = -8\pi G a^2 \mathcal{V}_i \tag{6.4}$$

and

$$\frac{1}{2}\triangle I_i - (2\mathcal{H}^2 - 2\mathcal{H}' + K)I_i = 16\pi G a^2(\bar{\rho} + \bar{p})\mathcal{X}_i\,. \tag{6.5}$$

To gain more insight into these equations, it is helpful to define the (gauge-invariant) quantity

$$\mathcal{Q}_i = (\bar{\rho} + \bar{p})(\mathcal{X}_i + I_i)\,, \tag{6.6}$$

which can be interpreted as the transverse momentum current. Combining the two equations above, one can then find the conservation equation

$$\mathcal{Q}_i' + 4\mathcal{H}\mathcal{Q}_i = -\frac{1}{2}\triangle\mathcal{V}_i - K\mathcal{V}_i\,, \tag{6.7}$$

which is simply the momentum conservation equation (4.6) we have found from the covariant energy-momentum conservation. This does not come as a surprise, since the Einstein tensor satisfies the Bianchi identity $\nabla_\mu G^\mu{}_\nu = 0$, and so any solution of Einstein's equations satisfies the energy-momentum conservation. Using this new variable, we can also write the remaining equation as

$$\triangle I_i + 2K I_i = 16\pi G a^2 \mathcal{Q}_i\,, \tag{6.8}$$

which is now a constraint equation for I_i.
Finally, we take a brief look at the scalar equations, which are the most lengthy to derive. These can be written in the form

$$3\mathcal{H}\Psi' - 3K\Psi - \triangle\Psi + 3\mathcal{H}^2\Phi = -4\pi G a^2\mathcal{E}\,, \tag{6.9a}$$

$$\Psi'' + 2\mathcal{H}\Psi' + \mathcal{H}\Phi' + (2\mathcal{H}' + \mathcal{H}^2)\Phi = 4\pi G a^2\left(\mathcal{P} + \frac{2}{3}\triangle\mathcal{S}\right)\,, \tag{6.9b}$$

$$\Psi' + \mathcal{H}\Phi = -4\pi G a^2(\bar{\rho} + \bar{p})\mathcal{L}\,, \tag{6.9c}$$

$$\Psi - \Phi = 8\pi G a^2\mathcal{S}\,. \tag{6.9d}$$

We will not discuss these equations in detail here. However, it is helpful to remark that they can be used to derive the conservation equations (4.4) and (4.5).