Cosmological perturbations of the metric

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1 Background spacetime

Various different conventions and notations are being used in the theory of cosmological perturbations. The aim of this section is to establish the conventions and notations we will use throughout this lecture, which shall reflect the most common literature. We start with the unperturbed metric on the spacetime manifold M, for which we assume a homogeneous and isotropic Friedmann-Lemaître-Robertson-Walker metric given by

$$\bar{g}_{\mu\nu}\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu = -\mathrm{d}t \otimes \mathrm{d}t + a^2\gamma_{ij}\mathrm{d}x^i \otimes \mathrm{d}x^j = a^2(-\mathrm{d}\eta \otimes \mathrm{d}\eta + \gamma_{ij}\mathrm{d}x^i \otimes \mathrm{d}x^j)\,, \tag{1.1}$$

where Greek indices are spacetime indices running from 0 to 3, while Latin indices are space indices running from 1 to 3. The two different time coordinates introduced here are the cosmological time t, which is the time measured by a co-moving observer (which is an observer at fixed spatial coordinates x^i), and the conformal time η. They are related by

$$\mathrm{d}t = a\,\mathrm{d}\eta\,. \tag{1.2}$$

Here a is the scale factor, which depends on time (one can express this dependence either using cosmological or conformal time). Further, we have the spatial background metric, which is in spherical coordinates r, ϑ, φ on a purely spatial manifold Σ written as

$$\gamma_{ij}\mathrm{d}x^j \otimes \mathrm{d}x^j = \frac{\mathrm{d}r \otimes \mathrm{d}r}{1 - Kr^2} + r^2(\mathrm{d}\vartheta \otimes \mathrm{d}\vartheta + \sin^2\vartheta\,\mathrm{d}\varphi \otimes \mathrm{d}\varphi)\,. \tag{1.3}$$

Here $K \in \{-1, 0, 1\}$ is the curvature parameter. Note that this spatial metric does not depend on time. Finally, for the metric and its derived quantities (connection, curvature), we use the notation that a bar denotes the background value.

Besides the split of the coordinates into time and space, it is also common to introduce a split of the metric in the form

$$\bar{g}_{\mu\nu} = -\bar{n}_\mu\bar{n}_\nu + \bar{h}_{\mu\nu}\,, \tag{1.4}$$

where

$$\bar{n}_\mu\mathrm{d}x^\mu = -\mathrm{d}t = -a\,\mathrm{d}\eta\,, \quad \bar{h}_{\mu\nu}\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu = a^2\gamma_{ij}\mathrm{d}x^i \otimes \mathrm{d}x^j \tag{1.5}$$

are the unit normal covector field, $\bar{g}^{\mu\nu}\bar{n}_\mu\bar{n}_\nu = -1$, and the spatial metric.

Further, one introduces a notation for time derivatives. It is most common to denote derivatives with respect to cosmological time t by a dot, and with respect to conformal time with a prime. With these conventions, one further defines

$$H = \frac{\dot{a}}{a} = \frac{1}{a}\frac{\mathrm{d}a}{\mathrm{d}t}\,, \quad \mathcal{H} = \frac{a'}{a} = \frac{1}{a}\frac{\mathrm{d}a}{\mathrm{d}\eta} = aH\,, \tag{1.6}$$

which are the Hubble parameter and conformal Hubble parameter, respectively.

Finally, since there are now several metrics defined, it is useful to introduce conventions which metric is to be used for raising and lowering indices. For spacetime indices, we will use the metric $\bar{g}_{\mu\nu}$, which is valid since we are considering only perturbations to linear order here. For raising and lowering space indices, however, we use the metric γ_{ij}, whose inverse we write as γ^{ij}. Hence, we will consider objects carrying Greek indices as tensors on spacetime M, equipped with metric $\bar{g}_{\mu\nu}$, but objects with Latin indices as tensors on Σ (with an additional time dependence), equipped with metric γ_{ij}. Also note that the Christoffel symbols of these metrics are related, and we find in particular the spatial Christoffel symbols of the spacetime metric to be given by

$$\begin{aligned}\bar{\Gamma}^i{}_{jk} &= \frac{1}{2}\bar{g}^{i\mu}(\partial_j\bar{g}_{\mu k} + \partial_k\bar{g}_{j\mu} - \partial_\mu\bar{g}_{jk}) \\ &= \frac{1}{2a^2}\gamma^{il}a^2(\partial_j\gamma_{lk} + \partial_k\gamma_{jl} - \partial_l\gamma_{jk}) \\ &= \frac{1}{2}\gamma^{il}(\partial_j\gamma_{lk} + \partial_k\gamma_{jl} - \partial_l\gamma_{jk})\,,\end{aligned} \tag{1.7}$$

and are thus just the Christoffel symbols of the spatial metric γ_{ij}, since the scale factor a, which depends only on time, but not on spatial coordinates, cancels. In the following, we will write $\bar{\nabla}_\mu$ for the covariant derivative of $\bar{g}_{\mu\nu}$, and d_i for the covariant derivative of γ_{ij}. Also it is convenient to write

$$\triangle = \gamma^{ij}\mathrm{d}_i\mathrm{d}_j \tag{1.8}$$

for the spatial Laplace operator.

2 Metric perturbation

We now consider a metric which is a linear perturbation around the homogeneous and isotropic background, and so can be written as

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu}\,. \tag{2.1}$$

For the perturbation, it is then common to introduce a space-time split in the form

$$\delta g_{\mu\nu}\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu = a^2\left[-2\phi\,\mathrm{d}\eta\otimes\mathrm{d}\eta + \tilde{B}_i(\mathrm{d}\eta\otimes\mathrm{d}x^i + \mathrm{d}x^i\otimes\mathrm{d}\eta) + 2(\tilde{E}_{ij} - \psi\gamma_{ij})\mathrm{d}x^i\otimes\mathrm{d}x^j\right]\,, \tag{2.2}$$

where $\tilde{E}_{ij}$ is trace-free,

$$\gamma^{ij}\tilde{E}_{ij} = 0\,, \tag{2.3}$$

so that the trace of the spatial part of the perturbation is described by ψ only. Note that while the background metric depends only on time, all metric perturbations depend on both space and time coordinates. The objects $\tilde{B}_i$ and $\tilde{E}_{ij}$ are then further decomposed as

$$\tilde{B}_i = \mathrm{d}_iB + B_i\,, \quad \tilde{E}_{ij} = \mathrm{d}_i\mathrm{d}_jE + 2\mathrm{d}_{(i}E_{j)} + E_{ij}\,, \tag{2.4}$$

which satisfy the relations

$$\gamma^{ij}\mathrm{d}_iB_j = \gamma^{ij}\mathrm{d}_iE_j = 0\,, \quad \gamma^{ij}\mathrm{d}_iE_{jk} = 0\,, \quad \gamma^{ij}E_{ij} = 0\,. \tag{2.5}$$

The perturbation variables are now the scalars ϕ, ψ, B, E, divergence-free vectors B_i, E_i and trace-free, divergence-free symmetric tensor E_{ij} on the spatial manifold Σ. The reason for this decomposition is that these three different types of perturbations turn out to decouple, i.e., they satisfy independent differential equations, and can thus be described separately.

3 Example: Einstein equations for tensor perturbation

To see how the field equations indeed simplify, and as a glimpse towards their full decomposition, we take a look at a pure tensor perturbation, which means that we assume that all perturbations except E_{ij} vanish. For reasons which will become apparent in another lecture, we start from the Einstein equations in the mixed indices form

$$G^\mu{}_\nu = 8\pi G T^\mu{}_\nu \quad \Rightarrow \quad \bar{G}^\mu{}_\nu = 8\pi G\bar{T}^\mu{}_\nu\,, \quad \delta[G^\mu{}_\nu] = 8\pi G\delta[T^\mu{}_\nu]\,. \tag{3.1}$$

For simplicity, we also assume that the matter perturbation vanishes, $\delta[T^\mu{}_\nu] = 0$, so that we can restrict ourselves to the equation

$$\delta[G^\mu{}_\nu] = 0\,. \tag{3.2}$$

Note that we used square brackets here to indicate that the object whose perturbation we calculate is the Einstein tensor with mixed indices. This distinction is important, since the Einstein tensor of the background does not vanish (in contrast to a Minkowski background), so that the perturbations are related by

$$\delta[G^{\mu}{}_{\nu}] = \delta[g^{\mu\rho}G_{\rho\nu}] = \bar{g}^{\mu\rho}\delta[G_{\rho\nu}] + \delta[g^{\mu\rho}]\bar{G}_{\rho\nu} = \bar{g}^{\mu\rho}\delta[G_{\rho\nu}] - \delta[g_{\omega\tau}]\bar{g}^{\mu\omega}\bar{g}^{\rho\tau}\bar{G}_{\rho\nu}\,. \tag{3.3}$$

We start by evaluating the first term. Recall that we can calculate the perturbation of the Einstein tensor with lower indices from the formula

$$\begin{aligned}\delta[G_{\mu\nu}] = &\frac{1}{2}\left(\bar{R}^{\rho\sigma}\delta g_{\rho\sigma} + \bar{\nabla}^{\rho}\bar{\nabla}_{\rho}\delta g^{\sigma}{}_{\sigma} - \bar{\nabla}^{\rho}\bar{\nabla}^{\sigma}\delta g_{\rho\sigma}\right)\bar{g}_{\mu\nu} \\ &+ \frac{1}{2}\left(\bar{\nabla}^{\rho}\bar{\nabla}_{\mu}\delta g_{\nu\rho} + \bar{\nabla}^{\rho}\bar{\nabla}_{\nu}\delta g_{\mu\rho} - \bar{\nabla}^{\rho}\bar{\nabla}_{\rho}\delta g_{\mu\nu} - \bar{\nabla}_{\mu}\bar{\nabla}_{\nu}\delta g^{\rho}{}_{\rho} - \bar{R}\delta g_{\mu\nu}\right)\,.\end{aligned} \tag{3.4}$$

To derive this result, it is helpful to perform a number of steps. First, we write the inverse background metric explicitly in all contractions, so that we can more easily calculate the space-time split:

$$\begin{aligned}\delta[G_{\mu\nu}] = &\frac{1}{2}\bar{g}^{\rho\sigma}\bar{g}^{\omega\tau}\left(\bar{R}_{\rho\omega}\delta g_{\tau\sigma} + \bar{\nabla}_{\rho}\bar{\nabla}_{\sigma}\delta g_{\omega\tau} - \bar{\nabla}_{\rho}\bar{\nabla}_{\omega}\delta g_{\tau\sigma}\right)\bar{g}_{\mu\nu} \\ &+ \frac{1}{2}\bar{g}^{\rho\sigma}\left(\bar{\nabla}_{\rho}\bar{\nabla}_{\mu}\delta g_{\nu\sigma} + \bar{\nabla}_{\rho}\bar{\nabla}_{\nu}\delta g_{\mu\sigma} - \bar{\nabla}_{\rho}\bar{\nabla}_{\sigma}\delta g_{\mu\nu} - \bar{\nabla}_{\mu}\bar{\nabla}_{\nu}\delta g_{\rho\sigma}\right) - \frac{1}{2}\bar{R}\delta g_{\mu\nu}\,.\end{aligned} \tag{3.5}$$

Next, we can simplify a few terms. Since we consider only the spatial tensor perturbation E_{ij}, it follows from

$$\delta g_{\mu\nu}\mathrm{d}x^{\mu}\otimes\mathrm{d}x^{\nu} = 2a^2E_{ij}\mathrm{d}x^i\otimes\mathrm{d}x^j \tag{3.6}$$

that $\delta g_{\mu\nu}$ has only spatial components. With this knowledge, we can rewrite the last term in brackets in the second line as

$$\bar{g}^{\rho\sigma}\bar{\nabla}_{\mu}\bar{\nabla}_{\nu}\delta g_{\rho\sigma} = \bar{\nabla}_{\mu}\bar{\nabla}_{\nu}(\bar{g}^{\rho\sigma}\delta g_{\rho\sigma}) = \bar{\nabla}_{\mu}\bar{\nabla}_{\nu}(\bar{g}^{ij}\delta g_{ij}) = \bar{\nabla}_{\mu}\bar{\nabla}_{\nu}(a^{-2}\gamma^{ij}\cdot 2a^2E_{ij}) = 0\,, \tag{3.7}$$

since the background metric is covariantly constant with respect to its Levi-Civita connection and the trace of E_{ij} vanishes. By the same argument, we can also eliminate the term

$$\bar{g}^{\rho\sigma}\bar{g}^{\omega\tau}\bar{\nabla}_{\rho}\bar{\nabla}_{\sigma}\delta g_{\omega\tau} = 0 \tag{3.8}$$

from the first line. Similarly, the first term yields

$$\bar{g}^{\rho\sigma}\bar{g}^{\omega\tau}\bar{R}_{\rho\omega}\delta g_{\tau\sigma} = \bar{g}^{ij}\bar{g}^{kl}\bar{R}_{ik}\delta g_{jl} = a^{-4}\gamma^{ij}\gamma^{kl}\cdot(\mathcal{H}' + 2\mathcal{H}^2 + 2K)\gamma_{ik}\cdot 2a^2E_{jl} = 0\,, \tag{3.9}$$

using the Ricci tensor components

$$\bar{R}_{ij} = (\mathcal{H}' + 2\mathcal{H}^2 + 2K)\gamma_{ij}\,. \tag{3.10}$$

The remaining terms are now given by

$$\delta[G_{\mu\nu}] = -\frac{1}{2}\bar{g}_{\mu\nu}\bar{g}^{\rho\sigma}\bar{g}^{\omega\tau}\bar{\nabla}_{\rho}\bar{\nabla}_{\omega}\delta g_{\tau\sigma} + \frac{1}{2}\bar{g}^{\rho\sigma}\left(\bar{\nabla}_{\rho}\bar{\nabla}_{\mu}\delta g_{\nu\sigma} + \bar{\nabla}_{\rho}\bar{\nabla}_{\nu}\delta g_{\mu\sigma} - \bar{\nabla}_{\rho}\bar{\nabla}_{\sigma}\delta g_{\mu\nu}\right) - \frac{1}{2}\bar{R}\delta g_{\mu\nu}\,. \tag{3.11}$$

We can eliminate the first term by evaluating the expression

$$\begin{aligned}\bar{g}^{\omega\tau}\bar{\nabla}_{\omega}\delta g_{\tau\sigma}\mathrm{d}x^{\sigma} &= \bar{g}^{\omega\tau}\left(\partial_{\omega}\delta g_{\tau\sigma} - \bar{\Gamma}^{\rho}{}_{\tau\omega}\delta g_{\rho\sigma} - \bar{\Gamma}^{\rho}{}_{\sigma\omega}\delta g_{\tau\rho}\right)\mathrm{d}x^{\sigma} \\ &= \gamma^{jk}\left(\partial_k E_{ij} - \bar{\Gamma}^{l}{}_{jk}E_{il} - \bar{\Gamma}^{l}{}_{ij}E_{kl}\right)\mathrm{d}x^i + \mathcal{H}\delta^l_j\gamma^{jk}E_{kl}\mathrm{d}\eta \\ &= \mathrm{d}^jE_{ij}\mathrm{d}x^i + \mathcal{H}E_i{}^i\mathrm{d}\eta \\ &= 0\,,\end{aligned} \tag{3.12}$$

since the tensor perturbations are trace-free and divergence-free, so that we are left with

$$\delta[G_{\mu\nu}] = \frac{1}{2}\bar{g}^{\rho\sigma}\left(\bar{\nabla}_{\rho}\bar{\nabla}_{\mu}\delta g_{\nu\sigma} + \bar{\nabla}_{\rho}\bar{\nabla}_{\nu}\delta g_{\mu\sigma} - \bar{\nabla}_{\rho}\bar{\nabla}_{\sigma}\delta g_{\mu\nu}\right) - \frac{1}{2}\bar{R}\delta g_{\mu\nu}\,. \tag{3.13}$$

Before we can apply the same divergence rule to the first and second term, we must reorder the covariant derivatives, which introduces the curvature of the background metric, since

$$\bar{\nabla}_\rho\bar{\nabla}_\mu\delta g_{\nu\sigma} = \bar{\nabla}_\mu\bar{\nabla}_\rho\delta g_{\nu\sigma} - \bar{R}^\omega{}_{\nu\rho\mu}\delta g_{\omega\sigma} - \bar{R}^\omega{}_{\sigma\rho\mu}\delta g_{\nu\omega}\,. \tag{3.14}$$

Here the first term vanishes, since it is contracted to a divergence, while the remaining terms yield

$$-\frac{1}{2}\bar{g}^{\rho\sigma}\bar{R}^\omega{}_{\nu\rho\mu}\delta g_{\omega\sigma}\mathrm{d}x^\mu\otimes\mathrm{d}x^\nu = (\mathcal{H}^2 + K)E_{ij}\mathrm{d}x^i\otimes\mathrm{d}x^j\,, \tag{3.15a}$$

$$-\frac{1}{2}\bar{g}^{\rho\sigma}\bar{R}^\omega{}_{\sigma\rho\mu}\delta g_{\nu\omega}\mathrm{d}x^\mu\otimes\mathrm{d}x^\nu = (\mathcal{H}' + 2\mathcal{H}^2 + 2K)E_{ij}\mathrm{d}x^i\otimes\mathrm{d}x^j\,. \tag{3.15b}$$

Note that each of these terms appears twice, since it appears again with swapped indices $\mu \leftrightarrow \nu$, in which it is symmetric. We then use the Ricci scalar for the background metric, which is given by

$$\bar{R} = 6a^{-2}(\mathcal{H}' + \mathcal{H}^2 + K)\,, \tag{3.16}$$

so that the last term becomes

$$-\frac{1}{2}\bar{R}\delta g_{\mu\nu}\mathrm{d}x^\mu\otimes\mathrm{d}x^\nu = -6(\mathcal{H}' + \mathcal{H}^2 + K)E_{ij}\mathrm{d}x^i\otimes\mathrm{d}x^j\,. \tag{3.17}$$

Finally, a rather tedious calculation which involves expanding the Christoffel symbols, yields

$$-\frac{1}{2}\bar{g}^{\rho\sigma}\bar{\nabla}_\rho\bar{\nabla}_\sigma\delta g_{\mu\nu}\mathrm{d}x^\mu\otimes\mathrm{d}x^\nu = \left(E''_{ij} + 2\mathcal{H}E'_{ij} - \triangle E_{ij} - 2\mathcal{H}^2E_{ij}\right)\mathrm{d}x^i\otimes\mathrm{d}x^j \tag{3.18}$$

We thus find that all components except the spatial ones vanish, and that the latter are given by

$$\delta[G_{ij}] = E''_{ij} + 2\mathcal{H}E'_{ij} - \triangle E_{ij} - (4\mathcal{H}' + 2\mathcal{H}^2)E_{ij}\,. \tag{3.19}$$

Returning to the original problem, we still need the background value of the Einstein tensor. Also here we need only the spatial components, since it will be contracted with the metric perturbation, for which we likewise consider only spatial components. These are given by

$$\bar{G}_{ij} = -(2\mathcal{H}' + \mathcal{H}^2 + K)\gamma_{ij}\,. \tag{3.20}$$

From this we find the term

$$-\delta[g_{ik}]\bar{g}^{kl}\bar{G}_{lj} = (4\mathcal{H}' + 2\mathcal{H}^2 + 2K)E_{ij}\,. \tag{3.21}$$

Combining all terms, we finally arrive at the result

$$0 = \bar{g}_{ik}\delta[G^k{}_j] = \delta[G_{ij}] - \delta[g_{ik}]\bar{g}^{kl}\bar{G}_{lj} = E''_{ij} + 2\mathcal{H}E'_{ij} - \triangle E_{ij} + 2KE_{ij}\,. \tag{3.22}$$

We now see that the only non-vanishing contribution to the field equations which arises from the spatial, divergence-free, trace-free tensor perturbation is itself a spatial, divergence-free, trace-free tensor. This is the characteristic property of the decomposition we introduced, that any of the three different types of metric perturbations only leads to the same type of perturbation in the field equations. Hence, decomposing the field equations in the same way as the metric, one finds that the different types of perturbations decouple.

The equation we obtained describes the propagation of gravitational waves on a homogeneous and isotropic, but not necessarily flat, Friedmann-Lemaître-Robertson-Walker background. Note that, properly taking into account the scale factor a for measuring time and distance, gravitational waves still propagate at the speed of light, as we also saw for the Minkowski background. However, there is an important difference: while the frequency and wavelength of a gravitational wave governed by the wave equation are constant, and this also holds here in the coordinates η and x^i, the actual, physical frequency and wavelength are related to these by the scale factor a, so that the frequency decreases and wavelength increases, as a grows. Hence, gravitational waves undergo the same redshift as electromagnetic waves. Further, note the appearance of a few additional terms. The term $2\mathcal{H}E'_{ij}$ is a damping term. Its appearance indicates that the amplitude of the gravitational wave decreases as $\mathcal{H} > 0$. Finally, the last term $2KE_{ij}$ accounts for the fact that on a curved spatial background, the eigenfunctions of the wave operator are not plane waves, but harmonics.