Differential geometry for physicists - Lecture 16

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54 Gauge transformation of matter fields

In physics we often encounter theories which are invariant under "local symmetries", i.e., "local actions" of a group. Many examples come from field theory, such as electromagnetism, which is locally invariant under U(1), or the strong interaction between quarks, which is locally invariant under SU(3). We will now discuss how these theories can be described using the geometrical notions we introduced in the previous lectures. In this section we start by introducing a few important notions in the way they are used in the context of gauge theories. One conventionally distinguishes two types of fields in these theories - *gauge fields* and *matter fields*. The latter is defined as follows.

Definition 54.1 (Matter field)**.** A *matter field* is a section $\Phi : M \to P \times_\rho F$ of a fiber bundle $\pi_\rho : P \times_\rho F \to M$ with fiber F associated to a principal G-bundle $\pi : P \to M$ with Lie group G.

Recall that the total space $P \times_\rho F$ is constituted by equivalence classes $[p, f]$, where $p \in P$, $f \in F$ and equivalence is defined by $(p, f) \sim (p \cdot g, \rho(g^{-1}, f))$. We now pose the question how matter fields change if we perform an operation on the principal bundle which preserves its fibers and the right action of the structure group G. We define this operation as follows.

Definition 54.2 (Gauge transformation)**.** Let $\pi : P \to M$ be a principal G-bundle with Lie group G. A *gauge transformation* is a vertical principal bundle automorphism of P, i.e., a diffeomorphism $\varphi : P \to P$ such that $\pi \circ \varphi = \pi$ and $R_g \circ \varphi = \varphi \circ R_g$ for all $g \in G$. The gauge transformations form a group, which is called the *gauge group*.

Every gauge transformation φ induces a transformation φ_ρ of the associated bundle $P \times_\rho F$ given by $\varphi_\rho([p, f]) = [\varphi(p), f]$. This is well defined, since

$$\varphi_\rho([p \cdot g, \rho(g^{-1}, f)]) = [\varphi(p \cdot g), \rho(g^{-1}, f)] = [\varphi(p) \cdot g, \rho(g^{-1}, f)] = [\varphi(p), f] = \varphi_\rho([p, f])\,. \tag{54.1}$$

This action is fiber preserving and thus induces an action on the space $\Gamma(P \times_\rho F)$ of sections, where

$$\varphi_\rho(\Phi) = \varphi_\rho \circ \Phi\,. \tag{54.2}$$

Further, we obtain an action on the jet bundles $J^r(P \times_\rho F)$, which is defined such that

$$\varphi_\rho(j^r_x\Phi) = j^r_x\varphi_\rho(\Phi)\,. \tag{54.3}$$

To see that this is well-defined, we have to check that $\varphi_\rho(j^r_x\Phi)$ is independent of the choice of the representative Φ, i.e., that $j^r_x\varphi_\rho(\Phi) = j^r_x\varphi_\rho(\Phi')$ for $j^r_x\Phi = j^r_x\Phi'$. This is indeed the case, which can easily be proven using the fact that $\varphi_\rho : P \times_\rho F \to P \times_\rho F$ is a bundle isomorphism.

55 Gauge transformation of gauge fields

The second ingredient we will need is the notion of a gauge field. Essentially, a gauge field is a principal Ehresmann connection, which is a G-equivariant section ω of the jet bundle $\pi_{1,0} : J^1(P) \to P$ over a principal bundle $\pi : P \to M$. Here we are in a similar situation as in the case of matter fields. Recall that matter fields, i.e., sections $\Phi \in \Gamma(P \times_\rho F)$, can also be understood as G-equivariant maps $\Phi \in C^\infty_G(P, F)$. The situation here is a bit different, since we do not consider arbitrary G-equivariant maps from P to $J^1(P)$, but only sections. However, it is indeed possible to consider gauge fields as sections of a bundle over M, which we construct as follows.

Definition 55.1 (Principal connection bundle)**.** Let $\pi : P \to M$ be a principal G-bundle with Lie group G and $J^1(P)$ the first jet space. The space $C = J^1(P)/G$ of G-orbits in $J^1(P)$ together with the canonical projection $\chi : C \to M$ defines a fiber bundle called the *principal connection bundle*.

To check that this construction is valid, first note that the right actions of G on both P and $J^1(P)$ are free, i.e., for each $p \in P$ the subgroup of G sending p to itself contains only the unit element of G, and analogously for $J^1(P)$. As a consequence, all group orbits in P and $J^1(P)$ are diffeomorphic to G. The orbits in P are simply the fibers of the bundle $\pi : P \to M$, so that the space P/G of orbits is canonically diffeomorphic to M. We denote by $C = J^1(P)/G$ the space of orbits in $J^1(P)$. Note that the projection $\pi_1 : J^1(P) \to M$ satisfies $\pi_1 \circ R_g = \pi_1$ for all $g \in G$, i.e., it sends all elements of an orbit to the same image in M. Thus, there is a unique projection $\chi : C \to M$. One easily checks that this defines a fiber bundle. Its sections should already be familiar, as the following theorem states.

Theorem 55.1. *There is a one-to-one correspondence between principal Ehresmann connections in a principal G-bundle $\pi : P \to M$ and sections of its principal connection bundle $\chi : C \to M$.*

Proof. Let $\omega : P \to J^1(P)$ be a principal Ehresmann connection. Since ω is an equivariant map, it preserves the orbits, i.e., if p and p' belong to the same orbit in P, then $\omega(p)$ and $\omega(p')$ belong to the same orbit in $J^1(P)$. Thus, ω defines a map $\Omega : M \to C$ sending orbits in P to orbits in $J^1(P)$, such that the diagram

$$\begin{array}{ccc} P & \xrightarrow{\ \omega\ } & J^1(P) \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\bullet\cdot G} \\ M & \xrightarrow{\ \Omega\ } & C \end{array} \tag{55.1}$$

commutes, where the map on the right is simply the canonical projection onto the space of orbits. Further, ω is a section of the bundle $\pi_{1,0} : J^1(P) \to P$, so that $\pi_{1,0} \circ \omega = \mathrm{id}_P$. Thus, for all $p \in P$ we have

$$\chi(\Omega(\pi(p))) = \chi(\omega(p) \cdot G) = \pi(\pi_{1,0}(\omega(p))) = \pi(p)\,, \tag{55.2}$$

which follows from the fact that $\pi_{1,0}$ is G-equivariant and thus $\pi_1 = \pi \circ \pi_{1,0} = \chi \circ (\bullet \cdot G)$. This shows that Ω is a section of the principal connection bundle $\chi : C \to M$.

Conversely, let $\Omega : M \to C$ be a section of the principal connection bundle. For $p \in P$, define $\omega(p)$ as the unique jet in $J^1(P)$ such that $\pi_{1,0}(\omega(p)) = p$ and $\omega(p) \cdot G = \chi(x)$. One easily checks that the jet $\omega(p)$ defined this way always exists, that it is unique and that the resulting map $\omega : P \to J^1(P)$ is a principal Ehresmann connection. □

With this statement at hand, we can now come to the following definition.

Definition 55.2 (Gauge field)**.** Let $\pi : P \to M$ be principal G-bundle with Lie group G. A *gauge field* is a section Ω of the principal connection bundle $\chi : C \to M$.

We finally discuss the question how gauge transformations act on sections of the principal connection bundle. The easiest way to construct such an action is to use G-equivariant connection forms $\theta : TP \to VP$. In this case we can simply define $\varphi(\theta) = \varphi_*^{-1} \circ \theta \circ \varphi_*$. To see that this is again a G-equivariant connection form, first note that by construction, $\varphi(\theta)$ is a vector bundle homomorphism covering the identity on P. Further, note that the pushforward of fundamental vector yields

$$\varphi_*(\tilde{X}(p)) = \varphi_*(R_*^p(X(e))) = (\varphi \circ R^p)_*(X(e)) = R_*^{\varphi(p)}(X(e)) = \tilde{X}(\varphi(p))\,, \tag{55.3}$$

so that the fundamental vector fields are invariant under the action of a gauge transformation. From this in particular follows that the vertical tangent bundle VP is invariant under φ, i.e., $\varphi_*(v) \in VP$ for all $v \in VP$. We thus have

$$\varphi(\theta)(v) = (\varphi_*^{-1} \circ \theta \circ \varphi_*)(v) = (\varphi_*^{-1} \circ \varphi_*)(v) = v\,, \tag{55.4}$$

so that $\varphi(\theta)$ restricts to the identity on VP. Finally, for all $g \in G$ and $w \in TP$ we find

$$\varphi(\theta)(R_{g*}(w)) = (\varphi_*^{-1} \circ \theta \circ \varphi_* \circ R_{g*})(w) = (R_{g*} \circ \varphi_*^{-1} \circ \theta \circ \varphi_*)(w) = R_{g*}(\varphi(\theta)(w))\,, \tag{55.5}$$

where we used the fact that all maps appearing above are G-equivariant, so that we can permute R_{g*} to the left. This shows that also $\varphi(\theta)$ is G-equivariant.

With this preliminary discussion we can now describe gauge transformations of principal Ehresmann connections, and thus of gauge fields. Let $\omega : P \to J^1(P)$ be a principal Ehresmann connection, which assigns to $p \in P$ with $\pi(p) = x$ a jet $\omega(p) = j_x^1\sigma_p \in J^1(P)$, and θ the corresponding connection form. We define $\varphi(\omega)$ as the principal Ehresmann connection corresponding to $\varphi(\theta)$. Then we have $\varphi(\omega)(p) = j_x^1(\varphi^{-1} \circ \sigma_{\varphi(p)})$. To check this, we calculate

$$\begin{aligned}\varphi(\theta)_p(w) &= w - \varphi_*^{-1}(\sigma_{\varphi(p)*}(\pi_*(w)))\\ &= \varphi_*^{-1}\left[\varphi_*(w) - \sigma_{\varphi(p)*}(\pi_*(\varphi_*(w)))\right]\\ &= \varphi_*^{-1}(\theta_{\varphi(p)}(\varphi_*(w)))\,,\end{aligned} \tag{55.6}$$

which shows that our formula is correct. Finally, taking the quotient by the group action of G yields the action of the gauge group on the space $\Gamma(C)$ of sections of the connection bundle $\chi : C \to M$.

56 Gauge fixing

Working with bundles, whose elements are equivalence classes of sections or orbits of a group action, can sometimes become rather involved. In order to construct local coordinates on these spaces one therefore usually constructs particular local trivializations of a bundle. Recall that for a principal bundle a local trivialization is given by a local section. We thus define the following notion.

Definition 56.1 (Gauge)**.** Let $\pi : P \to M$ be a principal G-bundle with Lie group G. A *gauge* on an open subset $U \subset M$ is a local section $\epsilon : U \to P$.

With the choice of a gauge we can now express matter and gauge fields in a simpler form, which is of course valid only locally, i.e., only on U, and depends on the choice of the gauge. For a matter field $\Phi : M \to P \times_\rho F$ we define

$$\begin{array}{cccc} \Phi^\epsilon & : & U & \to & F \\ & & x & \mapsto & [\epsilon(x)]^{-1}(\Phi(x)) \end{array} . \tag{56.1}$$

Here $[p]$ for $p \in P$ is the fiber diffeomorphism $[p] : F \to P_{\pi(p)} \times_\rho F, f \mapsto [p, f]$. Given coordinates on U and F, we thus obtain a coordinate description for Φ.
For a gauge field Ω we can proceed similarly. The easiest way is to view the gauge field in terms of the corresponding principal G-connection ϑ on P, and thus in particular as a $\mathfrak{g}$-valued one-form $\vartheta \in \Omega^1(P, \mathfrak{g})$ on P. Given a gauge ϵ we can thus define

$$\Omega^\epsilon = \epsilon^*(\vartheta) \in \Omega^1(M, \mathfrak{g}) . \tag{56.2}$$

Thus, the connection pulls back to a $\mathfrak{g}$-valued one-form on M. This is the description most often encountered in field theory.

57 Gauge invariance and Lagrangians

Finally, we pose the question how to treat theories involving gauge and matter fields using the Lagrangian formalism we introduced in a previous lecture. Here we restrict ourselves to theories in which there is only one Lie group G and one principal G-bundle $\pi : P \to M$ and summarize all matter fields within a single associated bundle $\pi_\rho : P \times_\rho F \to M$. Thus, a field configuration is given by a gauge field $\Omega : M \to C$ and a matter field $\Phi : M \to P \times_\rho F$. We can combine both into a section (Ω, Φ) of the Cartesian product bundle

$$E = C \times_M (P \times_\rho F) = \bigcup_{x \in M} C_x \times (P_x \times_\rho F) , \tag{57.1}$$

whose fibers are the Cartesian products of the fibers of C and $P \times_\rho F$, and which canonically inherits a bundle projection $\Pi : E \to M$. The gauge group acts on both Ω and Φ, and thus also on the pair (Ω, Φ). This defines an action of the gauge group on sections of E, and thus also on the jet bundles $J^r(E)$, and finally on the ∞-jet bundle $J^\infty(E)$. This allows us to define the following notion.

Definition 57.1 (Gauge invariant Lagrangian)**.** A Lagrangian $L \in \Omega^{n,0}(J^\infty(E))$ is called *gauge invariant* if it is invariant under the action of the gauge group on $J^\infty(E)$.

A Dictionary

English	Estonian
gauge	kalibratsioon
gauge transformation	kalibratsiooniteisendus
gauge field	kalibratsiooniväli
matter field	mateeriaväli