Differential geometry for physicists - Lecture 15

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51 Connections on fiber bundles

In this lecture we will discuss different types of *connections*. The most general type we discuss here is called an *Ehresmann connection*. There are many different, but equivalent ways to define Ehresmann connections. The one we use here is probably the most elegant and makes use of the notion of the jet bundle, which we discussed in a previous lecture.

Definition 51.1 (Ehresmann connection)**.** Let $\pi : E \to M$ be a fiber bundle. An *Ehresmann connection* is a section of the jet bundle $\pi_{1,0} : J^1(E) \to E$.

To better understand the geometric meaning of this definition, recall that an element of $J^1(E)$ is an equivalence class of local sections around a point $x \in M$, i.e., maps $\sigma : U_\sigma \to E$ with $x \in U_\sigma$ for an open subset $U_\sigma \subset M$, where two local sections σ, τ are considered equivalent if for all curves $\gamma \in C^\infty(\mathbb{R}, U_\sigma \cap U_\tau)$ with $\gamma(0) = x$ and all functions $f \in C^\infty(E, \mathbb{R})$ holds

$$(f \circ \sigma \circ \gamma)(0) = (f \circ \tau \circ \gamma)(0) \quad \text{and} \quad (f \circ \sigma \circ \gamma)'(0) = (f \circ \tau \circ \gamma)'(0)\,. \tag{51.1}$$

The first condition simply translates to $\sigma(x) = \tau(x)$, while the second condition can be written as $\sigma_*(u) = \tau_*(u)$ for all $u \in T_xM$. The equivalence class of σ, for which we introduced the notation $j^1_x\sigma$, is thus fully characterized by the following data:

- the point $\pi_1(j^1_x\sigma) = x \in M$,
- the image $\pi_{1,0}(j^1_x\sigma) = \sigma(x) \in E_x = \pi^{-1}(x)$,
- a linear map $\sigma_*|_x : T_xM \to T_{\sigma(x)}E$ such that $\pi_* \circ \sigma_*|_x = \mathrm{id}_{T_xM}$.

In the given case we are interested in sections ω of the bundle $\pi_{1,0} : J^1(E) \to E$. By definition of the section we have $\pi_{1,0} \circ \omega = \mathrm{id}_E$. For all $e \in E$ thus follows that $\omega(e) \in \pi_{1,0}^{-1}(e)$, so that $\omega(e)$ must be of the form $j^1_x\sigma$ with $\sigma(x) = e$ and $x = \pi(e)$. This requirement already uniquely fixes the first two items from the list above, so that in order to specify a section ω we only need to supply the last item. To see how much freedom we have for choosing this item, we consider two different jets $j^1_x\sigma, j^1_x\tau$ (where this time σ and τ should be sections of $\pi : E \to M$ which are *not* in the same equivalence class, i.e., define different jets). Since we are dealing with linear maps, we can take the difference

$$0 = (\pi_* \circ \sigma_*|_x) - (\pi_* \circ \tau_*|_x) = \pi_* \circ (\sigma_*|_x - \tau_*|_x)\,, \tag{51.2}$$

which shows that the image of $\sigma_*|_x - \tau_*|_x$ must be contained in the vertical tangent space V_eE. In other words, for any $u \in T_xM$, the image $\sigma_*|_x(u) - \tau_*|_x(u)$ is vertical. Note, however, that $\sigma_*|_x(u)$ is *not* vertical, so that it is *not* sufficient to specify a vertical vector only. The reason for this is that the condition $\pi_* \circ \sigma_*|_x = \mathrm{id}_{T_xM}$ specifies an *affine* space, i.e., the difference of any two such linear maps $\sigma_*|_x, \tau_*|_x$ lies in the vector space $\mathrm{Hom}(T_xM, V_eE)$, but the space of linear maps satisfying this condition is not a vector space.
To further illustrate this construction, we introduce coordinates (x^α) on a trivializing neighborhood $U \subset M$ and (y^a) on the fiber space of the bundle $\pi : E \to M$, so that we have coordinates (x^α, y^a) on E and the projection π simply discards the second part of these coordinates. We can denote the coordinates on the first jet space by $(x^\alpha, y^a, y^a_\alpha)$. In these coordinates a section of the bundle $\pi_{1,0} : J^1(E) \to E$ is thus expressed by a set of coordinate functions $y^a_\alpha(x, y)$.
A word of warning should be issued here. The coordinates y^a_α *look like* coordinates for a map $(u^\alpha\partial_\alpha \mapsto u^\alpha y^a_\alpha \bar{\partial}_a) \in \mathrm{Hom}(T_xM, V_eE)$, but as we have seen in our coordinate free introduction, they are *not*. The reason for this coordinate expression is simply that while introducing coordinates on E we have fixed a local trivialization. If we choose a different trivialization, these components will not transform as components of a vector space, but as components of an affine space, which is what they really are.
There exist a number of other, equivalent ways to describe an Ehresmann connection, which we will discuss in the following. For this purpose, we introduce another notion by the following definition.

Definition 51.2 (Connection form)**.** Let $\pi : E \to M$ be a fiber bundle. A *connection form* on E is a vector bundle homomorphism $\theta : TE \to VE$ covering the identity map id_E on E and restricting to the identity map on VE, i.e., $\theta|_{VE} = \mathrm{id}_{VE}$.

This definition requires a few explanations. A bundle morphism $\theta : TE \to VE$ *covers* a map $\varphi : E \to E$ if $\theta(w) \in V_{\varphi(e)}E$ for all $w \in T_eE$. In this case it covers the identity, so that $\varphi = \mathrm{id}_E$. Further, θ is a vector bundle homomorphism, which means that each restriction $\theta|_e : T_eE \to V_eE$ is linear. Further, it is in fact a projection onto VE, since $\theta(w) \in VE$ for all $w \in TE$ and θ restricts to the identity on VE, so that $\theta \circ \theta = \theta$.
We also illustrate this definition using the same local coordinates (x^α, y^a) on E which we introduced earlier. Writing a tangent vector $w \in T_eE$ in the form $w = u^\alpha\partial_\alpha + v^a\bar{\partial}_a$ we obtain coordinates (u^α, v^a) on T_eE, (v^a) on V_eE and thus $(x^\alpha, y^a, u^\alpha, v^a)$ on TE and (x^α, y^a, v^a) on VE. In these coordinates a connection form θ can be expressed in the form

$$\theta(x, y, u, v) = (u^\alpha\theta^a_\alpha(x, y) + v^a)\,\bar{\partial}_a \in V_{(x,y)}E\,. \tag{51.3}$$

A connection form is thus uniquely determined by the coordinate functions $\theta^a_\alpha(x, y)$. Note that these have the same index structure as the coordinate functions $y^a_\alpha(x, y)$ we constructed for an Ehresmann connection. This is not an arbitrary coincidence. On a more fundamental level, note that also connection forms form an affine space and not a vector space. The following theorem should thus not be too surprising.

Theorem 51.1. *For every fiber bundle $\pi : E \to M$ there is a one-to-one correspondence between Ehresmann connections and connection forms on E.*

Proof. We have seen that an Ehresmann connection assigns to each $e \in E$ with $\pi(e) = x$ a jet $j^1_x\sigma_e$ with $\pi_1(j^1_x\sigma_e) = x$ and $\pi_{1,0}(j^1_x\sigma_e) = \sigma_e(x) = e$ such that $\pi_* \circ \sigma_{e*}|_x = \mathrm{id}_{T_xM}$, and that the latter is the only ingredient that differs between different Ehresmann connections. Given this jet we can define for each $e \in E$ a linear function

$$\begin{array}{cccc} \theta_e & : & T_eE & \to & V_eE \\ & & w & \mapsto & w - \sigma_{e*}(\pi_*(w)) \end{array} . \tag{51.4}$$

It is clear that $\theta_e(w) \in V_eE$, since

$$\pi_*(\theta_e(w)) = \pi_*(w) - \pi_*(\sigma_{e*}(\pi_*(w))) = \pi_*(w) - \pi_*(w) = 0 . \tag{51.5}$$

Further, for $w \in V_eE$, we have $\pi_*(w) = 0$, and thus $\theta_e(w) = w$. Together with the linearity it follows that θ_e is a projection onto V_eE. Finally, since we have such a function θ_e for all $e \in E$, they constitute a map $\theta : TE \to VE$ which covers the identity on E. One easily checks that θ is a connection form.

Conversely, let θ be a connection form on E, i.e., a vector bundle homomorphism $\theta : TE \to VE$ covering the identity map id_E on E and restricting to the identity map on VE. For each $e \in E$ it thus defines a projection $\theta|_e : T_eE \to V_eE$. Let σ_e be a local section of the bundle $\pi : E \to M$ around $x = \pi(e)$ such that $\sigma_e(x) = e$ and $\theta(\sigma_{e*}(u)) = 0$ for all $u \in T_xM$. The latter condition means that $\sigma_{e*}(u)$ lies in the kernel of the projection $\theta|_e$. This completely fixes σ_{e*}, since for every local section σ_e we also have $\pi_* \circ \sigma_{e*} = \mathrm{id}_{T_xM}$. The set of all such local sections σ_e is thus simply the jet $j^1_x\sigma_e$. The jets for each $e \in E$ finally define a section $\omega : E \to J^1(E), e \mapsto j^1_x\sigma_e$ of the jet bundle, and thus an Ehresmann connection. □

We illustrate these constructions using the coordinates (x^α, y^a) on E. Recall that an Ehresmann connection ω in these coordinates can be written in the form $y^a_\alpha(x, y)$. For $w = u^\alpha\partial_\alpha + v^a\bar{\partial}_a \in T_eE$ we then have

$$\theta_e(w) = w - \sigma_{e*}(\pi_*(w)) = u^\alpha\partial_\alpha + v^a\bar{\partial}_a - u^\alpha(\partial_\alpha + y^a_\alpha(x, y)\bar{\partial}_a) = (v^a - u^\alpha y^a_\alpha(x, y))\bar{\partial}_a . \tag{51.6}$$

It thus follows that the coordinate expression for the connection form θ is simply given by $\theta^a_\alpha(x, y) = -y^a_\alpha(x, y)$.

Following the opposite construction, we start from a connection form θ, which assigns to a vector $w = u^\alpha\partial_\alpha + v^a\bar{\partial}_a \in T_{(x,y)}E$ the vertical vector $(u^\alpha\theta^a_\alpha(x, y) + v^a)\,\bar{\partial}_a \in V_{(x,y)}E$. This vector w lies in the kernel of θ if and only if $v^a = -u^\alpha\theta^a_\alpha(x, y)$. Consider a section σ of the bundle $\pi : E \to M$, which in coordinates is described as $y^a(x)$. Its differential σ_* maps a vector $u = u^\alpha\partial_\alpha \in T_xM$ to

$$\sigma_*(u) = u^\alpha \left(\partial_\alpha + \partial_\alpha y^a(x)\bar{\partial}_a\right) . \tag{51.7}$$

This lies in the kernel of θ for all u^α if and only if $\partial_\alpha y^a(x) = -\theta^a_\alpha(x, y)$. For the Ehresmann connection ω corresponding to θ we thus find the coordinate expression $y^a_\alpha(x, y) = -\theta^a_\alpha(x, y)$.

We finally discuss another equivalent description of an Ehresmann connection, which may appear more abstract in the beginning, but gives a deeper insight into the geometric structure of Ehresmann connection. For this purpose we introduce the following notion.

Definition 51.3 (Horizontal distribution)**.** Let $\pi : E \to M$ be a fiber bundle. A *horizontal distribution* on E is an assignment $e \mapsto H_eE$ of a *horizontal tangent space* $H_eE \subset T_eE$ to every $e \in E$ such that $T_eE = V_eE \oplus H_eE$ and for every $e \in E$ there exists a neighborhood U_e on which the horizontal tangent spaces are spanned by $n = \dim M$ vector fields $(X_1, \ldots, X_n)$.

A horizontal distribution thus assigns to each $e \in E$ a complement of the vertical tangent space V_eE. The condition that these horizontal vector spaces are locally spanned by vector fields can be understood geometrically as a requirement that this assignment is smooth. This means in particular that the union of the horizontal vector spaces forms a manifold HE, which is the total space of a vector bundle over E, called the horizontal tangent bundle. Note that while the vertical tangent bundle VE is canonically defined over the total space of every fiber bundle, a horizontal bundle is *not* canonically given and thus defines an additional structure. The following theorem states that this structure is essentially the same as an Ehresmann connection (and in fact the most common definition of an Ehresmann connection in the literature):

Theorem 51.2. *For every fiber bundle $\pi : E \to M$ there is a one-to-one correspondence between Ehresmann connections and horizontal distributions on E.*

Proof. We can make use of the preceding theorem that an Ehresmann connection is uniquely given by a connection form and vice versa. Given a connection form θ on E, the kernel of θ is a horizontal distribution. Conversely, given a horizontal distribution, we can uniquely split every vector $w \in TE$ in the form $w = w_V + w_H$, where $w_V \in VE$ is vertical and $w_H \in HE$ is horizontal. Then $\theta : w \mapsto w_V$ defines a connection form. □

The constructions used in this section work for every fiber bundle $\pi : E \to M$. However, we often have fiber bundles which have additional structure, such as principal fiber bundles or vector bundles. In this case we wish to consider only those connections which are compatible with this additional structure. We will discuss these connections in the following sections.

52 Connections on vector bundles

The first special case we consider is that of a vector bundle. In this case every fiber of the bundle carries the structure of a vector space. Thus, also the space of section is a vector space, where the addition and scalar multiplication are defined pointwise. From this follows that also the jet spaces $J^r_x(E)$ for $x \in M$ are vector spaces, since for any local sections σ, τ around x and $\mu, \nu \in \mathbb{R}$ the definition

$$\mu j^r_x\sigma + \nu j^r_x\tau = j^r_x(\mu\sigma + \nu\tau) \tag{52.1}$$

yields a vector space structure. Thus, $\pi_r : J^r(E) \to M$ is a vector bundle. (Note, however, that the bundles $\pi_{r,k} : J^r(E) \to J^k(E)$, and thus in particular $\pi_{r,0} : J^r(E) \to E$, are *not* vector bundles, since the fibers of these bundles are not vector (sub)spaces, but affine spaces.) This allows us to define the following notion.

Definition 52.1 (Linear Ehresmann connection)**.** Let $\pi : E \to M$ be a vector bundle. A *linear Ehresmann connection* on E is a vector bundle homomorphism $\omega : E \to J^1(E)$ such that $\pi_{1,0} \circ \omega = \mathrm{id}_E$.

This definition essentially consists of two parts. Being a map $\omega : E \to J^1(E)$ with $\pi_{1,0} \circ \omega = \mathrm{id}_E$ means that a linear Ehresmann connection is a section of the bundle $\pi_{1,0} : J^1(E) \to E$, and thus an Ehresmann connection. In addition, the restrictions $\omega|_x : E_x \to J^1_x(E)$ must be vector space homomorphisms for all $x \in M$.

To illustrate this definition, let (x^α, y^a) be local coordinates on E as in the previous section, where in addition we demand that the coordinates (y^a) on the fiber space F correspond to a basis $(e_1, \ldots, e_f)$ of F, where $f = \dim F$ and $y = y^a e_a$. Recall that a general Ehresmann connection on a fiber bundle is uniquely determined by a set $y^a_\alpha(x, y)$ of coordinate functions. For a linear Ehresmann connection these must be of the form $y^a_\alpha(x, y) = y^a{}_{b\alpha}(x) y^b$. On vector bundles one conventionally uses a different description for connections, which is given as follows.

Definition 52.2 (Koszul connection)**.** Let $\pi : E \to M$ be a vector bundle. A *Koszul connection* on E is an $\mathbb{R}$-linear function $\nabla : \Gamma(E) \to \Gamma(E \otimes T^*M)$ such that $\nabla(\epsilon f) = (\nabla\epsilon) f + \epsilon \otimes df$ for all $\epsilon \in \Gamma(E)$ and $f \in C^\infty(M, \mathbb{R})$.

We illustrate this definition using the same coordinates as above. In these coordinates a section $\epsilon \in \Gamma(E)$ is expressed in the form $y(x) = y^a(x) e_a(x)$, where $(e_a(x))$ is a basis of E_x and $y^a(x)$ are smooth functions. A Koszul connection assigns to ϵ a section $\nabla\epsilon \in \Gamma(E \otimes T^*M)$, whose coordinate expression follows from the Leibniz rule, which states that

$$\nabla\epsilon(x) = \partial_\alpha y^a(x) e_a(x) \otimes dx^\alpha + y^a(x) \nabla e_a(x)\,. \tag{52.2}$$

We can express ∇e_a in the basis $e_a \otimes dx^\alpha$ in the form $\nabla e^a(x) = \omega^a{}_{b\alpha}(x) e_b(x) \otimes dx^\alpha$ to finally obtain

$$\nabla\epsilon(x) = \left[\partial_\alpha y^a(x) + \omega^a{}_{b\alpha}(x) y^b(x)\right] e_a(x) \otimes dx^\alpha = y^a{}_{;\alpha}(x) e_a(x) \otimes dx^\alpha\,, \tag{52.3}$$

where we introduced the *semicolon notation*.

Koszul connections are very helpful as they can be used to define a number of operations on vector bundles. Although they are defined rather differently from Ehresmann connections, they are closely related. One may already get this impression from the coordinate expressions $y^a{}_{b\alpha}$ and $\omega^a{}_{b\alpha}$, which carry the same index structure. More formally, we formulate it as follows.

Theorem 52.1. *For every vector bundle $\pi : E \to M$ there is a one-to-one correspondence between linear Ehresmann connections and Koszul connections on E.*

Proof. Let $\epsilon : M \to E$ be a section. For $x \in M$ it defines a point $e = \epsilon(x) \in E$ and a linear map $\epsilon_*|_x : T_xM \to T_eE$ with $\pi_* \circ \epsilon_*|_x = \mathrm{id}_{T_xM}$. Also a linear Ehresmann connection $\omega : E \to J^1(E)$ defines a linear map $\sigma_{e*}|_x : T_xM \to T_eE$ with $\pi_* \circ \sigma_{e*}|_x = \mathrm{id}_{T_xM}$ through the jet $\omega(e) = j^1_x\sigma_e$. Their difference $\nabla^\omega\epsilon|_x = \epsilon_*|_x - \sigma_{e*}|_x$ therefore defines a linear map from T_xM to V_eE. Hence,

$$\nabla^\omega\epsilon|_x \in \mathrm{Hom}(T_xM, V_eE) \cong V_eE \otimes T^*_xM \cong E_x \otimes T^*_xM\,. \tag{52.4}$$

Doing this for each $x \in M$ we obtain a section $\nabla^\omega\epsilon \in \Gamma(E \otimes T^*M)$. The smoothness of this section can be proven using the smoothness of ω and ϵ. Further, given a function $f \in C^\infty(M, \mathbb{R})$ we find that

$$\nabla^\omega(\epsilon f)|_x = (\epsilon f)_*|_x - \sigma_{fe*}|_x = f\epsilon_*|_x + (\epsilon \otimes df)|_x - f\sigma_{e*}|_x = (f\nabla^\omega\epsilon)|_x + (\epsilon \otimes df)|_x\,. \tag{52.5}$$

This shows that ∇^ω satisfies the Leibniz rule and hence is a Koszul connection.
We will not prove the converse direction, but simply provide the construction how to obtain a linear Ehresmann connection from a Koszul connection $\nabla : \Gamma(E) \to \Gamma(E \otimes T^*M)$. For $e \in E$ with $\pi(e) = x \in M$ choose a section $\epsilon \in \Gamma(E)$ such that $\epsilon(x) = e$. Then $\epsilon^\nabla_x = \epsilon_*|_x - \nabla\epsilon|_x : T_xM \to T_eE$ is a linear map, which we can take as an ingredient to construct a section $\omega^\nabla : E \to J^1(E)$ as described in the previous section. Of course, to complete the proof one still needs to show that this is independent of the choice of the section ϵ. □

From the construction above one can derive how the coordinate expressions $y^a{}_{b\alpha}$ and $\omega^a{}_{b\alpha}$ we introduced earlier are related. A quick calculation shows that similarly to the case of general Ehresmann connections we have $y^a{}_{b\alpha} = -\omega^a{}_{b\alpha}$.
A Koszul connection allows us to perform another operation on vector bundles. Given a section of a vector bundle and a vector field on the base manifold, it allows us to take the derivative of this section along the vector field. This is defined as follows.

Definition 52.3 (Covariant derivative)**.** Let $\pi : E \to M$ be a vector bundle with a Koszul connection ∇. The *covariant derivative* of a section $\sigma \in \Gamma(E)$ with respect to a vector field $X \in \mathrm{Vect}(M)$ is the section $\nabla_X\epsilon = \iota_X(\nabla\epsilon)$.

It should be clear what the covariant derivative looks like in coordinates. If we write $\nabla\epsilon = y^a{}_{;\alpha}(x)e_a(x) \otimes dx^\alpha$ and $X(x) = X^\alpha(x)\partial_\alpha$, then $\nabla_X\epsilon = X^\alpha(x)y^a{}_{;\alpha}(x)e_a(x)$.

53 Connections on principal bundles

Recall that a principal G-bundle $\pi : P \to M$ is equipped with a right action of a Lie group G which is fiber preserving and free and transitive on the fibers. For $p \in P$ and $g \in G$ we can write this action in the form $R_g(p) = p \cdot g$. This right action also induces a right action on the space $\Gamma(P)$ of (local) sections given by $R_g(\sigma) = R_g \circ \sigma$ for $\sigma \in \Gamma(P)$. To see that this is indeed a right action and not a left action, one can explicitly calculate

$$R_{gh}(\sigma) = R_{gh} \circ \sigma = R_h \circ R_g \circ \sigma = R_h(R_g(\sigma))\,, \tag{53.1}$$

where the step $R_{gh} = R_h \circ R_g$ follows from the fact that we have a right action on P. Note that since R_g is a fiber preserving diffeomorphism for all $g \in G$, the r-jets for any $r \in \mathbb{N}$ of the images $R_g(\sigma), R_g(\tau)$ of two sections $\sigma, \tau \in \Gamma(P)$ agree if any only if σ and τ have the same r-jets. For all $x \in M$ thus holds

$$j^r_xR_g(\sigma) = j^r_xR_g(\tau) \quad\Leftrightarrow\quad j^r_x\sigma = j^r_x\tau\,. \tag{53.2}$$

This defines a right action on the jet spaces $J^r(P)$ given by $R_g(j^r_x\sigma) = j^r_x R_g(\sigma)$. Making use of this right action on $J^1(P)$ we can now define the following.

Definition 53.1 (Principal Ehresmann connection)**.** Let $\pi : P \to M$ be a principal G-bundle with Lie group G. A *principal Ehresmann connection* on P is a G-equivariant section of the jet bundle $\pi_{1,0} : J^1(P) \to P$.

In addition to the definition of a general Ehresmann connection we thus have the condition that the section $\omega : P \to J^1(P)$ must be G-equivariant. To study the consequences of this condition, recall that an Ehresmann connection assigns to every $p \in P$ a jet $\omega(p) = j^1_x\sigma_p$ with $x = \pi(p)$, where σ_p is a local section of $\pi : P \to M$ around x such that $\sigma_p(x) = p$. The condition of equivariance then takes the form

$$j^1_x\sigma_{p\cdot g} = \omega(p \cdot g) = \omega(R_g(p)) = R_g(\omega(p)) = R_g(j^1_x\sigma_p) = j^1_x(R_g \circ \sigma_p)\,. \tag{53.3}$$

Of course also principal Ehresmann connections can be expressed using connection forms or horizontal distributions. The most commonly used description makes use of connection forms. Recall that a connection form on a bundle $\pi : P \to M$ is a vector bundle homomorphism $\theta : TP \to VP$ covering the identity map on P and restricting to the identity on VP. Since both TP and VP carry right actions by the Lie group G, which are given by the differential R_{g*} of the right action on P, there is a well-defined notion of G-equivariant connection forms. The following statement should thus not be a big surprise.

Theorem 53.1. *For every principal G-bundle $\pi : P \to M$ with Lie group G there is a one-to-one correspondence between principal Ehresmann connections and G-equivariant connection forms on P.*

Proof. We have already proven that for general fiber bundles there is a one-to-one correspondence between Ehresmann connections ω and connection forms θ. We now have to show that ω is a principal Ehresmann connection if and only if θ is G-equivariant. We will thus start with a principal Ehresmann connection ω, which assigns to $e \in E$ with $\pi(e) = x$ the jet $\omega(e) = j^1_x\sigma_e$. This defines the connection form θ_e at e as $w \mapsto w - \sigma_{e*}(\pi_*(w))$, as shown for general Ehresmann connections. To see that θ is equivariant, we check that

$$\begin{aligned}
\theta_{e\cdot g}(R_{g*}(w)) &= R_{g*}(w) - \sigma_{e\cdot g*}(\pi_*(R_{g*}(w))) \\
&= R_{g*}(w) - (R_g \circ \sigma_e)_*(\pi_*(w)) \\
&= R_{g*}(w - \sigma_{e*}(\pi_*(w))) \\
&= R_{g*}(\theta_e(w))\,.
\end{aligned} \tag{53.4}$$

Thus, θ is equivariant. We also see from the derivation above that if ω is not a principal Ehresmann connection, then θ is not equivariant. □

We can thus describe any principal Ehresmann connection in terms of a G-equivariant connection form. However, it is more common to replace the target space VP of the connection form by the Lie algebra $\mathfrak{g}$. This is possible, since the fundamental vector fields establish a linear isomorphism between $\mathfrak{g}$ and every vertical tangent space V_pP. One thus often uses the following definition for a connection on a principal bundle.

Definition 53.2 (Principal G-connection)**.** Let $\pi : P \to M$ be a principal G-bundle with Lie group G. A *principal G-connection* on P is a $\mathfrak{g}$-valued one-form $\vartheta \in \Omega^1(P, \mathfrak{g})$ on P such that:

- ϑ is G-equivariant: $\vartheta = \mathrm{Ad}_g(R_g^*(\vartheta))$ for all $g \in G$.
- For all $X \in \mathfrak{g}$ and $p \in P$ the fundamental vector field $\tilde{X}$ yields $\iota_{\tilde{X}}\vartheta(p) = X$.

This definition requires a few explanations. The space $\Omega^1(P, \mathfrak{g})$ of Lie algebra valued one-forms is simply the tensor product space $\mathfrak{g} \otimes \Omega^1(P)$. The map $\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g})$ is called the *adjoint representation*. In order to define it, we need the following map.

Definition 53.3 (Inner automorphism)**.** For a Lie group G, the *inner automorphism* $\alpha : G \to \mathrm{Aut}(G)$ is defined such that for all $g, h \in G$ holds $\alpha_g(h) = ghg^{-1}$.

In other words, for all $g \in G$, the map $\alpha_g : G \to G$ is an automorphism, i.e., it satisfies $\alpha_g(hh') = \alpha_g(h)\alpha_g(h')$ for all $h, h' \in G$ and thus in particular $\alpha_g(e) = e$. Thus, the differential α_{g*} maps $T_eG \cong \mathfrak{g}$ to itself. We can thus define the following notion.

Definition 53.4 (Adjoint representation)**.** For a Lie group G, the *adjoint representation* is the map $\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g})$ defined by $\mathrm{Ad}_g = \alpha_{g*}$.

In other words, for all $g \in G$, the map $\mathrm{Ad}_g : \mathfrak{g} \to \mathfrak{g}$ is an automorphism, i.e., it satisfies $\mathrm{Ad}_g[X, Y] = [\mathrm{Ad}_g(X), \mathrm{Ad}_g(Y)]$ for all $X, Y \in \mathfrak{g}$. With this definition, we can now come to the following statement about principal G-connections.

Theorem 53.2. *For every principal G-bundle $\pi : P \to M$ with Lie group G there is a one-to-one correspondence between principal Ehresmann connections and principal G-connections on P.*

Proof. For every $p \in P$ there exists a vector space isomorphism $\tilde{\bullet}|_p : \mathfrak{g} \to V_pP$ defined by the fundamental vector fields. Via this isomorphism there exist isomorphisms between the following spaces:

$$\mathrm{Hom}(T_pP, V_pP) \cong \mathrm{Hom}(T_pP, \mathfrak{g}) \cong \mathfrak{g} \otimes T_p^*P\,. \tag{53.5}$$

Thus, a vector bundle homomorphism $\theta : TP \to VP$ covering the identity on E uniquely determines a section $\vartheta \in \Omega^1(P, \mathfrak{g})$. It is easy to see that θ restricts to the identity on VP if and only if $\iota_{\tilde{X}}\vartheta(p) = X$ for all $X \in \mathfrak{g}$ and $p \in P$. Further, it follows from the definition of the adjoint representation that θ is G-equivariant if and only if ϑ is G-equivariant. □

We finally discuss the relation between connections on a principal bundle and connections on an associated bundle. For our purpose it is enough to mention the following statement.

Theorem 53.3. *Let* $\pi : P \to M$ *be a principal* G*-bundle with Lie group* G *and* $\pi_\rho : P \times_\rho F \to M$ *an associated bundle with fiber* F*. A principal Ehresmann connection* $\omega : P \to J^1(P)$ *on* P*, which assigns to* $p \in P$ *with* $\pi(p) = x \in M$ *the jet* $j^1_x\sigma_p$*, induces a connection* $\omega_\rho : P \times_\rho F \to J^1(P \times_\rho F)$*, which assigns to* $[p, f] \in P \times_\rho F$ *the jet* $j^1_x[\sigma_p, f]$*.*

Proof. We first have to check that ω_ρ is well-defined. For this purpose, we have to check that it is independent of the representative (p, f) for $[p, f]$. Given another representative $(p \cdot g, \rho(g^{-1}, f))$ we find that

$$\omega_\rho([p \cdot g, \rho(g^{-1}, f)]) = j^1_x[\sigma_{p\cdot g}, \rho(g^{-1}, f)] = j^1_x[R_g \circ \sigma_p, \rho(g^{-1}, f)] = j^1_x[\sigma_p, f] = \omega_\rho([p, f])\,, \tag{53.6}$$

so that this is indeed satisfied. Here we used the fact that ω is a principal Ehresmann connection, so that a representative $\sigma_{p\cdot g}$ for the jet $\omega(p \cdot g) = j^1_x\sigma_{p\cdot g}$ is given by $R_g \circ \sigma_p$. Further, ω_ρ is a section, since

$$\pi_{\rho\,1,0}(\omega_\rho([p, f])) = [\sigma_p, f](x) = [\sigma_p(x), f] = [p, f]\,. \tag{53.7}$$

This shows that ω_ρ is an Ehresmann connection on $P \times_\rho F$. □

A Dictionary

English	Estonian
connection	seostus
covariant derivative	kovariantne tuletis
adjoint representation	adjungeeritud esitus