Differential geometry for physicists - Lecture 14

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47 Equivariant maps

For the constructions we discuss in this lecture it is useful to introduce a particular class of maps between manifolds which carry Lie group actions. These maps will be defined as follows.

Definition 47.1 (Equivariant map)**.** Let G be a Lie group and M, N manifolds which carry Lie group actions of G. A map $\varphi : M \to N$ is called *G-equivariant* if for all $g \in G$ and $x \in M$

- $\varphi(\rho_M(g, x)) = \rho_N(g, \varphi(x))$ if both $\rho_M : G \times M \to M$ and $\rho_N : G \times N \to N$ are left actions,
- $\varphi(\theta_M(x, g)) = \theta_N(\varphi(x), g)$ if both $\theta_M : M \times G \to M$ and $\theta_N : N \times G \to N$ are right actions,
- $\varphi(\rho_M(g, x)) = \theta_N(\varphi(x), g^{-1})$ if $\rho_M : G \times M \to M$ is a left action and $\theta_N : N \times G \to N$ is a right action,
- $\varphi(\theta_M(x, g)) = \rho_N(g^{-1}, \varphi(x))$ if $\theta_M : M \times G \to M$ is a right action and $\rho_N : G \times N \to N$ is a left action.

We denote the space of space of G-equivariant maps by $C^\infty_G(M, N)$.

This can be illustrated by a simple example.

Example 47.1. Consider the Lie group $G = \mathrm{SO}(3)$. Let $M = \mathbb{R}^3 \times \mathbb{R}^3$ with left action $\rho_M(g, (x, y)) = (gx, gy)$ and $N = \mathbb{R}^3$ with left action $\rho_N(g, x) = gx$, where gx denotes the multiplication of a matrix and a vector. Then the *cross product* $\times : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$ is an equivariant map.

48 Principal fiber bundles

We will now come to a class of fiber bundles which are of particular importance. They have the property that they carry the action of a Lie group, which is compatible with the fiber bundle structure, i.e., preserves the fibers. In addition, each fiber is diffeomorphic to the acting Lie group. We define this class of fiber bundles as follows.

Definition 48.1 (Principal fiber bundle)**.** Let G be a Lie group. A *principal G-bundle* is a fiber bundle $\pi : P \to M$ with a right Lie group action $\cdot : P \times G \to P$ which preserves the fibers and acts freely and transitively on them.

We clarify a few notions used in this definition. A group action is *fiber preserving* if for all $p \in P$ and $g \in G$ holds $\pi(p) = \pi(p \cdot g)$, i.e., p and $p \cdot g$ lie in the same fiber of P. Further, the action should be free and transitive on the fibers, which means that for each $p, p' \in P$ which lie in the same fiber, $\pi(p) = \pi(p')$, there exists a unique $g \in G$ such that $p' = p \cdot g$. An important example is that of a *coset space*, which is defined as follows.

Definition 48.2 (Coset space)**.** Let G be a Lie group and $H \subset G$ a closed subgroup. The (left) *coset* of $g \in G$ is the equivalence class

$$gH = \{gh, h \in H\}\,. \tag{48.1}$$

The (left) *coset space* G/H is the space of all (left) cosets, and is the base manifold of a principal H-bundle $\pi : G \to G/H, g \mapsto gH$.

There are many examples of coset spaces. One of the nicest examples is the following.

> *Example* 48.1. Let $G = \mathrm{SO}(n+1)$ and $H = \mathrm{SO(n)}$ for $n \in \mathbb{N}$. One can see $\mathrm{SO}(n+1)$ as the group of rotations in $n+1$ dimensions, i.e., linear maps $g : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ which map the unit sphere $S^n \subset \mathbb{R}^{n+1}$ to itself. Similarly, one can view $\mathrm{SO}(n) \subset \mathrm{SO}(n+1)$ as those linear maps $h : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ which in addition map the north pole $o \in S^n$ to itself. A coset gH can thus uniquely be identified by the image $g(o) \in S^n$. This identification defines a diffeomorphism, so that $\mathrm{SO}(n+1)/\mathrm{SO(n)} \cong S^n$.

As with any fiber bundle, we are of course interested in its (local) sections. In the case of principal fiber bundles, there is an interesting theorem about the existence of global sections, which can be stated as follows.

Theorem 48.1. *A principal fiber bundle is trivial if and only if it admits a global section.*

Proof. Let $\pi : P \to M$ be a principal G-bundle with Lie group G. If P is a trivial fiber bundle, then there exists a diffeomorphism $\phi : M \times G \to P$ such that $\pi \circ \phi = \mathrm{pr}_M$. Then $\sigma : M \to P, x \mapsto \phi(x, e)$ is a global section, where $e \in G$ is the unit element.
Conversely, let $\sigma : M \to P$ be a global section of an arbitrary principal G-bundle $\pi : P \to M$. Then every $p \in P$ can uniquely be written in the form $p = \sigma(x) \cdot g$ for some $x \in M$ and $g \in G$. One easily checks that the map $\phi : M \times G \to P, (x, g) \mapsto \sigma(x) \cdot g$ is a diffeomorphism, so that P is a trivial fiber bundle. □

We finally define a helpful class of vector fields on the total space of a principal bundle, which are defined by the right group action G. For this purpose, recall that the Lie algebra $\mathfrak{g} = \mathrm{Lie}\, G$ of a Lie group G is given by the left invariant vector fields on G. These vector fields define vector fields on P as follows.

Definition 48.3 (Fundamental vector fields)**.** Let G be a Lie group with Lie algebra $\mathfrak{g}$ and $\pi : P \to M$ a principal G-bundle. For a left invariant vector field $X \in \mathfrak{g}$ we define the *fundamental vector field* $\tilde{X}$ as the map

$$\begin{array}{rcccc} \tilde{X} & : & P & \to & TP \\ & & p & \mapsto & (R^p)_*(X(e)) \end{array}, \tag{48.2}$$

where $R^p : G \to P, g \mapsto p \cdot g$.

It is easy to see that $\tilde{X}(p) \in T_pP$, since $X(e) \in T_eG$ and $R^p(e) = p \cdot e = p$, so that $\tilde{X}$ is indeed a vector field. The fundamental vector fields satisfy an important relation.

Theorem 48.2. *The function* $\tilde{\bullet} : \mathfrak{g} \to \mathrm{Vect}(P)$ *is a Lie algebra homomorphism, i.e.,*

$$\widetilde{[X, Y]} = [\tilde{X}, \tilde{Y}] \tag{48.3}$$

for all $X, Y \in \mathfrak{g}$.

The proof is not difficult, but we will omit it here and instead state another helpful property.

Theorem 48.3. *For each* $p \in P$*, the fundamental vector fields on the principal* G*-bundle* $\pi : P \to M$ *define a vector space isomorphism between the Lie algebra* $\mathfrak{g}$ *of* G *and the vertical tangent space* V_pP.

49 Frame bundles

A prototypical example of a principal fiber bundle which exists on every manifold and is often encountered in physics is the *frame bundle.* In fact, there exists a whole class of frame bundles. For now we will restrict ourselves to the general linear frame bundle given by the following definition.

Definition 49.1 (Frame bundle)**.** Let M be a manifold of dimension $\dim M = n$ and $\mathrm{GL}(n)$ the group of bijective linear functions $g : \mathbb{R}^n \to \mathbb{R}^n$. A *frame* at $x \in M$ is a bijective linear function $p : \mathbb{R}^n \to T_xM$. The set of all frames constitutes the (general linear) *frame bundle* $\mathrm{GL}(M)$ with right action given by $p \cdot g = p \circ g$ for $p \in \mathrm{GL}(M)$ and $g \in \mathrm{GL}(n)$ and projection mapping $p : \mathbb{R}^n \to T_xM \in \mathrm{GL}(M)$ to $x \in M$.

A whole theory has been developed around the question whether the frame bundle of a given manifold is trivial or not. Those manifolds whose frame bundle is trivial deserve an own name.

Definition 49.2 (Parallelizable manifold)**.** A manifold M whose frame bundle $\mathrm{GL}(M)$ is trivial is called *parallelizable.*

Definition 49.3 (Parallelization)**.** A global section $\sigma \in \Gamma(\mathrm{GL}(M))$ of the frame bundle $\mathrm{GL}(M)$ of a parallelizable manifold M is called a *parallelization* of M.

A nice case for studying this property is given in the following example.

> *Example* 49.1. The only spheres S^n that are parallelizable are S^1, S^3 and S^7. A Cartesian product of at least two spheres is parallelizable if and only if at least one of them is odd.

We will not prove this here, since the proof is highly non-trivial. However, we can nicely prove the following.

Theorem 49.1. *Every Lie group G is parallelizable.*

Proof. Let G be a Lie group of dimension $\dim G = n$ and $(X_1, \ldots, X_n)$ a basis of the Lie algebra $\mathfrak{g}$. Every basis element X_i is a left invariant vector field on G. For all $g \in G$, the vectors $(X_1(g), \ldots, X_n(g))$ constitute a basis of T_gG, and thus define a frame at g. This yields a global section of the frame bundle. It thus follows that the frame bundle is trivial, so that G is parallelizable. □

50 Associated fiber bundles

In the last section of this lecture we introduce another notion, which is of particular importance in physics. The basic idea behind this construction is to replace the fiber of a principal bundle by another fiber which carries an action of the same Lie group. The resulting bundle, which we call *associated* to the original principal bundle, is defined as follows.

Definition 50.1 (Associated fiber bundle)**.** Let G be a Lie group and $\pi : P \to M$ a principal G-bundle. Further, let F be a manifold together with a left Lie group action $\rho : G \times F \to F$. Consider the right action on the Cartesian product $P \times F$ given by

$$(p, f) \cdot g = (p \cdot g, \rho(g^{-1}, f)) \,. \tag{50.1}$$

Let $P \times_\rho F$ be the set of orbits of this right action and denote the orbit of $(p, f) \in P \times F$ by $[p, f]$. Finally, let $\pi_\rho : P \times_\rho F \to M$ be the projection map given by $\pi_\rho([p, f]) = \pi(p)$. Then $P \times_\rho F$ is called the *fiber bundle associated* to P via the action ρ.

It should be clear that for every $x \in M$ the fiber $\pi_\rho^{-1}(x) = P_x \times_\rho F$ is diffeomorphic to F. We can explicitly construct suitable diffeomorphisms as follows.

Definition 50.2 (Fiber diffeomorphism)**.** Let $\pi : P \to M$ be a principal G-bundle and $\pi_\rho : P \times_\rho F \to M$ an associated bundle with fiber F. For $p \in P$ the diffeomorphism $[p] : F \to P_{\pi(p)} \times_\rho F, f \mapsto [p, f]$ is called the *fiber diffeomorphism* of p.

It is now easy to see the following property of the fiber diffeomorphisms.

Theorem 50.1. *For every $p \in P$, $g \in G$ and $f \in F$ the fiber diffeomorphism defined above satisfies*

$$[p \cdot g](f) = [p](\rho(g, f))\,. \tag{50.2}$$

Proof. By definition of the orbits $[p, f]$ we have

$$[p \cdot g](f) = [p \cdot g, f] = [p, \rho(g, f)] = [p](\rho(g, f))\,. \tag{50.3}$$

□

With the help of this property we can now understand the structure of the space $\Gamma(P \times_\rho F)$ of sections of an associated fiber bundle.

Theorem 50.2. *There is a one-to-one correspondence between sections $\sigma \in \Gamma(P \times_\rho F)$ of an associated fiber bundle $P \times_\rho F$ and G-equivariant maps $\phi \in C^\infty_G(P, F)$.*

Proof. Let $\sigma \in \Gamma(P \times_\rho F)$ be a section of $P \times_\rho F$. We then define a map ϕ by

$$\begin{array}{rcccc} \phi & : & P & \to & F \\ & & p & \mapsto & [p]^{-1}(\sigma(\pi(p))) \end{array}\,. \tag{50.4}$$

This map is well-defined, since $\sigma(\pi(p)) \in P_{\pi(p)} \times_\rho F$ and $[p] : F \to P_{\pi(p)} \times_\rho F$ is a diffeomorphism, and thus possesses an inverse. It is G-equivariant, since for all $g \in G$:

$$\phi(p \cdot g) = [p \cdot g]^{-1}(\sigma(\pi(p \cdot g))) = \rho(g^{-1}, [p]^{-1}(\sigma(\pi(p)))) = \rho(g^{-1}, \phi(p))\,. \tag{50.5}$$

Conversely, let $\phi \in C^\infty_G(P, F)$ be an equivariant map. For $x \in M$ choose $p \in \pi^{-1}(x) \subset P$ and define $\sigma(x) = [p, \phi(p)]$. This definition is independent of the choice of p, since for any other $p' = p \cdot g$ we have

$$[p', \phi(p')] = [p \cdot g, \phi(p \cdot g)] = [p \cdot g, \rho(g^{-1}, \phi(p))] = [p, \phi(p)]\,. \tag{50.6}$$

It is easy to check that σ defines a section of $P \times_\rho F$. □

A Dictionary

English	Estonian
equivariant map	ekvivariantne (?) kujutus
coset	kõrvalklass
fundamental vector field	fundamentaalne vektoriväli (?)
parallelizable manifold	paralleeliseeruv muutkond (?)
frame bundle	reeperi kihtkond (?)
principal fiber bundle	peakihtkond
associated fiber bundle	assotsieeritud kihtkond (?)