Differential geometry for physicists - Lecture 13

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44 Symmetries of Lagrangian systems

We have previously discussed Lagrangian systems in the language of the variational bicomplex, which is defined on the infinite jet space of a fiber bundle. We will now discuss a particular notion of symmetry of a Lagrangian in this formalism. Since Lagrangians are defined as differential forms on the infinite jet space $J^{\infty}(E)$ of a fiber bundle $\pi : E \to M$, we will describe symmetries in terms of (complete) vector fields on $J^{\infty}(E)$, whose flow leaves the Euler-Lagrange equations invariant.
However, we cannot consider arbitrary vector fields on $J^{\infty}(E)$. To see this, note that the flow of a vector field on $J^{\infty}(E)$ is a one-parameter group of diffeomorphisms of $J^{\infty}(E)$. Recall that the elements of $J^{\infty}(E)$ are jets of sections of $\pi : E \to M$. We are in particular interested in those diffeomorphisms of $J^{\infty}(E)$ which are generated by diffeomorphisms of the space $\Gamma(E)$ of sections. In other words, we are looking for diffeomorphisms ϕ whose action on a jet $j^{\infty}_r\sigma$ of a section $\sigma \in \Gamma(E)$ at a point $x \in M$ is given by $\phi(j^{\infty}_x\sigma) = j^{\infty}_x\varphi(\sigma)$ for some diffeomorphism φ of $\Gamma(E)$. This in particular means that ϕ should preserve the subspace $\pi_{\infty}(x) = J^{\infty}_x(E) \subset J^{\infty}(E)$ for every $x \in M$, from which follows that the generating vector field must be vertical. We will now construct these vector fields, starting with the following definition.

Definition 44.1 (Evolutionary vector field)**.** Let $\pi : E \to M$ be a fiber bundle and $\nu : VE \to E$ the vertical tangent bundle of E. An *evolutionary vector field* is a map $X \in C^{\infty}(J^{\infty}(E), VE)$ such that $\nu \circ X = \pi_{\infty,0}$.

First of all, note that an evolutionary vector field, despite its name, is not a vector field. In the literature one often finds the term *generalized vector field* for a map taking jets of sections to tangent vectors on E. Formally, it can be written as a "vector field on E with coefficients in $J^{\infty}(E)$.
To further understand the meaning of the definition, consider a section $\sigma \in \Gamma(E)$. At a point $x \in M$ this gives us the image $\sigma(x) \in E$ and the ∞-jet $j^{\infty}_x\sigma$, where $\pi_{\infty,0}(j^{\infty}_x\sigma) = \sigma(x)$. An evolutionary vector field X assigns to the jet $j^{\infty}_x\sigma$ a vertical tangent vector $X(j^{\infty}_x\sigma) \in V_{\sigma(x)}E$. This tangent vector will describe how much the value $\sigma(x)$ of the section σ changes under a certain type of flow.
An evolutionary vector field thus describes how much a section will change at each point. This tells us also how a section as a whole will change under this flow, and thus also how its jets will change. In other words, we can obtain a vertical vector field on $J^{\infty}(E)$, which we define as follows.

Definition 44.2 (Prolongation)**.** Let $\pi : E \to M$ be a fiber bundle and $X \in C^\infty(J^\infty(E), VE)$ an evolutionary vector field. Its *prolongation* is the unique vertical vector field $\operatorname{pr} X$ on $J^\infty(E)$ such that $X = \pi_{\infty,0*} \circ \operatorname{pr} X$ and $\mathcal{L}_{\operatorname{pr} X}\theta$ is a contact form for every contact form θ.

To clarify this definition, recall that a contact forms is defined as a differential form on $J^\infty(E)$ whose pullback along the ∞-jet $j^\infty\sigma$ of any section $\sigma \in \Gamma(E)$ vanishes. Since we wish that the flow of $\operatorname{pr} X$ maps the ∞-jets of sections again to ∞-jets of sections, it also maps contact forms to contact forms. However, it is easier to work with contact forms, which is why we used them in the definition above.
As a further illustration, we write the prolongation in terms of coordinates. Let (x^α) be coordinates on M and (x^α, y^a) coordinates on E corresponding to a local trivialization. In these coordinates an evolutionary vector field X can be written in the form $X = X^a\bar{\partial}_a$, where the coefficients X^a depend on the jet coordinates (x^α, y^a_Λ). One can show that the prolongation of X is then given by

$$\operatorname{pr} X = \sum_\Lambda d_\Lambda X^a \bar{\partial}^\Lambda_a\,. \tag{44.1}$$

Here d_Λ denotes the total derivative. The reason for this formula is intuitively clear: if the flow of X describes the transformation of a section σ, then we need to take all derivatives of X to see how the flow of $\operatorname{pr} X$ transforms the jet of a section.
The prolongations of evolutionary vector fields have a few nice properties, which we summarize here.

Theorem 44.1. *The prolongation* $\operatorname{pr} X$ *of an evolutionary vector field* X *satisfies* $\mathcal{L}_{\operatorname{pr} X} d_H = d_H \mathcal{L}_{\operatorname{pr} X}$ *and* $\mathcal{L}_{\operatorname{pr} X} = \iota_{\operatorname{pr} X} d_V + d_V \iota_{\operatorname{pr} X}$.

We will not prove these properties here, but use them later. Now we have found the class of vector fields on $J^\infty(E)$ which correspond to transformations of the space $\Gamma(E)$ of sections. We can now restrict ourselves to those vector fields from this class which leave the dynamics of the Lagrangian system, given by the Euler-Lagrange equations, invariant. We define them as follows.

Definition 44.3 (Symmetry)**.** Let $\pi : E \to M$ be a fiber bundle with $\dim M = n$ and $L \in \Omega^{n,0}(J^\infty(E))$ a Lagrangian. A *symmetry* of L is an evolutionary vector field X such that $\mathcal{L}_{\operatorname{pr} X} L \in \Omega^{n,0}(J^\infty(E))$ is d_H-exact.

The definition above states that the flow of $\operatorname{pr} X$ changes the Lagrangian L only by a d_H-exact form. This means that the pullback $(j^\infty\sigma)^*(L)$ of L to M along the ∞-jet of a section changes only by an exact form on M. This in turn means that the action functional

$$S[\sigma] = \int_M (j^\infty\sigma)^*(L) \tag{44.2}$$

is invariant. It also follows that $\mathcal{L}_{\operatorname{pr} X}\mathcal{E}L = 0$, i.e., the Euler-Lagrange equations are invariant.

45 Conserved currents

The task of finding the solutions to the Euler-Lagrange equations can often be simplified if the Lagrangian system contains something known as a *conserved current* in field theory, or a *constant of motion* in mechanics. Here we will use the term *conserved current* and the following definition.

Definition 45.1 (Conserved current). Let $\pi : E \to M$ be a fiber bundle with $\dim M = n$ and $L \in \Omega^{n,0}(J^\infty(E))$ a Lagrangian. A *conserved current* of L is an element $\psi \in \Omega^{n-1,0}(J^\infty(E))$ such that $d_H\psi = 0$ on the subspace of $J^\infty(E)$ where $\mathcal{E}L = 0$.

In order to understand the meaning of this, let $\sigma \in \Gamma(E)$ a solution of the Euler-Lagrange equations, i.e., $\mathcal{E}L \circ j^\infty\sigma = 0$. Then $(j^\infty\sigma)^*(\psi)$ is a $n-1$-form on M, where $n = \dim M$, with

$$d(j^\infty\sigma)^*(\psi) = (j^\infty\sigma)^*(d_H\psi) = 0\,. \tag{45.1}$$

In other words, for each solution σ, the pullback $(j^\infty\sigma)^*(\psi)$ is closed. This resembles the standard notion of a conserved current.

46 Noether's theorem

With the preliminary definitions made in the previous sections we can now come to the central topic of this lecture, which is Noether's theorem. In the formalism we use here, it is formulated as follows.

Theorem 46.1 (Noether's theorem). *Let X be a symmetry of a Lagrangian $L \in \Omega^{n,0}(J^\infty(E))$ on a fiber bundle $\pi : E \to M$ with $\dim M = n$. Then*

$$\psi = \iota_{\mathrm{pr}\,X}\eta - \sigma \in \Omega^{n-1,0}(J^\infty(E))\,, \tag{46.1}$$

where $d_H\sigma = \iota_{\mathrm{pr}\,X} d_V L$ and $d_H\eta = \mathcal{E}L - d_V L$, is a conserved current.

Proof. By definition of a symmetry, $\iota_{\mathrm{pr}\,X} d_V L$ is d_H-exact, i.e., there exists $\sigma \in \Omega^{n-1,0}(J^\infty(E))$ such that $d_H\sigma = \iota_{\mathrm{pr}\,X} d_V L$. Further, by the definition of the internal Euler operator ϱ, the difference $\mathcal{E}L - d_V L = \varrho(d_V L) - d_V L$ is also d_H-exact, i.e., there exists $\eta \in \Omega^{n-1,1}(J^\infty(E))$ such that $d_H\eta = \mathcal{E}L - d_V L$.
Using the fact that $\mathrm{pr}\,X$ is the prolongation of an evolutionary vector field, we can now evaluate $d_H\psi$ and find

$$\begin{aligned}
d_H\psi &= d_H\iota_{\mathrm{pr}\,X}\eta - d_H\sigma \\
&= d\iota_{\mathrm{pr}\,X}\eta - d_V\iota_{\mathrm{pr}\,X}\eta - \iota_{\mathrm{pr}\,X}d_V L \\
&= \mathcal{L}_{\mathrm{pr}\,X}\eta - \iota_{\mathrm{pr}\,X}d\eta - d_V\iota_{\mathrm{pr}\,X}\eta - \iota_{\mathrm{pr}\,X}d_V L \\
&= (d_V\iota_{\mathrm{pr}\,X} + \iota_{\mathrm{pr}\,X}d_V)\eta - \iota_{\mathrm{pr}\,X}d_H\eta - \iota_{\mathrm{pr}\,X}d_V\eta - d_V\iota_{\mathrm{pr}\,X}\eta - \iota_{\mathrm{pr}\,X}d_V L \\
&= -\iota_{\mathrm{pr}\,X}d_H\eta - \iota_{\mathrm{pr}\,X}d_V L \\
&= -\iota_{\mathrm{pr}\,X}\mathcal{E}L\,.
\end{aligned} \tag{46.2}$$

This obviously vanishes where $\mathcal{E}L = 0$, so that ψ is a conserved current. $\square$

The theorem is as elegant and as simple as a theorem could be. We will apply it to a few examples.

Example 46.1 (Momentum conservation). Let $M = \mathbb{R}$ and $E = \mathbb{R} \times Q$ with a manifold Q, so that the bundle $\pi : E \to M$ is a trivial bundle with $\pi = \mathrm{pr}_{\mathbb{R}}$ being the projection onto the first factor. Using the coordinate t on $\mathbb{R}$ and coordinates (q^a) on Q, we have coordinates (t, q^a) on E and $(t, q^a = q_0^a, \dot{q}^a = q_1^a, \ddot{q}^a = q_2^a, \ldots)$ on $J^\infty(E)$. This systems can be used to model, for example, the motion of a point mass on a manifold Q, with t measuring time and q^a the position of the point mass.
We now consider a Lagrangian of the form $L = \mathcal{L}(\dot{q})dt \in \Omega^{1,0}(J^\infty(E))$, where $\mathcal{L}$ depends only on the velocity $\dot{q}^a$, but not on the position q^a. Taking the vertical derivative we obtain

$$d_V L = \frac{\partial \mathcal{L}}{\partial \dot{q}^a} \dot{\theta}^a \wedge dt \,. \tag{46.3}$$

Further applying the internal Euler operator ϱ we obtain

$$\mathcal{E}L = \varrho d_V L = -d_t \frac{\partial \mathcal{L}}{\partial \dot{q}^a} \theta^a \wedge dt = -\frac{\partial^2 \mathcal{L}}{\partial \dot{q}^a \partial \dot{q}^b} \ddot{q}^b \theta^a \wedge dt \,, \tag{46.4}$$

where d_t is the total time derivative. The second derivative of $\mathcal{L}$ appearing here is also called the *Lagrange metric*, and is usually assumed to be non-degenerate, so that the Euler-Lagrange equations imply $\ddot{q}^a = 0$. From the expressions above it is easy to see that

$$\mathcal{E}L - d_V L = -\left(d_t \frac{\partial \mathcal{L}}{\partial \dot{q}^a} \theta^a + \frac{\partial \mathcal{L}}{\partial \dot{q}^a} \dot{\theta}^a \right) \wedge dt = d_H \left(\frac{\partial \mathcal{L}}{\partial \dot{q}^a} \theta^a \right) = d_H \eta \tag{46.5}$$

is indeed d_H-exact, by the definition of the internal Euler operator ϱ.
We now consider the evolutionary vector field $X = \xi^a \bar{\partial}_a$ on $J^\infty(E)$ with constant ξ^a. Its prolongation is simply the vector field itself, $\mathrm{pr}\, X = X$. One easily checks that it is a symmetry of the Lagrangian, since

$$\iota_{\mathrm{pr}\, X} d_V L = \iota_{\xi^a \bar{\partial}_a} \left(\frac{\partial \mathcal{L}}{\partial \dot{q}^a} \dot{\theta}^a \wedge dt \right) = 0 \,. \tag{46.6}$$

This ultimately follows from the fact that $\mathcal{L}$ does not depend on the position q^a, so that $d_V L$ does not contain the contact form θ^a, which would give a non-vanishing contribution with $\bar{\partial}_a$. We thus simply have $\sigma = 0$. This yields us the conserved current

$$\psi = \iota_{\mathrm{pr}\, X} \eta - \sigma = \xi^a \frac{\partial \mathcal{L}}{\partial \dot{q}^a} = \xi^a p_a \,. \tag{46.7}$$

The components p_a defined above are called *canonical momenta.* One can see that this is indeed a conserved current, since

$$d_H \psi = \xi^a d_t p_a dt = \xi^a \frac{\partial^2 \mathcal{L}}{\partial \dot{q}^a \partial \dot{q}^b} \ddot{q}^b dt \tag{46.8}$$

vanishes on solutions of the Euler-Lagrange equations.

Example 46.2 (Total energy conservation). We consider the same fiber bundle as in the previous example, but allow the Lagrangian $L = \mathcal{L}(q, \dot{q})dt \in \Omega^{1,0}(J^\infty(E))$ to depend also on the position q^a. The vertical derivative is then given by

$$d_V L = \left(\frac{\partial\mathcal{L}}{\partial q^a}\theta^a + \frac{\partial\mathcal{L}}{\partial\dot{q}^a}\dot{\theta}^a\right) \wedge dt\,. \tag{46.9}$$

Application of the internal Euler operator then yields

$$\mathcal{E}L = \varrho d_V L = \left(\frac{\partial\mathcal{L}}{\partial q^a} - d_t\frac{\partial\mathcal{L}}{\partial\dot{q}^a}\right)\theta^a \wedge dt = \left(\frac{\partial\mathcal{L}}{\partial q^a} - \frac{\partial^2\mathcal{L}}{\partial\dot{q}^a\partial q^b}\dot{q}^b - \frac{\partial^2\mathcal{L}}{\partial\dot{q}^a\partial\dot{q}^b}\ddot{q}^b\right)\theta^a \wedge dt\,. \tag{46.10}$$

From this we read off that

$$\mathcal{E}L - d_V L = -\left(d_t\frac{\partial\mathcal{L}}{\partial\dot{q}^a}\theta^a + \frac{\partial\mathcal{L}}{\partial\dot{q}^a}\dot{\theta}^a\right) \wedge dt = d_H\left(\frac{\partial\mathcal{L}}{\partial\dot{q}^a}\theta^a\right) = d_H\eta\,, \tag{46.11}$$

which actually yields the same expression for η as in the previous example.
We now consider the evolutionary vector field $X = \dot{q}^a\bar{\partial}_a$, whose prolongation is given by

$$\operatorname{pr} X = \sum_{\lambda=0}^{\infty} q^a_{\lambda+1}\bar{\partial}^\lambda_a\,. \tag{46.12}$$

This is a symmetry of the Lagrangian, since

$$\iota_{\operatorname{pr} X} d_V L = \left(\frac{\partial\mathcal{L}}{\partial q^a}\dot{q}^a + \frac{\partial\mathcal{L}}{\partial\dot{q}^a}\ddot{q}^a\right)dt = d_t\mathcal{L}dt = d_H\mathcal{L} = d_H\sigma \tag{46.13}$$

is d_H-exact with $\sigma = \mathcal{L}$. This gives us the conserved current

$$\psi = \iota_{\operatorname{pr} X}\eta - \sigma = \frac{\partial\mathcal{L}}{\partial\dot{q}^a}\dot{q}^a - \mathcal{L} = p_a\dot{q}^a - \mathcal{L} = \mathcal{H}\,, \tag{46.14}$$

which is called the *Hamiltonian* and describes the total energy of the system. This is a conserved current, since

$$\begin{aligned} d_H\psi = d_t\mathcal{H}dt &= \left(\frac{\partial^2\mathcal{L}}{\partial\dot{q}^a\partial q^b}\dot{q}^b + \frac{\partial^2\mathcal{L}}{\partial\dot{q}^a\partial\dot{q}^b}\ddot{q}^b\right)\dot{q}^a dt + \frac{\partial\mathcal{L}}{\partial\dot{q}^a}\ddot{q}^a dt - \frac{\partial\mathcal{L}}{\partial q^a}\dot{q}^a dt - \frac{\partial\mathcal{L}}{\partial\dot{q}^a}\ddot{q}^a dt \\ &= \left(\frac{\partial^2\mathcal{L}}{\partial\dot{q}^a\partial q^b}\dot{q}^b + \frac{\partial^2\mathcal{L}}{\partial\dot{q}^a\partial\dot{q}^b}\ddot{q}^b - \frac{\partial\mathcal{L}}{\partial q^a}\right)\dot{q}^a dt\,, \end{aligned} \tag{46.15}$$

which vanishes when the Euler-Lagrange equations are imposed.

A Dictionary

English	Estonian
evolutionary vector field	evolutsiooniline vektoriväli (?)
symmetry	sümmeetria
conserved current	jääv vool (?)