Differential geometry for physicists - Lecture 12

Manuel Hohmann

12. May 2015

39 Flows of vector fields

In the last lecture we have discussed the action of Lie groups on manifolds. We will now restrict ourselves to the action of a particular group, namely that of the real line $(\mathbb{R}, +)$. This is in fact an important special case, since any one-parameter subgroup $\varphi : \mathbb{R} \to G$ of a Lie group G acting on a manifold M induces an action of $(\mathbb{R}, +)$ on M, and all of these actions together describe (locally) the action of G. To study the local behavior of Lie group actions, and in particular those of $(\mathbb{R}, +)$, we start with a definition.

Definition 39.1 (Integral curve)**.** Let M be a manifold and X a vector field on M. An *integral curve* of X is a curve $\gamma \in C^{\infty}((a, b), M)$ such that $\dot{\gamma}(t) = X(\gamma(t))$.

One may ask whether such integral curves exist for any vector field. This is indeed the case, and is guaranteed by the following theorem, which comes from the theory of differential equations.

Theorem 39.1. *Let M be a manifold and X a vector field on M. For each $x \in M$ there exists an open set $U \subset M$ containing x, $\epsilon > 0$ and a map $\gamma : (-\epsilon, \epsilon) \times U \to M, (t, y) \mapsto \gamma_y(t)$ such that for all $y \in U$ the curve γ_y is an integral curve of X with $\gamma_y(0) = y$.*

The most interesting case for us is given when an integral curve can be defined for all on $\mathbb{R}$. For this case we define the following notion.

Definition 39.2 (Complete vector field)**.** A vector field X on a manifold M is called *complete* if for each $x \in M$ there exists an integral curve $\gamma \in C^{\infty}(\mathbb{R}, M)$ of X.

Given a complete vector field, we can define the following notion.

Definition 39.3 (Flow)**.** Let M be a manifold and X a complete vector field on M. The *flow* of X is the unique map $\phi : \mathbb{R} \times M \to M$ such that for each $x \in M$ the map $\phi_{\bullet}(x) : \mathbb{R} \to M$ is an integral curve of X and $\phi_0(x) = x$.

In fact, the flow can also be defined *locally* for a non-local vector field. In this case it is simply a map from an open subset $U \in \mathbb{R} \times M$ to M, where $\{0\} \times M \subset U$. The flow has a number of nice properties, one of which can be written most nicely for complete vector fields.

Theorem 39.2. *The flow of a complete vector field X is both a left and a right Lie group action of $(\mathbb{R}, +)$ on M.*

Proof. Since $(\mathbb{R}, +)$ is abelian, every left action is also a right action. We thus simply have to check that $\phi : \mathbb{R} \times M \to M$ is a smooth map such that $\phi_{s+t}(x) = \phi_s(\phi_t(x))$ for all $s, t \in \mathbb{R}$ and $x \in M$. We will not check the smoothness here. For fixed $t \in \mathbb{R}$ and $x \in M$ the maps $\gamma_1 : s \mapsto \phi_{s+t}(x)$ and $\gamma_2 : s \mapsto \phi_s(\phi_t(x))$ defines curves on M. For these curves we have

$$\dot{\gamma}_1(s) = X(\phi_{s+t}(x)) = X(\gamma_1(s))\,, \dot{\gamma}_2(s) = X(\phi_s(\phi_t(x))) = X(\gamma_2(s))\,, \tag{39.1}$$

so that both of them are integral curves of X. Further, they have the same initial point $\gamma_1(0) = \phi_t(x) = \gamma_2(0)$. Since integral curves are unique, it thus follows that $\gamma_1(s) = \gamma_2(s)$ for all $s \in \mathbb{R}$, and therefore $\phi_{s+t}(x) = \phi_s(\phi_t(x))$. □

In fact, the relation $\phi_{s+t}(x) = \phi_s(\phi_t(x))$ holds also for local flows, whenever both sides are well-defined. This will be sufficient for the constructions in this lecture. However, note that the flow is a group action only for complete vector fields.

40 The Lie derivative of tensor fields

Using the tools from the previous section we can now define a useful and important object in differential geometry.

Definition 40.1 (Lie derivative)**.** Let $T \in \Gamma(T^r_s M)$ be a tensor field and $X \in \mathrm{Vect}(M)$ a vector field on a manifold M. Let $\phi : \mathbb{R} \times M \supseteq U \to M$ be the flow of X. The *Lie derivative* of T with respect to X is the tensor field defined by

$$\mathcal{L}_X T = \lim_{t \to 0} \frac{\phi_t^* T - T}{t}\,. \tag{40.1}$$

We see that the Lie derivative can be seen as the infinitesimal change of the tensor field T along the flow of X: starting from a point $x \in M$ one follows the flow line of X, takes the tensor field at that point $\phi_t(x)$, pulls it back along ϕ_t to obtain a tensor at the original point x and then measures how much this tensor at x changes with t. Of course one has to show that this limit really exists an that it yields a smooth tensor field. Instead of proving this here in a rigorous, coordinate-free way, we only illustrate the definition and derive the coordinate expression of the Lie derivative.
Using coordinates (x^a) on M, let $X = X^a \partial_a$ be a vector field and

$$T = T^{a_1 \cdots a_r}{}_{b_1 \cdots b_s} \partial_{a_1} \otimes \ldots \otimes \partial_{a_r} \otimes dx^{b_1} \otimes \ldots \otimes dx^{b_s}\,. \tag{40.2}$$

Writing the pullback $\phi_t^* T$ in the same coordinates as

$$\phi_t^* T = T'_t = T'^{a_1 \cdots a_r}_t{}_{b_1 \cdots b_s} \partial_{a_1} \otimes \ldots \otimes \partial_{a_r} \otimes dx^{b_1} \otimes \ldots \otimes dx^{b_s}\,. \tag{40.3}$$

With this notation the coordinate expression for the Lie derivative takes the form

$$(\mathcal{L}_X T)^{a_1\cdots a_r}{}_{b_1\cdots b_s}(x) = \lim_{t\to 0} \frac{T_t'^{a_1\cdots a_r}{}_{b_1\cdots b_s}(x) - T^{a_1\cdots a_r}{}_{b_1\cdots b_s}(x)}{t} = \frac{d}{dt} T_t'^{a_1\cdots a_r}{}_{b_1\cdots b_s}(x)\bigg|_{t=0} . \tag{40.4}$$

To evaluate this derivative, recall that the pullback of a tensor field by a diffeomorphism is given by

$$T_t'^{a_1\cdots a_r}{}_{b_1\cdots b_s}(x) = T^{c_1\cdots c_r}{}_{d_1\cdots d_s}(x_t'(x)) \frac{\partial x^{a_1}}{\partial x_t'^{c_1}}(x_t'(x)) \cdots \frac{\partial x^{a_r}}{\partial x_t'^{c_r}}(x_t'(x)) \frac{\partial x_t'^{d_1}}{\partial x^{b_1}}(x) \cdots \frac{\partial x_t'^{d_s}}{\partial x^{b_s}}(x) , \tag{40.5}$$

where we wrote the flow ϕ of X in the form $x_t'(x)$. It is related to the vector field X via the flow equation

$$X^a(x) = \frac{d}{dt} x_t'^a(x)\bigg|_{t=0} . \tag{40.6}$$

This equation together with the chain rule is used to evaluate

$$\frac{d}{dt} T^{a_1\cdots a_r}{}_{b_1\cdots b_s}(x_t'(x))\bigg|_{t=0} = X^c(x)\partial_c T^{a_1\cdots a_r}{}_{b_1\cdots b_s}(x) . \tag{40.7}$$

From the fact that partial derivatives commute follows that

$$\frac{d}{dt}\frac{\partial x_t'^b}{\partial x^a}(x)\bigg|_{t=0} = \partial_a \frac{d}{dt} x_t'^b(x)\bigg|_{t=0} = \partial_a X^b(x) . \tag{40.8}$$

We further use the fact that $\phi_t^{-1} = \phi_{-t}$, from which follows that

$$\frac{d}{dt}\frac{\partial x^a}{\partial x_t'^b}(x_t'(x))\bigg|_{t=0} = \frac{d}{dt}\frac{\partial x_{-t}'^a}{\partial x^b}(x_t'(x))\bigg|_{t=0} = -\partial_b X^a(x) + X^c(x)\partial_c\delta_b^a = -\partial_b X^a(x) . \tag{40.9}$$

Putting everything together we finally find the coordinate expression for the Lie derivative as

$$\begin{aligned}(\mathcal{L}_X T)^{a_1\cdots a_r}{}_{b_1\cdots b_s} = {} & X^c\partial_c T^{a_1\cdots a_r}{}_{b_1\cdots b_s} \\ & - \partial_c X^{a_1} T^{c a_2\cdots a_r}{}_{b_1\cdots b_s} - \ldots - \partial_c X^{a_r} T^{a_1\cdots a_{r-1}c}{}_{b_1\cdots b_s} \\ & + \partial_{b_1} X^c T^{a_1\cdots a_r}{}_{c b_2\cdots b_s} + \ldots + \partial_{b_s} X^c T^{a_1\cdots a_r}{}_{b_1\cdots b_{s-1}c} .\end{aligned} \tag{40.10}$$

The Lie derivative of tensor fields has a few helpful and important properties, which we summarize below.

Theorem 40.1. *Let M be a manifold, S, T tensor fields on M, X, Y vector fields on M, $\mu, \nu \in \mathbb{R}$ and $k, l \in \mathbb{R}$. The Lie derivative satisfies:*

- *Linearity in the tensor fields:*

$$\mathcal{L}_X(\mu S + \nu T) = \mu\mathcal{L}_X S + \nu\mathcal{L}_X T . \tag{40.11}$$

- *Linearity in the vector fields:*

$$\mathcal{L}_{\mu X + \nu Y} T = \mu\mathcal{L}_X T + \nu\mathcal{L}_Y T . \tag{40.12}$$

- *Commutator:*

$$\mathcal{L}_{[X,Y]} T = \mathcal{L}_X\mathcal{L}_Y T - \mathcal{L}_Y\mathcal{L}_X T . \tag{40.13}$$

- *Leibniz rule:*

$$\mathcal{L}_X(S \otimes T) = (\mathcal{L}_X S) \otimes T + S \otimes (\mathcal{L}_X T)\,. \tag{40.14}$$

- *Compatibility with contraction:*

$$\mathcal{L}_X(\mathrm{tr}_l^k T) = \mathrm{tr}_l^k(\mathcal{L}_X T)\,. \tag{40.15}$$

We will not prove these properties here. Instead, we will discuss a few examples for the Lie derivative of particular tensor fields.

41 The Lie derivative of real functions

The simplest possible tensor field is of course a tensor field of type $(0,0)$, which is the same as a real function. In this case the Lie derivative has a very simple form.

Theorem 41.1. *For a function $f \in C^\infty(M,\mathbb{R})$ the Lie derivative with respect to a vector field X is given by*

$$\mathcal{L}_X f = Xf\,. \tag{41.1}$$

In other words, the Lie derivative of a function reduces to the action of a vector field. From this follow a few useful properties of the Lie derivative in this special case.

Theorem 41.2. *For a vector field $X \in \mathrm{Vect}(M)$ and real functions $f, g \in C^\infty(M,\mathbb{R})$ on a manifold M the Lie derivative satisfies:*

- *Leibniz rule:*

$$\mathcal{L}_X(fg) = \mathcal{L}_X f \cdot g + f \cdot \mathcal{L}_X g\,. \tag{41.2}$$

- *Multiplication of the vector field:*

$$\mathcal{L}_{gX} f = g \cdot \mathcal{L}_X f\,. \tag{41.3}$$

The Leibniz rule follows immediately from the Leibniz rule for tensor fields, since for real functions we simply have $f \otimes g = fg$. The second property holds only for functions and not for other tensor fields.

42 The Lie derivative of vector fields

As the next example we discuss the Lie derivative of vector fields. Also in this case it reduces to a familiar object as follows.

Theorem 42.1. *For a vector field $Y \in \mathrm{Vect}(M)$ the Lie derivative with respect to a vector field X is given by*

$$\mathcal{L}_X Y = [X, Y]\,. \tag{42.1}$$

From the fact that $\mathrm{Vect}(M)$ together with the Lie bracket forms a Lie algebra one can derive the following properties of the Lie derivative of vector fields.

Theorem 42.2. *For vector fields $X, Y, Z \in \mathrm{Vect}(M)$ on a manifold M the Lie derivative satisfies:*

- *Antisymmetry:*

$$\mathcal{L}_X Y = -\mathcal{L}_Y X\,. \tag{42.2}$$

- *Jacobi identity:*

$$\mathcal{L}_X [Y, Z] = [\mathcal{L}_X Y, Z] + [Y, \mathcal{L}_X Z]\,. \tag{42.3}$$

The second relation can be brought into various different forms.

43 The Lie derivative of differential forms

The last special case for the Lie derivative which we discuss in this lecture is the Lie derivative of differential forms. Also in this case there exists a helpful formula for the Lie derivative in terms of objects we have already previously encountered.

Theorem 43.1. *For a k-form $\omega \in \Omega^k(M)$ with $k \geq 1$ the Lie derivative with respect to a vector field X is given by "Cartan's magic formula"*

$$\mathcal{L}_X\omega = \iota_X(d\omega) + d(\iota_X\omega)\,. \tag{43.1}$$

Again we will omit the proof, since it is rather lengthy. One can directly use the formula above and the properties of the operations on differential forms to derive the following properties of the Lie derivative.

Theorem 43.2. *For vector fields $X, Y \in \mathrm{Vect}(M)$, differential forms $\omega \in \Omega^k(M), \sigma \in \Omega^l(M)$ and functions $f \in C^\infty(M, \mathbb{R})$ on a manifold M the Lie derivative satisfies:*

- *Compatibility with exterior derivative:*

$$d\mathcal{L}_X\omega = \mathcal{L}_X d\omega\,. \tag{43.2}$$

- *Leibniz rule with exterior product:*

$$\mathcal{L}_X(\omega \wedge \sigma) = (\mathcal{L}_X\omega) \wedge \sigma + \omega \wedge (\mathcal{L}_X\sigma)\,. \tag{43.3}$$

- *Relation with interior product:*

$$\iota_{[X,Y]}\omega = \mathcal{L}_X\iota_Y\omega - \iota_Y\mathcal{L}_X\omega = \iota_X\mathcal{L}_Y\omega - \mathcal{L}_Y\iota_X\omega\,. \tag{43.4}$$

- *Distribution law:*

$$\mathcal{L}_{fX}\omega = f\mathcal{L}_X\omega + df \wedge \iota_X\omega\,. \tag{43.5}$$

A Dictionary

English	Estonian
integral curve	integraaljoon
flow	voog
Lie derivative	Lie tuletis