Differential geometry for physicists - Lecture 11

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5. May 2015

36 Lie groups

In this lecture we will introduce manifolds which carry an additional algebraic structure, namely that of a group. In order to work with this structure, it must be compatible with the manifold structure. We make this precise in the following definition.

Definition 36.1 (Lie group)**.** A *Lie group* is a manifold G which carries the structure of a group, such that the group multiplication $\cdot : G \times G \to G$ and the inverse $\bullet^{-1} : G \to G$ are smooth maps.

This compatibility condition is a bit similar to the compatibility condition for vector bundles, where we wanted the vector space operations (addition and scalar multiplication) to be smooth operations. There are many examples for Lie groups which frequently appear in physics:

Example 36.1. The group $(\mathbb{R}, +)$ of real numbers with the addition as group operation is a Lie group of dimension 1.

Example 36.2. The complex numbers $z \in \mathbb{C}$ with $|z| = 1$ and group operation the multiplication is a Lie group of dimension 1 which is diffeomorphic to the circle S^1.

Example 36.3. The following matrix groups for $n \in \mathbb{N}$ are Lie groups, where the group multiplication is given by matrix multiplication:

- The *general linear group* $\mathrm{GL}(n)$ of real invertible $n \times n$ matrices is a Lie group of dimension n^2.
- The *special linear group* $\mathrm{SL}(n)$ of real $n \times n$ matrices with determinant 1 is a Lie group of dimension $n^2 - 1$.
- The *orthogonal group* $\mathrm{O}(n)$ of real $n \times n$ matrices such that $AA^t = \mathbb{1}$ is a Lie group of dimension $n(n-1)/2$.
- The *special orthogonal group* $\mathrm{SO}(n)$ of real $n \times n$ matrices with determinant 1 such that $AA^t = \mathbb{1}$ is a Lie group of dimension $n(n-1)/2$.

- The *unitary group* $\mathrm{U}(n)$ of complex $n \times n$ matrices such that $AA^\dagger = \mathbb{1}$ is a Lie group of dimension n^2.
- The *special unitary group* $\mathrm{SU}(n)$ of complex $n \times n$ matrices with determinant 1 such that $AA^\dagger = \mathbb{1}$ is a Lie group of dimension $n^2 - 1$.

In order to relate different Lie groups to each other, we need the same compatibility condition for homomorphisms between Lie groups.

Definition 36.2 (Lie group homomorphism / isomorphism)**.** Let G_1 and G_2 be Lie groups. A *Lie group homomorphism* from G_1 to G_2 is a smooth map $\varphi : G_1 \to G_2$ such that $\varphi(gh) = \varphi(g)\varphi(h)$ for all $g, h \in G_1$. If it is also a diffeomorphism, it is called a *Lie group isomorphism.*

There are numerous homomorphisms and isomorphisms between the groups given in the examples above.

Example 36.4. The map $\varphi : \mathbb{R} \to S^1 \subset \mathbb{C}, x \mapsto e^{ix}$ is a Lie group homomorphism.

Example 36.5. The map $\varphi : S^1 \subset \mathbb{C} \to \mathrm{SO}(2)$ defined by

$$\varphi(z) = \begin{pmatrix} \mathrm{Re}(z) & \mathrm{Im}(z) \\ -\mathrm{Im}(z) & \mathrm{Re}(z) \end{pmatrix} \tag{36.1}$$

is a Lie group isomorphism.

37 Lie group actions

A familiar concept from algebra is that of the *action* of a group on a set. Since we are working with Lie groups here, we are in particular interested how a Lie group can act on a manifold. Again we demand compatibility of the differentiable and algebraic structures, as in the following definition.

Definition 37.1 (Lie group action)**.** Let G be a Lie group and M a manifold. A *left Lie group action* is a smooth map $\phi : G \times M \to M$ such that $\phi(gh, x) = \phi(g, \phi(h, x))$ for all $g, h \in G$ and $x \in M$. A *right Lie group action* is a smooth map $\theta : M \times G \to M$ such that $\theta(x, gh) = \theta(\theta(x, g), h)$ for all $g, h \in G$ and $x \in M$.

We also say that a group G *acts from the left / right* on a manifold M. The following statement follows immediately from the definition above.

Theorem 37.1. *Let $\phi : G \times M \to M$ be a left Lie group action. For each $g \in G$ the map $x \mapsto \phi(g, x)$ is a diffeomorphism on M with inverse given by $x \mapsto \phi(g^{-1}, x)$. The same holds for right Lie group actions.*

We further distinguish between different types of Lie group actions.

Definition 37.2 (Types of Lie group actions)**.** Let G be a Lie group and M a manifold. A left Lie group action $\phi : G \times M \to M$ is called ...

- ... *transitive* if for all $x, y \in M$ there exists a $g \in G$ such that $\phi(g, x) = y$.
- ... *effective* (or *faithful*) if for all distinct $g, h \in G$ there exists $x \in M$ such that $\phi(g, x) \neq \phi(h, x)$.
- ... *free* if for all distinct $g, h \in G$ and for all $x \in M$ holds $\phi(g, x) \neq \phi(h, x)$.

The same naming is used for right Lie group actions.

It follows immediately that every free action is also effective. Of course there are many examples of group actions which appear in physics.

Example 37.1. Each of the matrix groups G from the previous section acts from the left on $\mathbb{R}^n$ via multiplication. This group action is effective, but neither transitive nor free.

Example 37.2. Every Lie group G acts on itself from the left by left multiplication $\phi(g, x) = gx$ and from the right by right multiplication $\theta(x, g) = xg$. Both actions are free and transitive.

The last example is of particular interest, because it is a property of every Lie group. The diffeomorphisms obtained from these actions deserve their own names.

Definition 37.3 (Translation maps)**.** Let G be a Lie group. For $g \in G$ the *left translation* is the map $L_g : G \to G, h \mapsto gh$, while the right translation is the map $R_g : G \to G, h \mapsto hg$.

We further introduce the following concepts, which will help us analyze the structure of Lie group actions.

Definition 37.4 (Orbit)**.** Let $\phi : G \times M \to M$ be a left Lie group action. For $x \in M$ the *orbit* is the set

$$\{\phi(g, x), g \in G\} \subset M\,. \tag{37.1}$$

The same is defined for a right Lie group action.

Example 37.3. Let $\phi : \mathrm{SO}(3) \times \mathbb{R}^3 \to \mathbb{R}^3$ be the left action given by matrix multiplication. For $x \in \mathbb{R}^3$ with $x \neq 0$ the orbit is the sphere with radius $\|x\|$ around the origin. For $x = 0$ the orbit contains only the origin itself.

Definition 37.5 (Stabilizer)**.** Let $\phi : G \times M \to M$ be a left Lie group action. For $x \in M$ the *stabilizer* is the subgroup

$$\{g \in G \,|\, \phi(g,x) = x\}\,. \tag{37.2}$$

The same is defined for a right Lie group action.

Example 37.4. Let $\phi : \mathrm{SO}(3) \times \mathbb{R}^3 \to \mathbb{R}^3$ be the left action given by matrix multiplication. For $x \in \mathbb{R}^3$ with $x \neq 0$ the stabilizer is the subgroup of rotations around the axis $x\mathbb{R}$. For $x = 0$ the stabilizer is $\mathrm{SO}(3)$ itself.

38 Lie algebras

So far we have introduced the basic structure of Lie groups and their actions on manifolds. We now consider particular classes of vector fields and differential forms on Lie groups, which play an important role in physics. We start with the following definition.

Definition 38.1 (Invariant vector field)**.** Let G be a Lie group. A vector field X on G is called *left invariant* if its pullback along the diffeomorphism L_g for all $g \in G$ satisfies $L_g^*(X) = X$. Similarly, it is called *right invariant* if $R_g^*(X) = X$ for all $g \in G$.

From the fact that diffeomorphisms preserve the Lie bracket follows the following property.

Theorem 38.1. *Let X, Y be left (right) invariant vector fields on a Lie group G. Then also their Lie bracket $[X, Y]$ is left (right) invariant.*

In the following we will use the standard convention and work with left invariant vector fields in order to be consistent with the literature. The statement above then tells us that the left invariant vector fields together with the Lie bracket form a Lie algebra, which plays a fundamental role.

Definition 38.2 (Lie algebra)**.** Let G be a Lie group. Its *Lie algebra* is the Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$ defined by the left invariant vector fields together with the Lie bracket of vector fields.

The question arises whether this Lie algebra is finite-dimensional, and what is its dimension. The following theorem answers both of these questions.

Theorem 38.2. *The Lie algebra $\mathfrak{g}$ of a Lie group G is canonically isomorphic as a vector space to the tangent space T_eG at the unit element $e \in G$.*

Proof. This is easy to see. Given a left-invariant vector field X, one can simply evaluate it at e to obtain $X(e) \in T_eG$. Conversely, given $v \in T_eG$, one can uniquely construct a left invariant vector field X as $X(g) = L_{g*}(v) \in T_gG$. □

It thus follows immediately that the dimension of the Lie algebra $\mathfrak{g}$ is the same as the dimension of the Lie group G, and we can simply identify $\mathfrak{g}$ and T_eG. This allows us to construct the Lie algebras of the matrix groups shown in the first section.

Example 38.1. The Lie algebras of the matrix groups for $n \in \mathbb{N}$ are as follows, where the Lie bracket $[A, B]$ is given by the matrix commutator $AB - BA$:

- The *general linear algebra* $\mathfrak{gl}(n)$ of real $n \times n$ matrices is a Lie algebra of dimension n^2.
- The *special linear algebra* $\mathfrak{sl}(n)$ of real $n \times n$ matrices with trace 0 is a Lie algebra of dimension $n^2 - 1$.
- The *orthogonal algebra* $\mathfrak{o}(n)$, which is the same as the *special orthogonal algebra* $\mathfrak{so}(n)$, of real, antisymmetric $n \times n$ matrices, $A = -A^t$, is a Lie algebra of dimension $n(n-1)/2$.
- The *unitary algebra* $\mathfrak{u}(n)$ of complex, anti-hermitian $n \times n$ matrices, $A = -A^\dagger$, is a Lie algebra of dimension n^2.
- The *special unitary algebra* $\mathfrak{su}(n)$ of complex, anti-hermitian $n \times n$ matrices, $A = -A^\dagger$, with trace 0 is a Lie algebra of dimension $n^2 - 1$.

To further explore the relationship between Lie groups and their Lie algebras, we define the following.

Definition 38.3 (One-parameter subgroup)**.** A *one-parameter subgroup* of a Lie group G is a Lie group homomorphism $\varphi : (\mathbb{R}, +) \to G$.

In other words, a one-parameter subgroup is a curve φ on G such that $\varphi(s+t) = \varphi(s)\varphi(t)$ for all $s, t \in \mathbb{R}$. In particular it follows that $\varphi(0) = e$ is the unit element of G. A one-parameter subgroup thus defines an element $\dot{\varphi}(0) \in T_eG \cong \mathfrak{g}$. The following theorem states that also the converse is true.

Theorem 38.3. *Let G be a Lie group and $X \in \mathfrak{g}$ a left invariant vector field. Then there exists a unique one-parameter subgroup φ_X such that $\dot{\varphi}_X(t) = X(\varphi_X(t))$ for all $t \in \mathbb{R}$.*

The proof is a bit lengthy, but simple, so we will omit it here. This theorem allows us to finally define another important concept.

Definition 38.4 (Exponential map)**.** Let G be a Lie group and $\mathfrak{g}$ its Lie algebra. The *exponential map* is the map

$$\begin{array}{rccc} \exp & : & \mathfrak{g} & \to & G \\ & & X & \mapsto & \varphi_X(1) \end{array}, \tag{38.1}$$

where φ_X is the unique one-parameter subgroup such that $\dot{\varphi}_X(t) = X(\varphi_X(t))$ for all $t \in \mathbb{R}$.

We will conclude with a few properties of the exponential map.

Theorem 38.4. *The exponential map satisfies:*

- *It maps the zero element* $0 \in \mathfrak{g}$ *to the unit* e *of the Lie group:* $\exp(0) = e$.
- *For all* $X \in \mathfrak{g}$ *holds* $\exp(-X) = \exp(X)^{-1}$.
- *For all* $X \in \mathfrak{g}$ *and* $s, t \in \mathbb{R}$ *holds* $\exp((s+t)X) = \exp(sX)\exp(tX)$.

A Dictionary

English	Estonian
Lie group	Lie rühm
Lie algebra	Lie algebra
Lie group action	Lie rühma toime
transitive group action	transitiivne rühma toime
effective group action	effektiivne rühma toime
free group action	vaba rühma toime
translation map	lükkekujutus (?)
orbit	orbiit
stabilizer	stabilisaator
exponential map	eksponentsiaalkujutus