Differential geometry for physicists - Lecture 10

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31 Action principle and variation

We finally come to the question how to derive equations of motion, and thus the space of solutions of the Lagrangian system introduced in the last section. The *principle of least action* states that solutions of a Lagrangian system are those sections $\sigma \in \Gamma(E)$ for which the action assumes a local minimum in the space of sections. This will now be clarified.

Definition 31.1 (Local minimum of the action)**.** Let $\pi : E \to M$ be a fiber bundle with action functional S. A section $\sigma \in \Gamma(E)$ is called a *local minimum of the action* if for all smooth families $\tilde{\sigma}_\bullet : \mathbb{R} \to \Gamma(E)$ of sections with $\tilde{\sigma}_0 = \sigma$ the function

$$\begin{array}{rccc} S[\tilde{\sigma}_\bullet] & : & \mathbb{R} & \to & \mathbb{R} \\ & & \epsilon & \mapsto & S[\tilde{\sigma}_\epsilon] \end{array} \tag{31.1}$$

has a local minimum at $\epsilon = 0$.

Here we call the family $\tilde{\sigma}_\bullet : \mathbb{R} \to \Gamma(E)$ of sections smooth if and only if the map $\tilde{\sigma}_\bullet(\bullet) : \mathbb{R} \times M \to E$ is smooth. It is clear that a *necessary* condition for a local minimum σ of the action is that the action is *stationary*, which is expressed by the equation

$$\delta S = \left.\frac{dS[\tilde{\sigma}_\epsilon]}{d\epsilon}\right|_{\epsilon=0} = 0 \tag{31.2}$$

for all smooth families of sections with $\tilde{\sigma}_0 = \sigma$. In the remainder of this lecture we will only discuss this necessary condition, and simplify it in several steps.

32 Infinitesimal change of ∞-jets

One may already guess that δS depends only on $\tilde{\sigma}_0 = \sigma$ and some object which one could denote $d\tilde{\sigma}_\epsilon/d\epsilon|_{\epsilon=0}$. This objects describes how the section $\tilde{\sigma}_\epsilon \in \Gamma(E)$ changes as ϵ varies. For fixed $p \in M$ it describes how the image $\tilde{\sigma}_\epsilon(p) \in E$ changes with ϵ, and thus yields the tangent vector $\xi(p)$ of the curve $\epsilon \mapsto \tilde{\sigma}_\epsilon(p)$ at $\epsilon = 0$. This tangent vector is vertical, i.e., $\pi_*(\xi(p)) = 0$, and thus an element of the vertical tangent space $V_{\sigma(p)}E$, since the curve $\epsilon \mapsto \tilde{\sigma}_\epsilon(p)$ must be contained in the fiber $\pi^{-1}(p) \subset E$, which follows from the fact that each $\tilde{\sigma}_\epsilon$ is a section. In summary, the object $\xi = d\tilde{\sigma}_\epsilon/d\epsilon|_{\epsilon=0}$ assigns to each $p \in M$ a vertical tangent vector $\xi(p) \in V_{\sigma(p)}E$. Writing the vertical bundle as $\nu : VE \to E$ we thus see that $\xi \in C^\infty(M, VE)$ with $\nu \circ \xi = \sigma$. Further, since σ is a section, we find that $\pi \circ \nu \circ \xi = \mathrm{id}_M$,

so that ξ is a section of the bundle $\pi \circ \nu : VE \to M$. Note that this bundle is *not* a vector bundle, since the vector space structure on VE is defined only on the fibers over E.
To illustrate the construction above, we introduce coordinates (x^α) on a trivializing subset $U \subset M$ and (x^α, y^a) on $\pi^{-1}(U) \subset E$. The tangent bundle TE is spanned by the coordinate vector fields $\partial_\alpha = \partial/\partial x^\alpha$ and $\bar{\partial}_a = \partial/\partial y^a$, where the latter only span the vertical tangent bundle VE. We thus have coordinates $(x^\alpha, y^a, u^\alpha, v^a)$ on TE and (x^α, y^a, v^a) on VE, where (x^α, y^a) specify a point in E and the remaining coordinates denote a vector $u^\alpha\partial_\alpha + v^a\bar{\partial}_a$ in $V_{x,y}E$. In these coordinates a point $p \in U$ can be expressed by its coordinates x, while a section σ can locally be expressed by the coordinate functions $y(x)$. Given a family σ_ϵ of sections, we thus have a family of functions $y_\epsilon(x)$. The coordinate expression for ξ then takes the form

$$x \mapsto v^a(x)\bar{\partial}_a = \left.\frac{dy^a_\epsilon(x)}{d\epsilon}\right|_{\epsilon=0} \bar{\partial}_a \in V_{y_0(x)}E\,. \tag{32.1}$$

We now consider the variation of $j^\infty\tilde{\sigma}_\epsilon \in \Gamma(J^\infty(E))$, which we aim to express in terms of ξ. This variation can be understood in the same way as the variation of $\tilde{\sigma}_\epsilon$. For a fixed $p \in M$ we have a curve

$$\epsilon \mapsto j^\infty\tilde{\sigma}_\epsilon(p) = j^\infty_p\tilde{\sigma}_\epsilon \in J^\infty_p(E) \subset J^\infty(E)\,. \tag{32.2}$$

One may intuitively want to take the tangent vector of this curve at $\epsilon = 0$. However, since $J^\infty(E)$ is not a manifold, we first have to define the space in which it lives and how to derive it.

Definition 32.1 ((Vertical) tangent bundle of $J^\infty(E)$)**.** Let $\pi : E \to M$ be a fiber bundle and $J^\infty(E)$ its infinite jet space. The *tangent bundle* of $J^\infty(E)$ is the projective limit

$$TJ^\infty(E) = \varprojlim TJ^r(E) = \left\{ (v_0, v_1, \ldots) \in \bigtimes_{r=0}^{\infty} TJ^r(E) \,\middle|\, \forall k \leqslant r : \pi_{r,k*}(v_r) = v_k \right\}. \tag{32.3}$$

The *vertical tangent bundle* of $J^\infty(E)$ is the projective limit $VJ^\infty(E) = \varprojlim VJ^r(E)$ defined analogously, where $VJ^r(E) = \ker \pi_{r*} \subset TJ^r(E)$.

In other words, an element of $VJ^\infty(E)$ is an infinite sequence of elements $v_r \in V^r(E)$ such that for all $k \leqslant r$ the condition $\pi_{r,k*}(v_r) = v_k$ is satisfied. For the curve above this infinite sequence is given by

$$\left.\frac{d}{d\epsilon}j^\infty\tilde{\sigma}_\epsilon(p)\right|_{\epsilon=0} = \left(\left.\frac{d}{d\epsilon}j^0\tilde{\sigma}_\epsilon(p)\right|_{\epsilon=0}, \left.\frac{d}{d\epsilon}j^1\tilde{\sigma}_\epsilon(p)\right|_{\epsilon=0}, \ldots \right) \in V_{j^\infty_p\sigma}J^\infty(E)\,. \tag{32.4}$$

One easily checks that the members of this sequence are indeed vertical tangent vectors and satisfy the condition $\pi_{r,k*}(v_r) = v_k$ for all $k \leqslant r$.
We also introduce suitable coordinate bases on $TJ^\infty(E)$ and $VJ^\infty(E)$. Recall that on $J^\infty(E)$ we used coordinates (x^α, y^a_Λ) derived from the coordinates (x^α) on M and (x^α, y^a) on E. An element of a tangent space $T_qJ^\infty(E)$ with $q \in J^\infty(E)$ can be written in the form

$$u^\alpha\frac{\partial}{\partial x^\alpha} + v^a_\Lambda\frac{\partial}{\partial y^a_\Lambda} = u^\alpha\partial_\alpha + v^a_\Lambda\bar{\partial}^\Lambda_a\,. \tag{32.5}$$

This yields us coordinates (u^α, v^a_Λ) on $T_qJ^\infty(E)$, and thus coordinates $(x^\alpha, y^a_\Lambda, u^\alpha, v^a_\Lambda)$ on $TJ^\infty(E)$, where the first half specifies the point q and the second half the tangent vector at q. On the vertical tangent bundle $VJ^\infty(E)$ one thus has coordinates $(x^\alpha, y^a_\Lambda, v^a_\Lambda)$.
We finally express these elements in terms of the section ξ we encountered earlier. For this purpose, note that since ξ is a section, one can construct is prolongation $j^\infty\xi$ to the infinite jet bundle over $\pi \circ \nu : VE \to M$. To understand the role of this section, we need the following statement.

Theorem 32.1. *For $r \in \mathbb{N}$, there exists a canonical isomorphism from $J^r(\pi \circ \nu : VE \to M)$ to $VJ^r(\pi : E \to M)$, which yields an isomorphism from $J^\infty(\pi \circ \nu : VE \to M)$ to $VJ^\infty(\pi : E \to M)$.*

We will not construct this isomorphism explicitly here. Using coordinates one can easily see that on $J^\infty(\pi\circ\nu : VE \to M)$ one can introduce the same coordinates $(x^\alpha, y^a_\Lambda, v^a_\Lambda)$ as on $VJ^\infty(\pi : E \to M)$. In these coordinates the isomorphism mentioned above simply amounts to the identity map. We can thus view the ∞-jet of the section $\xi : M \to VE$ as a section $j^\infty\xi : M \to VJ^\infty(E)$, which in coordinates takes the form

$$x \mapsto v^a_\Lambda(x)\bar{\partial}^\Lambda_a = \left.\frac{dy^a_{\Lambda,\epsilon}(x)}{d\epsilon}\right|_{\epsilon=0} \bar{\partial}^\Lambda_a \in V_{y_{\Lambda,0}(x)}E\,. \tag{32.6}$$

It is now easy to see that $j^\infty\xi$ is the object $dj^\infty\tilde{\sigma}_\epsilon/d\epsilon$ we wanted to construct.

33 Variation of forms on $J^\infty(E)$

In order to calculate the variation of the action we need to know how $(j^\infty\tilde{\sigma}_\epsilon)^*(L)$ varies with ϵ. This is given by the following theorem.

Theorem 33.1. *Let $\tilde{\sigma}_\epsilon : M \to E$ be a smooth family of sections of the fiber bundle $\pi : E \to M$ and $\omega \in \Omega^{k,0}(J^\infty)(E))$ a horizontal k-form on the infinite jet bundle $J^\infty(E)$. Then the pullback of ω along $j^\infty\tilde{\sigma}_\epsilon$ satisfies*

$$\left.\frac{d}{d\epsilon}(j^\infty\tilde{\sigma}_\epsilon)^*(\omega)\right|_{\epsilon=0} = (j^\infty\sigma)^*\left(\iota_{j^\infty\xi}(d_V\omega)\right)\,, \tag{33.1}$$

where $\xi = d\tilde{\sigma}_\epsilon/d\epsilon|_{\epsilon=0}$ and $\sigma = \tilde{\sigma}_0$.

We will not prove this here, but rather illustrate it using the coordinates we introduced in the previous section. Recall that any horizontal k-form can be written in the form

$$\omega(x, y_\Lambda) = \omega_{\alpha_1\cdots\alpha_k}(x, y_\Lambda)dx^{\alpha_1} \wedge \ldots \wedge dx^{\alpha_k}\,. \tag{33.2}$$

Its pullback to M along the ∞-jet of a section $\tilde{\sigma}_\epsilon$ is obtained by replacing the coordinate arguments y^a_Λ by the partial derivatives $\partial_\Lambda y^a_\epsilon(x)$ of $\tilde{\sigma}_\epsilon$, so that one obtains

$$(j^\infty\tilde{\sigma}_\epsilon)^*(\omega)(x) = \omega_{\alpha_1\cdots\alpha_k}(x, \partial_\Lambda y_\epsilon(x))dx^{\alpha_1} \wedge \ldots \wedge dx^{\alpha_k}\,. \tag{33.3}$$

Taking the derivative with respect to ϵ and using the chain rule yields the left hand side

$$\begin{aligned}\left.\frac{d}{d\epsilon}(j^\infty\tilde{\sigma}_\epsilon)^*(\omega)\right|_{\epsilon=0} &= \left[\left.\frac{d}{d\epsilon}\partial_\Lambda y^a_\epsilon(x)\right|_{\epsilon=0}\bar{\partial}^\Lambda_a\omega_{\alpha_1\cdots\alpha_k}(x, \partial_\Lambda y(x))\right]dx^{\alpha_1} \wedge \ldots \wedge dx^{\alpha_k}\\ &= \left[v^a_\Lambda(x)\bar{\partial}^\Lambda_a\omega_{\alpha_1\cdots\alpha_k}(x, \partial_\Lambda y(x))\right]dx^{\alpha_1} \wedge \ldots \wedge dx^{\alpha_k}\,,\end{aligned} \tag{33.4}$$

where we used the coordinate expression $v^a_\Lambda(x)\bar{\partial}^\Lambda_a$ for the ∞-jet $j^\infty\xi$ of ξ. To compare with the right hand side, we calculate the vertical derivative

$$d_V\omega(x, y_\Lambda) = \bar{\partial}^\Lambda_a \omega_{\alpha_1\cdots\alpha_k}(x, y_\Lambda)\theta^a_\Lambda \wedge dx^{\alpha_1} \wedge \ldots \wedge dx^{\alpha_k}\,. \tag{33.5}$$

After inserting the vertical vector field $j^\infty\xi$, which satisfies $\theta^a_\Lambda(j^\infty\xi) = v^a_\Lambda(x)$ and $dx^\alpha(j^\infty\xi) = 0$, and taking the pullback via $j^\infty\sigma$ we finally arrive at the same coordinate expression as for the left hand side.
If $\omega = L$ is a Lagrangian, we thus find that the action is stationary at the section $\sigma : M \to E$ if and only if

$$0 = \delta S = \left.\frac{dS[\tilde{\sigma}_\epsilon]}{d\epsilon}\right|_{\epsilon=0} = \int_M (j^\infty\sigma)^* \left(\iota_{j^\infty\xi}(d_V L)\right) \tag{33.6}$$

for every $\xi : M \to VE$ with $\nu \circ \xi = \omega$, i.e., if and only if

$$(j^\infty\sigma)^* \left(\iota_{j^\infty\xi}(d_V L)\right) \tag{33.7}$$

is exact every $\xi : M \to VE$ with $\nu \circ \xi = \omega$. This already brings us closer to our goal. However, this expression is still rather cumbersome, as it requires calculating the ∞-jet $j^\infty\xi$ for every possible ξ, and checking whether the result is an exact form. To get rid of this calculation, we need another step.

34 Integration by parts

We will now further simplify the condition for a stationary action. In this section we discuss the question under which circumstances the pullback $(j^\infty\sigma)^*(\omega)$ of a horizontal k-form $\omega \in \Omega^{k,0}(J^\infty(E))$ is exact. The answer to this question is given by the following statement.

Theorem 34.1. *The pullback $(j^\infty\sigma)^*(\omega)$ of a d_H-exact horizontal k-form $\omega \in \Omega^{k,0}(J^\infty(E))$ is exact.*

We will not prove this here, but illustrate it using coordinates. Let ω be the k-form given by

$$\omega(x, y_\Lambda) = \omega_{\alpha_1\cdots\alpha_k}(x, y_\Lambda)dx^{\alpha_1} \wedge \ldots \wedge dx^{\alpha_k}\,. \tag{34.1}$$

For its pullback along the ∞-jet of a section σ we write

$$(j^\infty\sigma)^*(\omega)(x) = \omega_{\alpha_1\cdots\alpha_k}(x, \partial_\Lambda y_(x))dx^{\alpha_1} \wedge \ldots \wedge dx^{\alpha_k}\,. \tag{34.2}$$

In the case that $\omega = d_h\eta$ is d_h-exact, we have

$$d_h\eta(x, y_\Lambda) = \left[\frac{\partial\eta_{\alpha_1\cdots\alpha_{k-1}}}{\partial x^\beta} + \sum_\Lambda y^a_{(\lambda_1,\ldots,\lambda_\beta+1,\ldots,\lambda_n)}\frac{\partial\eta_{\alpha_1\cdots\alpha_{k-1}}}{\partial y^a_\Lambda}\right] dx^\beta \wedge dx^{\alpha_1} \wedge \ldots \wedge dx^{\alpha_{k-1}}\,, \tag{34.3}$$

whose pullback along $j^\infty\sigma$ is just the total derivative

$$(j^\infty\sigma)^*(d_h\eta)(x) = d(j^\infty\sigma)^*(\eta)(x)\,. \tag{34.4}$$

This yields the statement above for our chosen coordinates - of course we would have to show it in an coordinate independent fashion if we wanted a proof.
Returning to our original problem, we may thus add an arbitrary d_H-exact form $d_H\eta \in \Omega^{n,0}(J^\infty(E))$ to $\iota_{j^\infty\xi}(d_V L)$ without changing the exactness of $(j^\infty\sigma)^* \left(\iota_{j^\infty\xi}(d_V L)\right)$. Here also the following statement will help.

Theorem 34.2. *For any horizontal k-form η and section $\xi : M \to VE$ holds*

$$d_H(\iota_{j^\infty\xi}\eta) = -\iota_{j^\infty\xi}(d_H\eta) \,. \tag{34.5}$$

The proof is straightforward. This in particular means that if η is d_H-exact, then also $\iota_{j^\infty\xi}\eta$ is d_H-exact. For our problem thus follows that we may add an arbitrary d_H-exact form $d_H\eta \in \Omega^{n,1}(J^\infty(E))$ to $d_V L$. We define the following operator, which will yield us this form.

Definition 34.1 (Internal Euler operator)**.** Let $\pi : E \to M$ be a fiber bundle with $\dim(M) = n$ and $\Omega^{n,s}(J^\infty(E))$ with $s \geqslant 1$ the space of forms of type (n,s) on the infinite jet bundle $J^\infty(E)$. The *internal Euler operator* is the unique function $\varrho : \Omega^{n,s}(J^\infty(E)) \to \Omega^{n,s}(J^\infty(E))$ such that:

- ϱ is a projector: $\varrho^2 = \varrho$.
- For $\omega \in \Omega^{n,s}(J^\infty(E))$, the difference $\omega - \varrho(\omega)$ is d_H-exact, i.e. there exists $\eta \in \Omega^{n-1,s}(J^\infty(E))$ such that $d_H\eta = \omega - \varrho(\omega)$.
- ϱ vanishes on d_H-exact forms: $\varrho \circ d_H = 0$.
- $\iota_X \circ \varrho = 0$ for all vector fields X on $J^\infty(E)$ with $\pi_{\infty,0*} \circ X = 0$.

We will not prove the existence and uniqueness of the internal Euler operator here, and we will not construct it explicitly. Instead, we will only provide the coordinate expression, which is given by

$$\begin{array}{cccc} \varrho & : & \Omega^{n,s}(J^\infty(E)) & \to & \Omega^{n,s}(J^\infty(E)) \\ & & \omega & \mapsto & \dfrac{1}{s}\sum_\Lambda (-1)^{|\Lambda|}\theta^a_{(0,\dots,0)} \wedge d_\Lambda\left(\iota_{\bar\partial^\Lambda_a}\omega\right) \end{array} . \tag{34.6}$$

Here we introduced the total derivative operator

$$d_\Lambda = (d_1)^{\lambda_1}\cdots(d_n)^{\lambda_n} \,, \tag{34.7}$$

which acts on functions $f \in \Omega^0(J^\infty(E))$ as

$$d_\alpha f = \partial_\alpha f + \sum_\Lambda y^a_{(\lambda_1,\dots,\lambda_\alpha+1,\dots,\lambda_n)}\bar\partial^\Lambda_a f \,. \tag{34.8}$$

To construct its action on higher degree forms, one uses the rules

$$d_\alpha(\omega \wedge \eta) = d_\alpha(\omega) \wedge \eta + \omega \wedge d_\alpha(\eta) \,, \quad d_\alpha(d\omega) = d(d_\alpha\omega) \,. \tag{34.9}$$

From these follows in particular the action on the coordinate one-forms as

$$d_\alpha dx^\beta = 0 \,, \quad d_\alpha dy^a_\Lambda = dy^a_{(\lambda_1,\dots,\lambda_\alpha+1,\dots,\lambda_n)} \,. \tag{34.10}$$

Note that the total derivative is *not* an exterior derivative - it does not change the degree of a form, does not square to zero and is not an antiderivation.

Since ϱ is a projector, $\varrho(\Omega^{n,s}(J^\infty(E)))$ is an invariant subspace which we denote $\mathcal{F}^s(J^\infty(E))$, which we can describe in coordinates as follows. Since a vector field X on $J^\infty(E)$ with $\pi_{\infty,0*} \circ X = 0$ has the coordinate expression

$$X = \sum_{|\Lambda| \geqslant 1} X_a^\Lambda \bar{\partial}_\Lambda^a \,, \tag{34.11}$$

it follows from the last condition in the definition of the internal Euler operator that the elements of $\mathcal{F}^s(J^\infty(E))$ are of the form

$$\omega = \omega_{a_1 \cdots a_s} \theta^{a_1}_{(0,\ldots,0)} \wedge \ldots \wedge \theta^{a_s}_{(0,\ldots,0)} \wedge dx^1 \wedge \ldots \wedge dx^n \,. \tag{34.12}$$

For a section $\xi : M \to VE$, which we expressed by the coordinate functions $y^a(x)$ and $v^a(x)$, we thus find that $\iota_{j^\infty\xi}\omega$ does not depend on the derivatives of the coordinate functions $v^a(x)$. We can thus simplify the task of finding sections $\sigma : M \to E$ for which the action is stationary by replacing $d_V L$ with $\varrho(d_V L)$.

35 Euler operator and Euler-Lagrange equations

We now finally use the results we derived so far and put them together. For this purpose we first introduce another helpful shorthand notation.

Definition 35.1 (Euler operator)**.** The *Euler operator* is the function $\mathcal{E} = \varrho \circ d_V : \Omega^{n,0}(J^\infty(E)) \to \mathcal{F}^1(J^\infty(E))$.

Writing a Lagrangian $L \in \Omega^{n,0}(J^\infty(E))$ in coordinates as $L = \mathcal{L} dx^1 \wedge \ldots \wedge dx^n$, we can write the Euler operator as

$$\mathcal{E}L = \mathcal{E}_a\mathcal{L}\, \theta^a_{(0,\ldots,0)} \wedge dx^1 \wedge \ldots \wedge dx^n \,, \tag{35.1}$$

where

$$\mathcal{E}_a\mathcal{L} = \sum_\Lambda (-1)^{|\Lambda|} d_\Lambda \left(\bar{\partial}_a^\Lambda \mathcal{L}\right) \,. \tag{35.2}$$

For completeness, we introduce another operator.

Definition 35.2 (Augmented vertical derivative)**.** For $s \geqslant 1$, the *augmented vertical derivative* is the function $\delta_V = \varrho \circ d_V : \mathcal{F}^s(J^\infty(E)) \to \mathcal{F}^{s+1}(J^\infty(E))$.

With this definition we can now extend the variational bicomplex introduced in the last lecture as follows.

$$
\begin{array}{ccccccccc}
\vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
\Omega^{0,2} & \xrightarrow{d_H} & \Omega^{1,2} & \dashrightarrow & \Omega^{n-1,2} & \xrightarrow{d_H} & \Omega^{n,2} & \xrightarrow{\varrho} & \mathcal{F}^2 \\
{\scriptstyle d_V}\uparrow & & {\scriptstyle d_V}\uparrow & & {\scriptstyle d_V}\uparrow & & {\scriptstyle d_V}\uparrow & & {\scriptstyle \delta_V}\uparrow \\
\Omega^{0,1} & \xrightarrow{d_H} & \Omega^{1,1} & \dashrightarrow & \Omega^{n-1,1} & \xrightarrow{d_H} & \Omega^{n,1} & \xrightarrow{\varrho} & \mathcal{F}^1 \\
{\scriptstyle d_V}\uparrow & & {\scriptstyle d_V}\uparrow & & {\scriptstyle d_V}\uparrow & & {\scriptstyle d_V}\uparrow & \overset{E}{\nearrow} & \\
\Omega^{0,0} & \xrightarrow{d_H} & \Omega^{1,0} & \dashrightarrow & \Omega^{n-1,0} & \xrightarrow{d_H} & \Omega^{n,0} & &
\end{array}
\tag{35.3}
$$

Here we omitted the part $J^\infty(E)$ in the notation. The edge

$$\Omega^{0,0} \xrightarrow{d_H} \cdots \xrightarrow{d_H} \Omega^{n,0} \xrightarrow{E} \mathcal{F}^1 \xrightarrow{\delta_V} \cdots \tag{35.4}$$

is called the *Euler-Lagrange complex.*
To proceed with our problem of stationary actions, we come to another very helpful statement.

Theorem 35.1. *Let $\xi : M \to VE$ be a section and $\sigma = \nu \circ \xi : M \to E$. The pullback $(j^\infty\sigma)^* \left(\iota_{j^\infty\xi}(\mathcal{E}L)\right)$ is exact if and only if it vanishes.*

This greatly simplifies our task. Instead of determining whether a differential form is exact, we need to check whether it vanishes. But we can simplify our task even more by the help of the following statement.

Theorem 35.2. *Let $\sigma : M \to E$ be a section. The pullback $(j^\infty\sigma)^* \left(\iota_{j^\infty\xi}(\mathcal{E}L)\right)$ vanishes for all $\xi : M \to VE$ with $\sigma = \nu \circ \xi$ if and only if $(\mathcal{E}L) \circ \sigma = 0$.*

We have now found an amazingly simple condition. Given a Lagrangian $L \in \Omega^{n,0}(J^\infty(E))$ we can now determine the sections $\sigma : M \to E$ as the solutions of the equation

$$(\mathcal{E}L) \circ \sigma = 0\,, \tag{35.5}$$

which in coordinates turns into a differential equation for σ. This equation is called the *Euler-Lagrange equation.*