Differential geometry for physicists - Lecture 9

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28 The infinite jet space

We have seen in the last lecture that for every fiber bundle $\pi : E \to M$ the jet spaces $J^r(E)$ for $r \in \mathbb{N}$ form an inverse sequence

$$M \xleftarrow{\pi} E \xleftarrow{\pi_{1,0}} J^1(E) \xleftarrow{\pi_{2,1}} J^2(E) \xleftarrow{\pi_{3,2}} \dots, \tag{28.1}$$

where the maps $\pi_{r,k} : J^r(E) \to J^k(E)$ are the projections of fiber bundles. For the purpose of this lecture we need to discuss what happens in the limit $r \to \infty$. We define this limit as follows.

Definition 28.1 (Infinite jet space)**.** Let $\pi : E \to M$ be a fiber bundle. Its *infinite jet space* is the projective limit

$$J^\infty(E) = \varprojlim J^r(E) = \left\{ (z_0, z_1, \dots) \in \bigtimes_{r=0}^{\infty} J^r(E) \middle| \forall k \leqslant r : \pi_{r,k}(z_r) = z_k \right\}. \tag{28.2}$$

An element of $J^\infty(E)$ is thus an infinite sequence of elements $z_r \in J^r(E)$ such that for all $k \leqslant r$ the condition $\pi_{r,k}(z_r) = z_k$ is satisfied. To understand the meaning of this, recall that a jet $z_r \in J^r(E)$ is an equivalence class of local sections of E such that their "partial derivatives up to order r" agree. The condition $\pi_{r,k}(z_r) = z_k$ here simply means that if $z_r = j^r_p\sigma$ for some point $p \in M$ and some local section $\sigma \in \Gamma_p(E)$, then $z_k = j^k_p\sigma$. In other words, any lower element z_k of this sequence is uniquely defined by any higher element z_r by throwing away any derivatives of order higher than k. Naively, we could thus just forget about almost all elements of the sequence and only look at the last one, which contains all derivatives - but of course, there is no such last element in an infinite sequence. So the only way to describe a section and "all of its infinitely many derivatives" is by an infinite sequence like the one above, and these sequences form the infinite jet space $J^\infty(E)$.
Given coordinates (x^α) on a trivializing neighborhood $U \in M$ and (y^a) on the fiber F of the bundle $\pi : E \to M$, so that we have coordinates (x^α, y^a) on $\pi^{-1}(U) \cong U \times F$ we have previously introduced coordinates (x^α, y^a_Λ) with $0 \leqslant |\Lambda| \leqslant r$ on $\pi_r^{-1}(U) \subset J^r(E)$. We get (infinitely many) coordinates on $J^\infty(E)$ by dropping the upper bound and allowing all multiindices Λ with $|\Lambda| \in \mathbb{N}$.
Note that $J^\infty(E)$ is *not* a manifold in the sense we defined manifolds - it is not locally diffeomorphic to any finite-dimensional Euclidean space $\mathbb{R}^n$. It has *some* properties of a manifold, so that *some* operations on manifolds can be generalized to $J^\infty(E)$, but not all

of them, so we have to be careful when working with this object. The following notions can nicely be generalized.

Definition 28.2 (∞-jet projection)**.** Let $\pi : E \to M$ be a fiber bundle. For $r \in \mathbb{N}$ we define the *jet projection*

$$\begin{array}{rcccc} \pi_{\infty,r} & : & J^\infty(E) & \to & J^r(E) \\ & & (z_0, z_1, \ldots) & \mapsto & z_r \end{array} . \tag{28.3}$$

The function $\pi_{\infty,0} : J^\infty(E) \to E$ is called the *target projection*, while $\pi_\infty = \pi \circ \pi_{\infty,0} : J^\infty(E) \to M$ is called the *source projection*.

As it is also the case for finite jet bundles, these projections throw away all derivatives of higher order that a fixed $r \in \mathbb{N}$.

Definition 28.3 (∞-jet of a section)**.** Let $\pi : E \to M$ be a fiber bundle, $p \in M$ and $\sigma \in \Gamma_p(E)$ a local section whose domain contains p. We define the *∞-jet* $j^\infty_p\sigma$ of σ at p as the infinite sequence

$$(j^0_p\sigma, j^1_p\sigma, \ldots) \in J^\infty(E) . \tag{28.4}$$

The ∞-jet is the object which captures "all derivatives" of a local section σ at some point $p \in M$.

Definition 28.4 (∞-jet prolongation)**.** Let $\pi : E \to M$ be a fiber bundle and σ a local section with domain $U \subset M$. Its ∞-jet prolongation is the function

$$\begin{array}{rcccc} j^\infty\sigma & : & U & \to & J^\infty(E) \\ & & p & \mapsto & j^\infty_p\sigma \end{array} . \tag{28.5}$$

As in the finite-dimensional case, taking the ∞-jet at each point p in the domain of σ yields its prolongation into $J^\infty(E)$.

29 The variational bicomplex

Another concept that can nicely be generalized to $J^\infty(E)$ is that of differential forms. Note that the pullbacks along the projection maps define a sequence

$$\Omega^k(M) \xrightarrow{\pi^*} \Omega^k(E) \xrightarrow{\pi^*_{1,0}} \Omega^k(J^1(E)) \xrightarrow{\pi^*_{2,1}} \Omega^k(J^2(E)) \xrightarrow{\pi^*_{3,2}} \ldots \tag{29.1}$$

for all $k \in \mathbb{N}$. Here it makes sense to consider k-forms with arbitrarily high k, since the dimension of the manifolds $J^r(E)$ is growing with r, so there will be non-trivial k-forms for any k. We can use this sequence to define the following object.

Definition 29.1 (Pullback to $J^\infty(E)$)**.** Let $\pi : E \to M$ be a fiber bundle and $J^\infty(E)$ its infinite jet space. For $k \in \mathbb{N}$ we define the space $\Omega^k(J^\infty(E))$ as the direct limit

$$\Omega^k(J^\infty(E)) = \varinjlim \Omega^k(J^r(E)) = \biguplus_{r=0}^{\infty} \Omega^k(J^r(E)) \Big/ \sim \,, \tag{29.2}$$

where two k-forms $\omega \in \Omega^k(J^r(E))$ and $\chi \in \Omega^k(J^{r'}(E))$ are considered equivalent, $\omega \sim \chi$, if and only if there exists $r'' \geqslant \max(r, r')$ such that $\pi^*_{r'',r}(\omega) = \pi^*_{r'',r'}(\chi)$. The equivalence class of $\omega \in \Omega^k(J^r(E))$ is denoted $\pi^*_{\infty,r}(\omega) \in \Omega^k(J^\infty(E))$ and called the *pullback* of ω to $J^\infty(E)$.

Note that despite the notation, the elements of $\Omega^k(J^\infty(E))$ are *not* differential forms, and $J^\infty(E)$ is not a manifold, so we cannot immediately use any operations which we defined on differential forms. Instead, they are equivalence classes of differential forms on finite jet spaces. Note that since for $r' \geqslant r$ the pullbacks $\pi^*_{r',r} : \Omega^k(J^r(E)) \to \Omega^k(J^{r'}(E))$ are injective functions (which is a consequence of the fact that the maps $\pi_{r',r} : J^{r'}(E) \to J^r(E)$ are surjective), two k-forms $\omega \in \Omega^k(J^r(E))$ and $\chi \in \Omega^k(J^{r'}(E))$ are equivalent if and only if $\pi^*_{r',r}\omega = \chi$. In other words, we identify all elements of $\Omega^k(J^r(E))$ with their images in $\Omega^k(J^{r'}(E))$. We thus obtain a sequence of inclusions

$$\Omega^k(J^0(E)) \tilde{\subset} \Omega^k(J^1(E)) \tilde{\subset} \ldots \tilde{\subset} \Omega^k(J^\infty(E)) \,, \tag{29.3}$$

where $\tilde{\subset}$ should be read as "the set formed by equivalence classes of elements contained in the set on the left is a subset of the set formed by equivalence classes of elements contained in the set on the right".

To illustrate this definition, we discuss how to write the elements of $\Omega^k(J^\infty(E))$ using the coordinates (x^α, y^a_Λ) we introduced on (finite and infinite) jet bundles. Any k-form $\omega \in \Omega^k(J^r(E))$ on a finite jet bundle $J^r(E)$ can be written as a finite linear combination of k-fold wedge products of the coordinate one-forms dx^α, dy^a_Λ, where $|\Lambda| \leqslant r$. The pullback $\pi^*_{r',r}(\omega) \in \Omega^k(J^{r'}(E))$ of ω is a k-form on $J^{r'}(E)$ which has the same coordinate representation. Hence, the equivalence relation we introduced above simply identifies k-forms if and only if their coordinate representations in the coordinates (x^α, y^a_Λ) agree. We can thus formally write an element of $\Omega^k(J^\infty(E))$ as a finite linear combination of k-fold wedge products of the coordinate one-forms dx^α, dy^a_Λ, where $|\Lambda| \in \mathbb{N}$. Note, however, that so far this is only a notation - we have not defined a wedge product of such equivalence classes yet. But actually we can do so.

Definition 29.2 (Exterior product)**.** Let $\pi : E \to M$ be a fiber bundle and $\omega \in \Omega^k(J^\infty(E))$, $\chi \in \Omega^l(J^\infty(E))$. By definition, we can find $r \in \mathbb{N}$, $\bar{\omega} \in \Omega^k(J^r(E))$ and $\bar{\chi} \in \Omega^l(J^r(E))$ such that $\omega = \pi^*_{\infty,r}(\bar{\omega})$ and $\chi = \pi^*_{\infty,r}(\bar{\chi})$. We define the *exterior product* (or wedge product)

$$\omega \wedge \chi = \pi^*_{\infty,r}(\bar{\omega} \wedge \bar{\chi}) \in \Omega^{k+l}(J^\infty(E)) \,. \tag{29.4}$$

We need a few remarks on this definition. First, note that we can always pick representatives $\bar{\omega}, \bar{\chi}$ of the equivalence classes ω, χ which are differential forms on the same jet space $J^r(E)$.

If we had picked one of them to be on a different jet space $J^{r'}(E)$ with $r' < r$, we could just obtain another representative on $J^r(E)$ by applying the pullback $\pi^*_{r,r'}$. Further, the wedge product above is well-defined, i.e., independent of the choice of the jet space $J^r(E)$ from which we take the representatives, since the pullback distributes over wedge products. Note that in coordinates the wedge product just looks as it always looks like for ordinary differential forms, so we can just calculate it as usual. The same applies to the exterior derivative, which we define as follows.

Definition 29.3 (Exterior derivative)**.** Let $\pi : E \to M$ be a fiber bundle and $\omega \in \Omega^k(J^\infty(E))$. By definition, we can find $r \in \mathbb{N}$ and $\bar{\omega} \in \Omega^k(J^r(E))$ such that $\omega = \pi^*_{\infty,r}(\bar{\omega})$. We define the *exterior derivative*

$$d\omega = \pi^*_{\infty,r}(d\bar{\omega}) \in \Omega^{k+1}(J^\infty(E)) \,. \tag{29.5}$$

Also this is well-defined, since pullbacks and the exterior derivative commute. Also the exterior derivative looks in coordinates just as in the finite case. Finally, it is also easy to prove that the exterior derivative and exterior product satisfy all their nice properties which they also have for ordinary differential forms on finite-dimensional manifolds. We can therefore just use them as we would naturally do. In particular, the exterior derivative satisfies $d^2 = 0$, so that we have an infinite sequence

$$\Omega^0(J^\infty(E)) \xrightarrow{d} \Omega^1(J^\infty(E)) \xrightarrow{d} \dots \,, \tag{29.6}$$

where the image of each function lies inside the kernel of the next one. This structure is called a *complex*. We will further refine this structure, and for this purpose need to decompose it further as follows.

Definition 29.4 (Horizontal form)**.** Let $\pi : E \to M$ be a fiber bundle. An element $\omega \in \Omega^k(J^\infty(E))$ is called *horizontal* if it is the pullback $\omega = \pi^*_{\infty,r}(\bar{\omega})$ of a horizontal k-form $\bar{\omega} \in \Omega^k(J^r(E))$, i.e., such that $\bar{\omega}$ vanishes on the kernel $\ker \pi_{r*}$ of $\pi_{r*} : TJ^r(E) \to TM$. The subspace of horizontal elements of $\Omega^k(J^\infty(E))$ is denoted $\Omega^{k,0}(J^\infty(E))$.

The kernel of π_{r*} is defined as the set of tangent vectors $v \in TJ^r(E)$ for which $\pi_{r*}(v) = 0$. These tangent vectors are tangent to the fibers $\pi_r^{-1}(p) \cong J^r_p(E)$ for $p \in M$. In coordinates (x^α, y^a_Λ) on $J^r(E)$ the space of vertical vectors is spanned by the vector fields $\partial^\Lambda_a = \partial/\partial y^a_\Lambda$. A k-form $\bar{\omega} \in \Omega^k(J^r(E))$ vanishes on these vectors if its coordinate representation contains only wedge products of dx^α, but no dy^a_Λ. The same holds for the coordinate representation of $\omega \in \Omega^k(J^\infty(E))$.
We also define a suitable counterpart.

Definition 29.5 (Contact form)**.** Let $\pi : E \to M$ be a fiber bundle. An element $\omega \in \Omega^k(J^\infty(E))$ is called a *contact form* if its pullback $(j^\infty\sigma)^*(\omega) \in \Omega^k(M)$ vanishes for every local section σ of $\pi : E \to M$. The subspace of contact forms of $\Omega^k(J^\infty(E))$ is denoted $\Omega^{0,k}(J^\infty(E))$.

The meaning of the pullback $(j^\infty\sigma)^*(\omega)$ for $\omega \in \Omega^k(J^\infty(E))$ should be almost clear. We can pick a representative $\bar{\omega} \in \Omega^k(J^r(E))$, and take its pullback $(j^r\sigma)^*(\bar{\omega}) \in \Omega^k(M)$ along the map $j^r\sigma : M \to J^r(E)$. This pullback is independent of the choice of the representative, and so defines a unique pullback $(j^\infty\sigma)^* : \Omega^k(J^\infty(E)) \to \Omega^k(M)$.
Using coordinates (x^α, y^a_Λ) on $J^\infty(E)$ it is easy to write down a basis for the space $\Omega^{0,1}(J^\infty(E))$ of contact one-forms. We define the *basic contact forms* θ^a_Λ as

$$\theta^a_\Lambda = dy^a_\Lambda - y^a_{(\lambda_1+1,\lambda_2,\ldots,\lambda_n)}dx^1 - y^a_{(\lambda_1,\lambda_2+1,\ldots,\lambda_n)}dx^2 - \ldots - y^a_{(\lambda_1,\lambda_2,\ldots,\lambda_n+1)}dx^n\,, \tag{29.7}$$

where $n = \dim(M)$. Any contact one-form θ can be written in the form $\theta = f^\Lambda_a\theta^a_\Lambda$. Further, they can be used to generate higher contact k-forms. This is indeed the case, due to the following property.

Theorem 29.1. *The wedge product of horizontal forms is horizontal. The wedge product of contact forms is a contact form.*

This is not difficult to prove - it follows immediately from the fact that the pullback distributes over wedge products. Using horizontal and contact forms we can now generate all of $\Omega^k(J^\infty(E))$ as a consequence of the following property.

Theorem 29.2. *For each $k \in \mathbb{N}$, the space $\Omega^k(J^\infty(E))$ splits into a direct sum*

$$\Omega^k(J^\infty(E)) = \bigoplus_{m=0}^{k} \Omega^{m,k-m}(J^\infty(E))\,, \tag{29.8}$$

where $\Omega^{k,l}(J^\infty(E))$ denotes the space spanned by wedge products of horizontal k-forms and contact l-forms.

In coordinates, the space $\Omega^{k,l}(J^\infty(E))$ is spanned by wedge products of the form

$$dx^{\alpha_1} \wedge \ldots \wedge dx^{\alpha_k} \wedge \theta^{a_1}_{\Lambda_1} \wedge \ldots \wedge \theta^{a_l}_{\Lambda_l}\,. \tag{29.9}$$

It is now easy to see how the exterior product and exterior derivative behave under this splitting.

Theorem 29.3. *Let $\omega \in \Omega^{k,l}(J^\infty(E))$ and $\chi \in \Omega^{k',l'}(J^\infty(E))$. Then $\omega\wedge\chi \in \Omega^{k+k',l+l'}(J^\infty(E))$ and $d\omega \in \Omega^{k+1,l}(J^\infty(E)) \oplus \Omega^{k,l+1}(J^\infty(E))$.*

This property of the exterior product is immediately clear. For the exterior derivative it means that $d\omega$ can be uniquely written as the sum of two terms, one of them belonging to $\Omega^{k+1,l}(J^\infty(E))$, the other one to $\Omega^{k,l+1}(J^\infty(E))$. This allows us to decompose the exterior derivative in the following way.

Definition 29.6 (Horizontal and vertical differentials)**.** Let $\pi : E \to M$ be a fiber bundle. For $m, n \in \mathbb{N}$ the *horizontal* (or total) differential $d_H : \Omega^{m,n}(J^\infty(E)) \to \Omega^{m+1,n}(J^\infty(E))$ and *vertical* differential $d_V : \Omega^{m,n}(J^\infty(E)) \to \Omega^{m,n+1}(J^\infty(E))$ are the unique functions such that $d_H\omega + d_V\omega = d\omega$ for all $\omega \in \Omega^{m,n}(J^\infty(E))$.

In order to work with these differentials, we first state a few of their properties, which will then allow us to write them using coordinates.

Theorem 29.4. *For each $\omega \in \Omega^{m,n}(J^\infty(E))$ and $\chi \in \Omega^{m',n'}(J^\infty(E))$ the horizontal and vertical differentials d_H and d_V satisfy:*

- *d_H and d_V are antiderivations:*

$$d_H(\omega \wedge \chi) = d_H\omega \wedge \chi + (-1)^{m+n}\omega \wedge d_H\chi\,, \quad d_V(\omega \wedge \chi) = d_V\omega \wedge \chi + (-1)^{m+n}\omega \wedge d_V\chi\,. \tag{29.10}$$

- *$d_H^2 = 0$, $d_V^2 = 0$ and $d_H d_V = -d_V d_H$.*

With these properties we can now construct the coordinate expressions for d_H and d_V by applying them to functions (zero-forms) and one-forms, since all differential forms can be construct from these simplest forms. For $f \in \Omega^{0,0}(J^\infty(E))$ we have the vertical differential given by

$$d_V f = \frac{\partial f}{\partial y^a_\Lambda}\theta^a_\Lambda\,. \tag{29.11}$$

For the horizontal differential then follows

$$d_H f = df - d_V f = D_\alpha f\, dx^\alpha\,, \tag{29.12}$$

where we introduced the total derivative

$$D_\alpha f = \frac{\partial f}{\partial x^\alpha} + \sum_\Lambda y^a_{(\lambda_1,\ldots,\lambda_\alpha+1,\ldots,\lambda_n)}\frac{\partial f}{\partial y^a_\Lambda}\,. \tag{29.13}$$

For the horizontal coordinate differentials we have

$$d_H(dx^\alpha) = 0\,, \quad d_V(dx^\alpha) = 0\,. \tag{29.14}$$

Finally, the basic contact forms satisfy

$$d_H\theta^a_\Lambda = dx^1 \wedge \theta^a_{(\lambda_1+1,\lambda_2,\ldots,\lambda_n)} + dx^2 \wedge \theta^a_{(\lambda_1,\lambda_2+1,\ldots,\lambda_n)} + \ldots + dx^n \wedge \theta^a_{(\lambda_1,\lambda_2,\ldots,\lambda_n+1)}\,, \quad d_V\theta^a_\Lambda = 0\,. \tag{29.15}$$

Since any differential form on $J^\infty(E)$ can be constructed as a linear combination of wedge products of the forms above, we can thus explicitly calculate the vertical and horizontal differentials for all differential forms.

We now return to the sequence induced by the exterior derivative $d : \Omega^k(J^\infty(E)) \to \Omega^{k+1}(J^\infty(E))$. Using the horizontal and vertical differentials we can construct a similar structure, which is not a complex, but a *bicomplex*. For $\dim(M) = n$ it has the form

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \vdots \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
\Omega^{0,1}(J^\infty(E)) & \xrightarrow{d_H} & \Omega^{1,1}(J^\infty(E)) & \dashrightarrow & \Omega^{n-1,1}(J^\infty(E)) & \xrightarrow{d_H} & \Omega^{n,1}(J^\infty(E)) \\
d_V\uparrow & & d_V\uparrow & & d_V\uparrow & & d_V\uparrow \\
\Omega^{0,0}(J^\infty(E)) & \xrightarrow{d_H} & \Omega^{1,0}(J^\infty(E)) & \dashrightarrow & \Omega^{n-1,0}(J^\infty(E)) & \xrightarrow{d_H} & \Omega^{n,0}(J^\infty(E))
\end{array} \tag{29.16}$$

This structure is called the *variational bicomplex*, and we will apply it to describe physical systems in the Lagrangian formulation.

30 Lagrangians and action functionals

We now come to a physical application of the formalism introduced in the previous section. The physical system we consider here is called a *Lagrangian system*. It is modeled by a fiber bundle $\pi : E \to M$, where physical solutions of the system are a subset of the space of sections $\Gamma(E)$. This set of solutions is obtained from an *action principle*. In order to clarify these terms, we start with a few definitions.

Definition 30.1 (Lagrangian)**.** Let $\pi : E \to M$ be a fiber bundle with $\dim M = n$. A *Lagrangian* on E is a horizontal n-form $L \in \Omega^{n,0}(J^\infty(E))$.

Recall that the elements of $J^\infty(E)$ describe sections $\sigma \in \Gamma(E)$ by choosing a point $p \in M$ and evaluating the section to $\sigma(p)$ and its partial derivatives of any order. A Lagrangian thus depends on p, $\sigma(p)$ and its derivatives at p, i.e., on the *local* behavior of the section σ. We know that we can obtain a *global* property if we integrate a differential form. This will be done in the next definition.

Definition 30.2 (Action functional)**.** Let $\pi : E \to M$ be a fiber bundle and L a Lagrangian on E. The *action functional* of L is the function

$$\begin{array}{rcccc} S & : & \Gamma(E) & \to & \mathbb{R} \\ & & \sigma & \mapsto & \displaystyle\int_M (j^\infty\sigma)^*(L) \end{array} . \tag{30.1}$$

Note that while the Lagrangian L is *not* a differential form on a manifold, its pullback $(j^\infty\sigma)^*(L) \in \Omega^n(M)$ is a n-form on M, and as such can be integrated. To illustrate this further, we give an example from classical mechanics.

Example 30.1 (First order Lagrangian of a point mass on a metric manifold with potential). Let $M = \mathbb{R}$ and Q a manifold of dimension n. Let $E = \mathbb{R} \times Q$ be the trivial fiber bundle with projection $\pi : \mathbb{R} \times Q \to \mathbb{R}$ onto the first factor. Sections of this bundle are uniquely expressed by maps $\gamma \in C^\infty(\mathbb{R}, Q)$, i.e., by curves on Q. We use the one-dimensional Euclidean coordinate t on $\mathbb{R}$ and arbitrary coordinates (q^a) on Q, so that we have coordinates (t, q^a) on $\mathbb{R} \times Q$. From these coordinates we derive the coordinates $(t, q^a_{(0)}, q^a_{(1)})$ on $J^1(E) \cong \mathbb{R} \times TQ$.
To construct a particular Lagrangian, let further $g \in \Gamma(T^0_2 Q)$ be a non-degenerate, positive definite, symmetric tensor field of type $(0, 2)$ (the *metric*) and $V \in C^\infty(Q, \mathbb{R})$ (the *potential*). To illustrate this, we do this in five steps, each of which we explain in our geometric language:

- We take an element of $J^1(E) \cong \mathbb{R} \times TQ$ and project it onto the second factor. This yields a tangent vector $q_{(1)} = q^a_{(1)}\partial_a \in T_{q_{(0)}}Q$, where $q_{(0)} \in Q$ is the result of using the bundle map of TQ on $q_{(1)}$. For convenience, we write $q = q_{(0)}, \dot{q} = q_{(1)}$ and also the coordinates $q^a = q^a_{(0)}, \dot{q}^a = q^a_{(1)}$.

- We take the metric g, which is a section of the tensor bundle $T_2^0 Q$, and maps q to $g(q) \in T_2^0{}_q Q$. This is a covariant tensor, so we can contract it with two copies of the vector $\dot{q} \in T_q Q$ and obtain a real number. In coordinates we thus get $g_{ab}(q)\dot{q}^a\dot{q}^b \in \mathbb{R}$. Doing this for all elements of $J^1(E)$ gives us a real function on $J^1(E)$.

- We take the potential V, which is a real function on Q, and evaluate it at q, so we obtain another real number $V(q)$. Doing this for all elements of $J^1(E)$ gives us another real function on $J^1(E)$.

- We take the canonical one-form $\omega = dt \in \Omega^1(\mathbb{R})$ on $\mathbb{R}$ and pull it back via the projection $\pi_\infty : J^\infty(E) \to \mathbb{R}$. This yields us a horizontal one-form $\pi_\infty^*(\omega) = dt \in \Omega^{1,0}(J^\infty(E))$.

- We combine the two real functions and the one-form constructed above to the Lagrangian

$$L(t, q, \dot{q}) = \left(\frac{1}{2} g_{ab}(q)\dot{q}^a\dot{q}^b - V(q)\right) dt \in \Omega^{1,0}(J^\infty(E)) \,. \tag{30.2}$$

Finally, we construct the action functional. For this purpose we consider a section, which in our chosen coordinates is described by functions $q^a(t)$. The pullback along this section then simply amounts to replacing the coordinates q^a and $\dot{q}^a$ in the Lagrangian by $q^a(t)$ and $dq^a(t)/dt$. This yields a one-form on $M = \mathbb{R}$, which can be integrated to the action

$$S[q] = \int_{\mathbb{R}} \left(\frac{1}{2} g_{ab}(q(t)) \frac{dq^a(t)}{dt} \frac{dq^b(t)}{dt} - V(q(t))\right) dt \tag{30.3}$$

One now easily recognizes the action of a point mass, with all function arguments explicitly written out in order to clarify that this is now truly an object on M. Of course one may ask why we use this particular Lagrangian - for now the answer is simply: "Because it yields us the correct physics in the end." But we still need to arrive at the reason for this.

We discuss another example from field theory.

Example 30.2 (First order Lagrangian of a massive scalar field on a metric manifold). Let M be a manifold of dimension n and $E = M \times \mathbb{R}$ the trivial line bundle with projection $\pi : M \times \mathbb{R} \to M$ onto the first factor. Sections of this bundle are uniquely expressed by maps $\varphi \in C^\infty(M, \mathbb{R})$, i.e., by real functions on M. We use arbitrary coordinates (x^a) on M and the one-dimensional Euclidean coordinate ϕ on $\mathbb{R}$, so that we have coordinates (x^a, ϕ) on $M \times \mathbb{R}$. From these coordinates we derive the coordinates

$$(x^a, \phi, \phi_{,a}) = (x^a, \phi_{(0,\dots,0)}, \phi_{(1,0,\dots,0)}, \dots, \phi_{(0,\dots,0,1)}) \tag{30.4}$$

on $J^1(E) \cong T^*M \times \mathbb{R}$.
To construct a particular Lagrangian, let further $g \in \Gamma(T_2^0 M)$ be a non-degenerate, symmetric tensor field of type $(0, 2)$ (the *metric*) and $V \in C^\infty(\mathbb{R}, \mathbb{R})$ (the *potential*). To illustrate this, we do this in five steps, each of which we explain in our geometric language:

- From an element $(x^a, \phi, \phi_{,a})$ we obtain elements $\phi \in \mathbb{R}$, $\phi_{,a}dx^a \in T_x^*M$ and $x \in M$ by applying suitable projections as in the previous example.

- Since the metric is non-degenerate, it possesses an inverse $g^{-1} \in \Gamma(T^2_0 M)$, which is also non-degenerate and symmetric. If we evaluate it at $x \in M$, we get an element $g^{-1}(x) \in T^2_{0\,x}M$. Contracting this element with two copies of $\phi_{,a}dx^a$ yields a real number $g^{ab}(x)\phi_{,a}\phi_{,b}$.
- The potential $V \in C^{\infty}(\mathbb{R}, \mathbb{R})$ can be applied to $\phi \in \mathbb{R}$, which yields another real number $V(\phi) \in \mathbb{R}$.
- The metric induces a volume form $\sqrt{|\det g(x)|}d^n x$ on M. The pullback of this volume form along $\pi_\infty : J^\infty(E) \to M$ is a horizontal n-form on $J^\infty(E)$.
- From the objects constructed above we compose the Lagrangian

$$\left(\frac{1}{2}g^{ab}(x)\phi_{,a}\phi_{,b} - V(\phi)\right)\sqrt{|\det g(x)|}d^n x\,. \tag{30.5}$$

To obtain the action, one finally considers a section, which is described in coordinates by a function $\phi(x)$, and replaces the coordinates ϕ and $\phi_{,a}$ by $\phi(x)$ and $\partial\phi(x)/\partial x^a$. The resulting one-form on M then yields the action

$$S[\phi] = \int_M \left(\frac{1}{2}g^{ab}(x)\frac{\partial\phi(x)}{\partial x^a}\frac{\partial\phi(x)}{\partial x^b} - V(\phi(x))\right)\sqrt{|\det g(x)|}d^n x\,. \tag{30.6}$$

Also here we have explicitly written out every dependence on the point x to illustrate that we are indeed integrating over a n-form on M. Note that in this example we have treated only ϕ as a dynamical field and assumed a fixed background metric g. Both of these objects are sections of vector bundles, and normally one would consider the sum of these vector bundles as the starting point of the construction to make both fields dynamical.

We see that we can formulate these two classical examples in terms of differential geometric objects (sections of bundles) without using coordinates. The coordinates are used here only to illustrate the process and to provide explicit formulas. However, the Lagrangian formulation presented here is independent of the choice of coordinates.

A Dictionary

English	Estonian
projective limit	projektiivne piir
direct limit	direktne piir
contact form	puutevorm (?)
action functional	mõjufunktsionaal