Differential geometry for physicists - Lecture 8

Manuel Hohmann

14. April 2015

26 Jets

In the previous lecture we have learned that tangent and cotangent vectors generalize the notions of the derivative of a function. Tangent vectors naturally appear as derivatives of curves $\gamma \in C^\infty(\mathbb{R}, M)$, while cotangent vectors appear as total derivatives of real functions $f \in C^\infty(M, \mathbb{R})$. We have also seen that the differential φ_* of a map $\varphi \in C^\infty(M, N)$ further generalizes this notion to maps between arbitrary manifolds. We now wish to generalize this notion to higher derivatives. In other words, we will generalize the notion of Taylor polynomials. These generalizations are called *jets*. In the most simple case of functions $f \in C^\infty(\mathbb{R}, \mathbb{R})$ they are exactly the Taylor polynomials, which we formally define as follows.

Definition 26.1 (Jets of $C^\infty(\mathbb{R}, \mathbb{R})$)**.** Let $f \in C^\infty(\mathbb{R}, \mathbb{R})$ be a real function of one variable and $p \in \mathbb{R}$. For $r \in \mathbb{N}$, we define the *r-jet* $j_p^r f$ of f at p as the equivalence class

$$j_p^r f = \left\{ g \in C^\infty(\mathbb{R}, \mathbb{R}) \,\middle|\, f(p) = g(p) \wedge f'(p) = g'(p) \wedge \ldots \wedge f^{(r)}(p) = g^{(r)}(p) \right\} \tag{26.1}$$

of functions $g \in C^\infty(\mathbb{R}, \mathbb{R})$ whose Taylor polynomials at p agree with the Taylor polynomial of f up to order r. The space of all r-jets at p is denoted $J_p^r(\mathbb{R}, \mathbb{R})$, while the space of all r-jets is denoted $J^r(\mathbb{R}, \mathbb{R})$.

In this simple case we can of course identify the jet $j_p^r f$ with the Taylor polynomial

$$f(p) + f'(p)x + \ldots + \frac{f^{(r)}(p)}{r!} x^r \,, \tag{26.2}$$

since there is a one-to-one correspondence between these polynomials and the equivalence classes we used in our definition. In fact, in the literature one also finds this definition in terms of polynomials instead of equivalence classes. However, we will stick to equivalence classes here, because the more general jets will always be equivalence classes of maps and not possess the algebraic structure suggested by polynomials. To see how this works, we will introduce jets of curves.

Definition 26.2 (Jets of $C^\infty(\mathbb{R}, M)$)**.** Let M be a manifold, $\gamma \in C^\infty(\mathbb{R}, M)$ a curve on M and $p \in \mathbb{R}$. For $r \in \mathbb{N}$, we define the *r-jet* $j_p^r \gamma$ of γ at p as the equivalence class

$$j_p^r \gamma = \left\{ \beta \in C^\infty(\mathbb{R}, M) \,\middle|\, \forall f \in C^\infty(M, \mathbb{R}) : j_p^r(f \circ \gamma) = j_p^r(f \circ \beta) \right\} \tag{26.3}$$

of curves $\beta \in C^\infty(\mathbb{R}, M)$ such that for all functions $f \in C^\infty(M, \mathbb{R})$ the r-jets $j^r_p(f \circ \gamma)$ and $j^r_p(f \circ \beta)$ agree. The space of all r-jets at p is denoted $J^r_p(\mathbb{R}, M)$, while the space of all r-jets is denoted $J^r(\mathbb{R}, M)$.

In this definition the composition $f \circ \gamma \in C^\infty(\mathbb{R}, \mathbb{R})$ is simply a real function of one variable, for which we have defined jets already earlier in this section. We say that two curves β, γ belong to the same equivalence class, and thus have the same r-jet $j^r_p\gamma = j^r_p\beta$, if *for all* $f \in C^\infty(M, \mathbb{R})$ the compositions $f \circ \gamma$ and $f \circ \beta$ have the same r-jets. To illustrate this, we will explicitly construct the first order jets.

Example 26.1 (First order jets of $C^\infty(\mathbb{R}, M)$). Let M be a manifold, $\gamma \in C^\infty(\mathbb{R}, M)$ a curve on M and $p \in \mathbb{R}$. The 1-jet $j^1_p\gamma$ is the equivalence class of curves $\beta \in C^\infty(\mathbb{R}, M)$ such that for all functions $f \in C^\infty(M, \mathbb{R})$ we have $(f \circ \gamma)(p) = (f \circ \beta)(p)$ and $(f \circ \gamma)'(p) = (f \circ \beta)'(p)$. The first condition implies that $\gamma(p) = \beta(p)$, which together with the second condition implies $\dot\gamma(p) = \dot\beta(p)$. In other words, each equivalence class is uniquely described by the tangent vector $\dot\gamma(p) \in TM$, so that we have $J^1_p(\mathbb{R}, M) \cong TM$. This holds for all $p \in \mathbb{R}$, so that we have $J^1(\mathbb{R}, M) \cong \mathbb{R} \times TM$.

This shows that 1-jets are in this case simply tangent vectors, so that jets generalize the concept of tangent vectors. To see that we can also generalize cotangent vectors, we define jets of real functions as follows.

Definition 26.3 (Jets of $C^\infty(M, \mathbb{R})$)**.** Let M be a manifold, $f \in C^\infty(M, \mathbb{R})$ a real function on M and $p \in M$. For $r \in \mathbb{N}$, we define the *r-jet* j^r_pf of f at p as the equivalence class

$$j^r_pf = \left\{g \in C^\infty(M, \mathbb{R}) \;\middle|\; \forall \gamma \in C^\infty(\mathbb{R}, M)|_{\gamma(0)=p} : j^r_0(f \circ \gamma) = j^r_0(g \circ \gamma)\right\} \qquad (26.4)$$

of functions $g \in C^\infty(M, \mathbb{R})$ such that for all curves $\gamma \in C^\infty(\mathbb{R}, M)$ with $\gamma(0) = p$ the r-jets $j^r_0(f \circ \gamma)$ and $j^r_0(g \circ \gamma)$ agree. The space of all r-jets at p is denoted $J^r_p(M, \mathbb{R})$, while the space of all r-jets is denoted $J^r(M, \mathbb{R})$.

The construction is very similar to that of $J^r(\mathbb{R}, M)$. We have simply reversed the order of composition in order to obtain a function $f \circ \gamma \in C^\infty(\mathbb{R}, \mathbb{R})$. As an example, we construct the first order jets.

Example 26.2 (First order jets of $C^\infty(M, \mathbb{R})$). Let M be a manifold, $f \in C^\infty(M, \mathbb{R})$ a real function on M and $p \in M$. The 1-jet j^1_pf is the equivalence class of functions $g \in C^\infty(M, \mathbb{R})$ such that for all curves $\gamma \in C^\infty(\mathbb{R}, M)$ with $\gamma(0) = p$ we have $(f \circ \gamma)(0) = (g \circ \gamma)(0)$ and $(f \circ \gamma)'(0) = (g \circ \gamma)'(0)$. The first condition implies that $f(p) = g(p)$, while the second condition implies $df(p) = dg(p)$. In other words, each equivalence class is uniquely described by the function value $f(p)$ and the value $df(p) \in T^*_pM$ of its total derivative at p, so that $J^1_p(M, \mathbb{R}) \cong T^*_pM \times \mathbb{R}$. This holds for all $p \in M$, so that we have $J^1(M, \mathbb{R}) \cong T^*M \times \mathbb{R}$.

This shows that jets also generalize the concept of cotangent vectors. But the most powerful property of jets is the fact that we can also extend the definition to jets of maps between arbitrary manifolds. This can be done as follows.

Definition 26.4 (Jets of $C^\infty(M,N)$)**.** Let M,N be manifolds, $\varphi \in C^\infty(M,N)$ a map and $p \in M$. For $r \in \mathbb{N}$, we define the *r-jet* $j^r_p\varphi$ of φ at p as the equivalence class

$$j^r_p\varphi = \{\vartheta \in C^\infty(M,N) \,|\, \forall \gamma \in C^\infty(\mathbb{R},M)|_{\gamma(0)=p}, f \in C^\infty(N,\mathbb{R}) : \\ j^r_0(f \circ \varphi \circ \gamma) = j^r_0(f \circ \vartheta \circ \gamma)\} \quad (26.5)$$

of maps $\vartheta \in C^\infty(M,N)$ such that for all curves $\gamma \in C^\infty(\mathbb{R},M)$ with $\gamma(0) = p$ and functions $f \in C^\infty(N,\mathbb{R})$ the r-jets $j^r_0(f \circ \varphi \circ \gamma)$ and $j^r_0(f \circ \vartheta \circ \gamma)$ agree. The space of all r-jets at p is denoted $J^r_p(M,N)$, while the space of all r-jets is denoted $J^r(M,N)$.

One should add a word of warning here. In the previous examples it appeared that the jet spaces would be vector spaces, which may seem logical, since polynomials form vector spaces. However, this is in general *not* the case. This false intuition comes from the fact that functions $\varphi \in C^\infty(\mathbb{R}^m,\mathbb{R}^n)$ for a vector space, whose structure comes from the vector space structure of $\mathbb{R}^n$. For maps between general manifolds there is no such structure. But there is another nice structure on the jet spaces, which is that of a manifold.

Theorem 26.1. *Let M,N be manifolds of dimensions $\dim M = m, \dim N = n$ and $r \in \mathbb{N}$. For each $p \in M$ the space $J^r_p(M,N)$ is a manifold of dimension $n\binom{m+r}{r}$, while $J^r(M,N)$ is a manifold of dimension $m + n\binom{m+r}{r}$.*

Instead of giving a complete proof we simply construct charts (coordinates) on $J^r(M,N)$, but not prove the compatibility of these charts. Let (x^α) be coordinates on M and (y^a) coordinates on N, with Greek indices in the range $1,\ldots,\dim M$ and Latin indices in the range $1,\ldots,\dim N$. In these coordinates a map $\varphi : M \to N$ can be expressed by the coordinate functions $y(x)$. The r-jet of φ is then given by those maps $\vartheta : M \to N$ which have the same Taylor polynomial

$$\sum_{\lambda_1+\ldots+\lambda_m \leq r} \frac{(x^1 - x^1_0)^{\lambda_1} \cdot \ldots \cdot (x^m - x^m_0)^{\lambda_m}}{\lambda_1! \cdot \ldots \cdot \lambda_m!} \frac{\partial^{\lambda_1+\ldots+\lambda_m}}{(\partial x^1)^{\lambda_1} \cdots (\partial x^m)^{\lambda_m}} y^a(x_0) \quad (26.6)$$

up to order r around a chosen point p with coordinates x^α_0. A r-jet $j^r_p\varphi$ is thus uniquely determined by the values of the coordinate functions $y^a(x_0)$ and their derivatives of order at most r at x_0. We will use these values as coordinates on $J^r_p(M,N)$. To simplify the notation, we define a *multiindex* $\Lambda = (\lambda_1,\ldots,\lambda_m)$ to be an m-tuple of natural numbers $\lambda_\alpha \in \mathbb{N}$ and denote their sum by $|\Lambda|$. For the $|\Lambda|$'th order derivative appearing in the Taylor polynomial we simply write $\partial_\Lambda y^a(x_0)$. In this notation, a r-jet $j^r_p\varphi$ is uniquely determined by the values $\partial_\Lambda y^a(x_0)$ for $0 \leq |\Lambda| \leq r$. This allows us to use them as coordinates $y^a_\Lambda = \partial_\Lambda y^a(x_0)$ on $J^r_p(M,N)$. In order to construct coordinates on $J^r(M,N)$ we need to specify the point p as well, using the coordinates (x^α) on M. Suitable coordinates on $J^r(M,N)$ are thus given by (x^α, y^a_Λ) with $0 \leq |\Lambda| \leq r$.
Note that some authors also use the notations (y^a, y^a_Λ) for coordinates on $J^r_p(M,N)$ and $(x^\alpha, y^a, y^a_\Lambda)$ for coordinates on $J^r(M,N)$ instead, where $1 \leq |\Lambda| \leq r$. In other words, the

coordinates $y^a_{(0,\ldots,0)}$ are instead denoted y^a. This is of course equivalent to our choice of coordinates.
Now the dimension of $J^r_p(M,N)$ follows directly from counting the number of terms in Taylor polynomials. The vector space of homogeneous polynomials of degree r in m variables has dimension $\binom{m+r-1}{r}$. The vector space of polynomials of degree at most r is thus

$$\sum_{r'=0}^{r} \binom{m+r'-1}{r'} = \binom{m+r}{r}. \tag{26.7}$$

The Taylor polynomials of n functions of m variables up to order r thus form a vector space of dimension $n\binom{m+r}{r}$, which is the dimension of $J^r_p(M,N)$. In $J^r(M,N)$ there is one such space for each point $p \in M$, so that its dimension is by m larger.
Once again it should be noted that despite their nice coordinate form, which maps jets into a vector space of polynomials, there is no vector space structure on jets, i.e., there is no way to treat them as vectors. It is only their coordinate representation we used here that has this structure, but it is not defined on the jets themselves without using coordinates.

Example 26.3. Let $\dim M = 2$ and $\dim N = 1$. We use coordinates (x^1, x^2) on M and the coordinate y on N in order to construct coordinates on $J^3_p(M,N)$ and $J^3(M,N)$. Here we need to consider the multiindices

$$\Lambda \in \{(0,0),(1,0),(0,1),(2,0),(1,1),(0,2),(3,0),(2,1),(1,2),(0,3)\}. \tag{26.8}$$

On $J^3_p(M,N)$ we have thus coordinates (y_Λ) with Λ taking the values above, so that $\dim J^3_p(M,N) = 10$. On $J^3(M,N)$ we have coordinates (x^1, x^2, y_Λ) with Λ taking the values above, so that $\dim J^3(M,N) = 12$. This agrees with the dimension formulas above.

27 Jet bundles

We have already learned that a particularly useful class of maps are sections of fiber bundles, and that many useful objects such as vector or tensor fields fall into this category. We will now study the jets of these maps. Since jets only depend on the *local* behavior of a map, we define the following helpful object.

Definition 27.1 (Local section)**.** Let $\pi : E \to M$ be a fiber bundle. A *local section* on an open subset $U \subset M$ (its *domain*) is a map $\varphi : U \to E$ such that $\pi \circ \varphi = \mathrm{id}_U$.

The main reason for using local instead of global sections is the fact that there are fiber bundles which do not have any global sections, but are still interesting objects for constructing jet bundles. We will (probably) not encounter any such examples in this lecture course, so for our purposes we can simply drop the word “local” from the following definition and work with global sections instead.

Definition 27.2 (Jets of local sections)**.** Let $\pi : E \to M$ be a fiber bundle, $p \in M$ and $\Gamma_p(E)$ the space of all local sections whose domain contains p. For $r \in \mathbb{N}$ and a local section $\sigma \in \Gamma_p(E)$ with domain U_σ we define the *r-jet* $j_p^r\sigma$ of σ at p as the equivalence class

$$j_p^r\sigma = \{\tau \in \Gamma_p(E) \,|\, \forall \gamma \in C^\infty(\mathbb{R}, U_\sigma \cap U_\tau)|_{\gamma(0)=p}, f \in C^\infty(E, \mathbb{R}) : \\ j_0^r(f \circ \sigma \circ \gamma) = j_0^r(f \circ \tau \circ \gamma)\} \quad (27.1)$$

of local sections $\tau \in \Gamma_p(E)$ with domain U_τ such that for all curves $\gamma \in C^\infty(\mathbb{R}, U_\sigma \cap U_\tau)$ with $\gamma(0) = p$ and functions $f \in C^\infty(E, \mathbb{R})$ the r-jets $j_0^r(f \circ \sigma \circ \gamma)$ and $j_0^r(f \circ \tau \circ \gamma)$ agree. The space of all r-jets at p is denoted $J_p^r(E)$, while the space of all r-jets is denoted $J^r(E)$.

The main difference between this definition and the definition from the previous sections is that we do not consider arbitrary maps from M to E, but only sections. This restriction also reduces the number of dimensions of the jet space, which we can state as follows.

Theorem 27.1. *Let $\pi : E \to M$ be a fiber bundle with fiber F and dimensions $\dim M = m, \dim F = n$ and $r \in \mathbb{N}$. For each $p \in M$ the space $J_p^r(E)$ is a manifold of dimension $n\binom{m+r}{r}$, while $J^r(E)$ is a manifold of dimension $m + n\binom{m+r}{r}$.*

We see that instead of the dimension $\dim E$ of the target manifold we only have the dimension $\dim F$ which enters the formula of the dimension. To see why this is the case, we can construct coordinates in the same way as we did for the jet manifolds of arbitrary maps. By definition, every fiber bundle is locally trivial, i.e., for every $p \in M$ there exists an open set $U \subset M$ containing p such that $U \times F \cong \pi^{-1}(U) \subset E$. Given coordinates (x^α) on U and (y^a) on F we can thus use coordinates (x^α, y^a) on $\pi^{-1}(U)$. Let now $\sigma : U \to \pi^{-1}(U)$ be a local section, whose domain we also assume to be U. (If it had a different domain $U' \ni p$ instead, we could simply replace U by $U \cap U'$ in the remainder of this construction.) This section is described by assigning coordinates (x^α, y^a) of the target space to coordinates (x^α) of the domain. However, the first part of these target coordinates is already fixed by the condition that σ is a section, and thus $\pi \circ \sigma = \mathrm{id}_U$. Hence, σ is uniquely determined by the coordinate functions $y^a(x)$. In other words, a section σ looks *locally* just like a map from U to F. Using the coordinate functions $y^a(x)$ we can use the same construction as in the previous section to construct coordinates (y^a_Λ) on $J_p^r(E)$ and (x^α, y^a_Λ) on $J^r(E)$.
Now it is also easy to see the following.

Theorem 27.2. *Let $\pi : E \to M$ be a fiber bundle and $p \in M$. Then $J_p^0(E) \cong \pi^{-1}(p) \cong F$ and $J^0(E) \cong E$.*

Proof. Recall that a 0-jet $j_p^0\sigma$ of a local section σ is uniquely determined by the value $\sigma(p) \in \pi^{-1}(p) \cong F$, which proves the first statement. The second statement follows from the fact that $J^0(E)$ is simply the union of $J_p^r(E)$ for all $p \in M$, while E is the union of all $\pi^{-1}(p)$. One can easily show that the maps $J_p^0(E) \to F$ and $J^0(E) \to E$ derived from these identifications are diffeomorphisms. □

Given now a number of jet manifolds, we may consider maps between them. A very useful class of maps is defined as follows.

Definition 27.3 (Jet projection)**.** Let $\pi : E \to M$ be a fiber bundle and $0 \leq k \leq r$. The *k-jet projection* is the map $\pi_{r,k} : J^r(E) \to J^k(E)$ which assigns to the r-jet $j^r_p\sigma$ of any local section σ its k-jet $j^k_p\sigma$ for every $p \in M$. The map $\pi_{r,0} : J^r(E) \to E$ is also called the *target projection*, while $\pi_r = \pi \circ \pi_{r,0} : J^r(E) \to M$ is called the *source projection*.

Of course we must check that the projections given above are indeed well-defined. This is the case, since any two local sections σ, τ which have the same r-jet also have the same k-jet for $k \leq r$, which follows immediately from the definition of jets. Therefore, the k-jet $j^k_p\sigma$ of a local section σ is uniquely determined by its r-jet $j^r_p\sigma$, as we presumed in the definition above. We will not prove here that the jet projections are smooth maps - the proof is lengthy, but simple. In coordinates (x^α, y^a_Λ) on $J^r(E)$ one can easily see that the projection $\pi_{r,k}$ simply discards all coordinates y^a_Λ with $|\Lambda| > k$ and keeps only the coordinates on $J^k(E)$. These maps have even more nice properties.

Theorem 27.3. *The triples $(J^r(E), J^k(E), \pi_{r,k})$, $(J^r(E), E, \pi_{r,0})$ and $(J^r(E), M, \pi_r)$ are fiber bundles.*

Also this can easily be proven, but we will not do it here. The last bundle of this list will be of particular interest for us, and has its own name.

Definition 27.4 (Jet bundle)**.** Let $\pi : E \to M$ be a fiber bundle and $r \in \mathbb{N}$. The bundle $(J^r(E), M, \pi_r)$ is called the *r'th jet bundle* of E.

Once we have constructed a fiber bundle, we are of course interested in its sections. For the jet bundle of a fiber bundle E there is a particular way to construct sections of $J^r(E)$ from the sections of E, which we define as follows.

Definition 27.5 (Jet prolongation)**.** Let $\pi : E \to M$ be a fiber bundle and $\sigma \in \Gamma(E)$ a section. For $r \in \mathbb{N}$ the *r-jet prolongation* $j^r\sigma$ of σ is the section of the bundle $\pi_r : J^r(E) \to M$ such that $(j^r\sigma)(p) = j^r_p\sigma$ for all $p \in M$.

It is once again easy to check that this construction is well-defined and indeed yields a section of the jet bundle. We illustrate this construction using coordinates (x^α) on $U \subset M$, (y^a) on F and (x^α, y^a) on $\pi^{-1}(U)$, from which we derive coordinates (x^α, y^a_Λ) on $\pi_r^{-1}(U)$. In these coordinates a section σ is locally expressed by the coordinate functions $y^a(x)$. Its r-jet prolongation $j^r\sigma$ is then expressed by the coordinate functions $y^a_\Lambda(x) = \partial_\Lambda y^a(x)$.
Now we have constructed an important and helpful tool which we will apply to physics. We can now make precise what it means that some function "depends on the value and derivatives up to order r of some section at some point". Such a function will simply be a function on $J^r(E)$, and if we feed it with a jet prolongation of some section, it will have exactly the dependence we need.

A Dictionary

English	Estonian
jet	juga
jet manifold	joamuutkond (?)
jet bundle	joakihtkond (?)
jet projektion	joaprojektsioon
jet prolongation	joapikendus
local section	lokaalne lõige