Differential geometry for physicists - Lecture 7

Manuel Hohmann

7. April 2015

22 Integrals over curve segments

We start our discussion of integrals with the simplest possible case, which should remind us to an integral over a single variable. From elementary calculus we know the meaning of integrals of the form

$$F(b) - F(a) = \int_a^b f(x)dx\,. \tag{22.1}$$

This looks like an integral of a function $f(x)$ over the one-dimensional manifold $\mathbb{R}$, and so one may be tempted to define a way to integrate functions over (one-dimensional) manifolds. However, doing so would very soon cause a lot of trouble. To see this, consider the famous change-of-variable formula for integrals. Let $p : \mathbb{R} \to \mathbb{R}$ be a smooth function with $p'(x) > 0$ for all $x \in \mathbb{R}$. Then we have

$$\int_{p(a)}^{p(b)} f(y)dy = \int_a^b f(p(x))p'(x)dx\,. \tag{22.2}$$

Note the appearance of the factor $p'(x) = dy/dx$, which tells us that $f(y)dy$ transforms as a 1-form. In other words, the objects we can and should integrate over a curve segments will not be functions, but 1-forms. This leads us to the following definition.

Definition 22.1 (Integration on $\mathbb{R}$)**.** Let $\omega = f\,dx$ be a 1-form on the one-dimensional manifold $\mathbb{R}$. Its *integral* over the interval $[a, b] \subset \mathbb{R}$ is defined as

$$\int_{[a,b]} \omega = \int_a^b f(x)dx\,, \tag{22.3}$$

where the right hand side is to be interpreted in the obvious way.

Note that the integral above of course depends on the order of the bounds a and b. Changing them would reverse the sign of the integral. For an (oriented) interval one has $a < b$, so that there is a unique prescription how the bounds on the integral must be ordered.
To see how this definition works together with the change-of-variable formula, note that the function $p : \mathbb{R} \to \mathbb{R}$ defined above is simply a diffeomorphism from the manifold $\mathbb{R}$ to itself. It then follows that

$$\int_{p([a,b])} \omega = \int_{p(a)}^{p(b)} f(y)dy = \int_a^b f(p(x))p'(x)dx = \int_{[a,b]} p^*(\omega)\,, \tag{22.4}$$

which shows that the integral as we defined it is invariant under a pullback by p. This is an important property which enters the proofs of many of the statements we encounter during this lecture. It is also important that we demanded $p' > 0$, since otherwise the order of the bounds would get reversed.
Of course we want to perform integration not only on $\mathbb{R}$ (or intervals), but also on general manifolds. To see how we can get there, we first take something we already know (an interval) and stick it into a manifold.

Definition 22.2 (Singular curve segment)**.** A *singular curve segment* on a smooth manifold M is a smooth function $c : [0,1] \to M$. We denote the space of singular curve segments on M by $\mathcal{K}_1(M)$.

By "singular" we mean that we make no assumption on c being injective, but leave it arbitrary. We already know that we can integrate 1-forms on an interval, so we need a prescription which yields us a 1-form on $[0,1]$ if we have a curve segment. If we have a 1-form on the target manifold M, then we can simply take its pullback. This leads us to the following definition.

Definition 22.3 (Integral over a curve segment)**.** Let M be a manifold and $\omega \in \Omega^1(M)$. For a curve segment $c \in \mathcal{K}_1(M)$, the *integral* of ω over c is defined as

$$\int_c \omega = \int_{[0,1]} c^*(\omega)\,. \tag{22.5}$$

This prescription now allows us to integrate 1-forms on a manifold along a curve segment. Again we raise the question how the change-of-variable formula and diffeomorphisms work together with this definition. Since we have fixed the interval of integration to be $[0,1]$, we will consider only diffeomorphisms of $\mathbb{R}$ which leave this interval unchanged. We define them as follows.

Definition 22.4 (Reparameterization of $[0,1]$)**.** A reparameterization of the unit interval $[0,1]$ is a smooth function $p : [0,1] \to [0,1]$ such that $p'(x) > 0$ for all $x \in [0,1]$.

We have restricted ourselves to *orientation preserving* reparameterizations here by demanding $p' > 0$ everywhere. If we would also allow the case $p' < 0$ everywhere, this would be an *orientation reversing* reparameterization, which would also swap the endpoints of the interval, as we discussed above. We keep things more simple by not considering this case.
Now it is easy to see what happens with our integral if we apply a reparameterization.

Theorem 22.1. *The integral over a curve segment is invariant under reparameterization,*

$$\int_c \omega = \int_{c \circ p} \omega \tag{22.6}$$

for all curve segments c, 1-forms ω and reparameterizations p.

Proof. By definition we have

$$\int_{c\circ p} \omega = \int_{[0,1]} (c \circ p)^*(\omega) = \int_{[0,1]} p^*(c^*(\omega)) = \int_{[0,1]} c^*(\omega) = \int_c \omega\,, \tag{22.7}$$

where we used the fact that $p([0,1]) = [0,1]$ by definition. □

Intuitively this means that the integral depends only on the path traced out by the curve segment, but not on the velocity with which this path is transversed.

23 Integrals over k-cubes

We now generalize our knowledge from the previous section from curve segments to k-cubes, and from intervals on the real line to boxes in Euclidean space $\mathbb{R}^k$. It should be clear from what we have learned that the objects which we can integrate on $\mathbb{R}^k$ must be k-forms, since they have the correct transformation behavior under a change of integration variables. We define integrals in analogy to the one-dimensional case.

Definition 23.1 (Integration on $\mathbb{R}^k$)**.** Let $\omega = f\, dx^1 \wedge \ldots \wedge dx^k$ be a k-form on the k-dimensional manifold $\mathbb{R}^k$. Its *integral* over the rectangular box $[a^1, b^1] \times \ldots \times [a^k, b^k] \subset \mathbb{R}^k$ is defined as

$$\int_{[a^1,b^1]\times\ldots\times[a^k,b^k]} \omega = \int_{a^k}^{b^k} \cdots \int_{a^1}^{b^1} f(x) dx^1 \cdots dx^k\,, \tag{23.1}$$

where the integrals on the right hand side are evaluated from the inside outwards.

In order to apply this knowledge to the integration of k-forms on manifolds, we first need to transfer the box over which we integrate into the manifold. In other word, we need to define the analogy of a singular curve segment. We will use cubes here, because the formulas will become easier. Other authors prefer simplices instead.

Definition 23.2 (Singular k-cube)**.** A *singular k-cube* on a smooth manifold M is a smooth function $c : [0,1]^k \to M$. We denote the space of singular curve segments on M by $\mathcal{K}_k(M)$.

Again by singular mean that c will not necessarily be injective. It should now be clear how to integrate a k-form over a k-cube, so we just provide the definition.

Definition 23.3 (Integral over a k-cube)**.** Let M be a manifold and $\omega \in \Omega^k(M)$. For a k-cube $c \in \mathcal{K}_k(M)$, the *integral* of ω over c is defined as

$$\int_c \omega = \int_{[0,1]^k} c^*(\omega)\,. \tag{23.2}$$

We finally come to the question how this definition of integrals behaves under reparameterizations. For this purpose we first need to generalize our definition of reparameterizations.

Definition 23.4 (Reparameterization of $[0,1]^k$)**.** A reparameterization of the unit cube $[0,1]^k$ is a smooth function $p : [0,1]^k \to [0,1]^k$ such that $\det p'(x) > 0$ for all $x \in [0,1]^k$.

Again we restrict ourselves to orientation preserving reparameterizations, which in this case means that the Jacobi determinant $\det p'$ must be everywhere positive. Without this restriction the sign of the integral would change. We finally come to the important result of this section.

Theorem 23.1. *The integral over a k-cube is invariant under reparameterization,*

$$\int_c \omega = \int_{c \circ p} \omega \tag{23.3}$$

for all k-cubes c, k-forms ω and reparameterizations p.

We will not prove this here. The proof is not too difficult and involves the change-of-variables formula for k variables and the transformation behavior of k-forms under diffeomorphisms.

24 Integrals over cubical chains

So far we have learned how to integrate k-forms over regions which can be parameterized by k-cubes. We now wish to further generalize this concept to regions which may be composed of several k-cubes, some of which may be transversed multiple times or in different orientation. The mathematical object corresponding to such a composite integration region is a chain, which we define as follows.

Definition 24.1 (Cubical chain)**.** A *cubical k-chain* on a smooth manifold M is an element of the free abelian group $\mathcal{C}_k(M)$ generated by the set $\mathcal{K}_k(M)$ of singular k-cubes on M.

We first need to clarify the notion of a free abelian group. The elements of $\mathcal{C}_k(M)$ are finite formal sums of elements of $\mathcal{K}_k(M)$ with integer coefficients, i.e., a chain $C \in \mathcal{C}_k(M)$ can be written as

$$C = \sum_{c \in \mathcal{K}_k(M)} C_c c \tag{24.1}$$

with integer coefficients C_c such that only finitely many C_c are non-zero. The group operation is the addition of formal sums and the group inversion is the negative,

$$C + C' = \sum_{c \in \mathcal{K}_k(M)} (C_c + C'_c)c\,, \quad -C = \sum_{c \in \mathcal{K}_k(M)} (-C_c)c\,. \tag{24.2}$$

A chain can thus be interpreted as a prescription which k-cubes should be transversed, how often and with which orientation. Note that every k-cube can also be interpreted as a k-chain which simply prescribes to transverse only this k-cube and exactly once with positive orientation. It is clear that this chain with $c \in \mathcal{K}_k(M) \subset \mathcal{C}_k(M)$ takes the form

$$c = \sum_{c' \in \mathcal{K}_k(M)} c' \cdot \begin{cases} 1 & \text{if } c = c' \\ 0 & \text{otherwise} \end{cases} . \tag{24.3}$$

Integration over a cubic chain is then simply the integration over all of its cubes.

Definition 24.2 (Integral over a cubical chain)**.** Let M be a manifold and $\omega \in \Omega^k(M)$. For a k-chain $C \in \mathcal{C}_k(M)$, the *integral* of ω over C is defined as

$$\int_C \omega = \sum_{c \in \mathcal{K}_k(M)} C_c \int_c \omega \,. \tag{24.4}$$

An interesting property of k-chains is the existence of a boundary, which is a $(k-1)$-chain. This boundary can be obtained as follows.

Definition 24.3 (Boundary operator)**.** Let $C \in \mathcal{C}_k(M)$ be a k-chain on a manifold M. Its *boundary* is the $(k-1)$-chain $\partial C \in \mathcal{C}_{k-1}(M)$ given by

$$\partial C = \sum_{c \in \mathcal{K}_k(M)} C_c \partial c \,, \tag{24.5}$$

where the boundary of a k-cube $c \in \mathcal{K}_k(M)$ is the $(k-1)$-chain $\partial c \in \mathcal{K}_{k-1}(M)$ is defined as

$$\partial c = \sum_{i=1}^{k} (-1)^i (c_{(i,0)} - c_{(i,1)}) \,, \tag{24.6}$$

where $c_{(i,y)} \in \mathcal{K}_{k-1}(M)$ is the face of the c defined by

$$c_{(i,y)}(x^1, \ldots, x^{k-1}) = c(x^1, \ldots, x^{i-1}, y, x^i, \ldots, x^{k-1}) \,. \tag{24.7}$$

The boundary operator has an interesting property.

Theorem 24.1. *The boundary of a boundary vanishes,* $\partial^2 = 0$.

We will not prove this here. Intuitively, it means that a boundary must always be "closed" in the sense that it does by itself not have a boundary. Otherwise it could not enclose its interior.

The boundary operator $\partial : \mathcal{C}_k(M) \to \mathcal{C}_{k-1}(M)$ is in some sense similar to the exterior derivative $d : \Omega^k(M) \to \Omega^{k+1}(M)$. Both satisfy $\partial^2 = 0$ resp. $d^2 = 0$. This already suggests that both operators are related to each other. Indeed there exists a close relationship, which is given by the following famous theorem.

Theorem 24.2 (Stokes' theorem)**.** *The integral of a $(k-1)$-form ω over the boundary ∂C of a k-chain C equals the integral of its exterior derivative $d\omega$ over the k-chain C,*

$$\int_{\partial C} \omega = \int_C d\omega \,. \tag{24.8}$$

25 Integrals over manifolds

In the the previous sections we have now learned how to integrate k-forms over regions of manifolds parameterized be k-chains. We finally come to the point of integrating over all of a manifold. In order to do this, we must decompose the integral into pieces which we can map into $\mathbb{R}^k$. This decomposition can be done using a partition of unity, which we define as follows.

Definition 25.1 (Partition of unity)**.** A *partition of unity* on a manifold M is a set R of smooth functions $\rho : M \to [0,1]$ such that for each $x \in M$ only a finite number of function values are non-zero and their sum equals 1.

Given a partition and unity and an atlas which are chosen to fit together in a suitable way, we can define integration on manifolds as follows.

Definition 25.2 (Integration on manifolds)**.** Let M be an orientable manifold of dimension n together with an oriented atlas $\mathcal{A}$ and a partition of unity R, such that for every $\rho \in R$ the support $\operatorname{supp}\rho$ is compact and there exists a chart (U_ρ, ϕ_ρ) such that $\operatorname{supp}\rho \subset U_\rho$. Let $B_\rho \subset \mathbb{R}^n$ be a box such that $\phi_\rho(\operatorname{supp}\rho) \subset B_\rho$. For a compactly supported n-form $\omega \in \Omega^n(M)$ the *integral* over M is defined as

$$\int_M \omega = \sum_{\rho \in R} \int_{B_\rho} (\phi_\rho^{-1})^*(\rho\omega)\,. \tag{25.1}$$

There are a few remarks regarding this definition. First of all, note that a partition of unity of this type does not exist on every manifold (using our definition of manifolds). One needs additional conditions (such as metrizability or paracompactness) to guarantee the existence of such a partition of unity. We will not go into these details, because all examples of manifolds we consider in this lecture course have these properties and suitable partitions of unity.
We further remark that although R will in general be an infinite set, the sum in the definition above is actually finite and thus well-defined. The reason is that the compact support of ω intersects only the supports of a finite number of $\rho \in R$, so that $\rho \cdot \omega$ will be non-zero only for these finitely many ρ.
Of course the important question arises how the value of the integral depends on the choice of the partition of unity and the atlas. This is answered by the following theorem.

Theorem 25.1. *Let M be a manifold and $(R, \mathcal{A})$ and $(R', \mathcal{A}')$ two choices of a partition and unity and an oriented atlas as given in the definition above, such that $\mathcal{A} \cup \mathcal{A}'$ is oriented. Then the integrals defined by $(R, \mathcal{A})$ and $(R', \mathcal{A}')$ are the same.*

The condition that the union of both atlases is oriented is important. It means that both atlases define the same orientation on M. If they define opposite orientation, their integrals will differ by a minus sign.