Differential geometry for physicists - Lecture 6

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18 Maps between vector bundles

In this lecture we discuss how maps can be used to transport objects living on the tangent, cotangent, tensor and exterior bundles between manifolds. All of these bundles are vector bundles, and we will construct particular classes of maps between the total spaces of these bundles. We start with a general definition of these classes and discuss a few of their properties, which will be useful in the examples we discuss later.

Definition 18.1 (Vector bundle homomorphism)**.** Let (E, M, π_E) and (F, N, π_F) be vector bundles. A *vector bundle homomorphism* is a smooth map $\theta : E \to F$ such that for each $x \in M$ there exists $y \in N$ such that the restriction of θ to a fiber E_x is a linear function which maps E_x into the fiber F_y.

The definition in particular says that a vector bundle homomorphism θ must be fiber-preserving: for two elements $v, w \in E_x$ of the same fiber over a point $x \in M$ there exists a point $y \in N$ such that also the images $\theta(v)$ and $\theta(w)$ lie in the same fiber F_y over y. This point y obviously cannot depend on the particular choice of $v \in E_x$, because it must be the same all over E_x, and thus can depend only on x. In other words, a fiber-preserving map $\theta : E \to F$ between the total spaces of a bundle automatically induces also a map $\tilde{\theta} : M \to N$ between their base spaces, which is the unique map such that the diagram

$$\begin{CD} E @>{\theta}>> F \\ @V{\pi_E}VV @VV{\pi_F}V \\ M @>>{\tilde{\theta}}> N \end{CD} \tag{18.1}$$

commutes. This map is smooth as a consequence of the smoothness of θ. One also calls θ a vector bundle homomorphism *over* $\tilde{\theta}$. We get even more for a more restricted class of maps:

Definition 18.2 (Vector bundle isomorphism)**.** A *vector bundle isomorphism* is a vector bundle homomorphism which is invertible and whose inverse is also a vector bundle homomorphism. Two vector bundles between which a vector bundle isomorphism exists are called *isomorphic*.

Any vector bundle isomorphism θ is also a diffeomorphism, and the map $\tilde{\theta}$ constructed above is also a diffeomorphism in this case. The following statement is easy to prove.

Theorem 18.1. *Let (E, M, π_E), (F, N, π_F) and (G, O, π_G) be vector bundles and $\theta_1 : E \to F$ and $\theta_2 : F \to G$ vector bundle homomorphisms (isomorphisms). Then also $\theta_2 \circ \theta_1$ is a vector bundle homomorphism (isomorphism).*

19 The pushforward

We now apply the abstract constructions to vector bundles we have already encountered, starting with the tangent bundle, where we can construct a map as follows.

Definition 19.1 (Differential and pushforward)**.** Let M and N be manifolds and $\varphi : M \to N$ a smooth map. The *differential* of φ is the smooth map $\varphi_* : TM \to TN$ which assigns to a tangent vector $v \in TM$ (which is a derivation acting on functions on M) its *pushforward* $\varphi_*(v) \in TN$ (derivation acting on functions on N) along φ defined by

$$\varphi_*(v)(f) = v(f \circ \varphi) \tag{19.1}$$

for $f \in C^\infty(N, \mathbb{R})$.

To see that this definition makes sense and indeed yields a map $\varphi_* : TM \to TN$ one of course needs to check that $\varphi_*(v)$ as defined above is a derivation and that φ_* is smooth. It is not very difficult to check this, so we will omit it here. Instead, we continue with an even stronger statement.

Theorem 19.1. *The differential $\varphi_* : TM \to TN$ of a smooth map $\varphi : M \to N$ is a vector bundle homomorphism over φ.*

Also this is not difficult to prove, and we will not do it here. To get a better picture of the differential and the pushforward, we can write them in coordinates. Let (x^a) be coordinates on M and (y^α) coordinates on N. We use different indices (Latin for M and Greek for N) here to distinguish between objects living on different manifolds, and to make clear that Latin indices run from 1 to $\dim M$, while Greek indices run from 1 to $\dim N$. In these coordinates a map $\varphi : M \to N$ can simply be written as $y(x)$. A tangent vector $v \in T_xM$ takes the form $v^a\partial_a$ and acts on a function $g \in C^\infty(M, \mathbb{R})$ by $v(g) = v^a\partial_a g(x)$. If this function is given by $g = f \circ \varphi$ for some $f \in C^\infty(N, \mathbb{R})$, we find

$$\varphi_*(v)(f) = v(f \circ \varphi) = v^a\partial_a f(y(x)) = v^a \frac{\partial y^\alpha}{\partial x^a}\partial_\alpha f(y(x)) \,, \tag{19.2}$$

using the chain rule for functions of several variables. It follows that the coordinate expression for $\varphi_*(v)$ is given by

$$\varphi_*(v) = v^a \frac{\partial y^\alpha}{\partial x^a}\partial_\alpha \,. \tag{19.3}$$

This also explains the name differential for the map φ_*, as it is basically some kind of derivative of φ. This also also suggests that the differential itself satisfies a chain rule, which we state as follows.

Theorem 19.2. *Let M, N, O be manifolds and $\varphi_1 : M \to N$ and $\varphi_2 : N \to O$ smooth maps. Then their differentials satisfy*

$$(\varphi_2 \circ \varphi_1)_* = \varphi_{2*} \circ \varphi_{1*} \,. \tag{19.4}$$

Proof. Let $f \in C^\infty(O, \mathbb{R})$ be a function on O and $v \in TM$. It follows that

$$\begin{aligned} \varphi_{2*}(\varphi_{1*}(v))(f) &= \varphi_{1*}(v)(f \circ \varphi_2) \\ &= v((f \circ \varphi_2) \circ \varphi_1) \\ &= v(f \circ (\varphi_2 \circ \varphi_1)) \\ &= (\varphi_2 \circ \varphi_1)_*(v)(f) \,, \end{aligned} \tag{19.5}$$

using the fact that map composition $\circ$ is associative. □

20 The pullback

While the pullback transfers objects (vectors) along a map in the same direction as the map points, the pullback works in the opposite direction and transfers objects (sections of bundles) from the target manifold to the source manifold. In fact, there are different notions of pullbacks, depending on the type of object to which it is applied. The simplest possible case is the pullback of a function.

Definition 20.1 (Pullback of a function)**.** Let M and N be manifolds and $\varphi : M \to N$ a smooth map. The *pullback* of a function $f \in C^\infty(N, \mathbb{R})$ to M along φ is the function $\varphi^*(f) = f \circ \varphi \in C^\infty(M, \mathbb{R})$.

It is clear that $\varphi^*(f)$ is a smooth function on M, since the composition of smooth maps is smooth. A slightly more sophisticated type of pullback is defined as follows.

Definition 20.2 (Pullback of a covector field)**.** Let M and N be manifolds and $\varphi : M \to N$ a smooth map. The *pullback* of a covector field $\omega \in \Omega^1(N)$ to M along φ is the covector field $\varphi^*(\omega) \in \Omega^1(M)$ such that for all $x \in M$ and $v \in T_xM$ holds

$$\langle v, \varphi^*(\omega)(x) \rangle = \langle \varphi_*(v), \omega(\varphi(x)) \rangle \,. \tag{20.1}$$

Note that there is a fundamental difference between the pullback and the pushforward, besides the fact that they transfer objects in different directions: while the pushforward takes *single* tangent vectors from TM to TN, the pullback takes whole *sections* of T^*N to sections of T^*N. This can be understood as follows.

A map $\varphi : M \to N$ assigns to each point $x \in M$ a point $y = \varphi(x) \in N$, but this map is in general not surjective or injective. Given a single vector $v \in T_xM$, the pushforward yields a single vector $\varphi_*(v) \in T_yN$. However, we cannot use the pushforward and apply it to a vector field $X \in \mathrm{Vect}(M)$ to obtain a vector field on $Y \in \mathrm{Vect}(N)$, because this would be a map $Y : N \to TM$ which assigns a unique vector to each $y \in N$. But the pushforward

does not yield any vector at points $y \in M$ which lie outside the image of φ. Further, if φ is not injective, it maps different vectors $X(x)$ and $X(x')$ with $\varphi(x) = \varphi(x') = y$ into T_yN. The converse holds for the pullback. We cannot pull a single covector $p \in T^*_yN$ back to M, because y may lie outside the image of φ and thus have no preimage at all, or may have multiple preimages. But if we have a covector field $\omega \in \Omega^1(N)$, which assigns a covector $\omega(y)$ to each point $y \in N$, we can obtain a covector field $\varphi^*(\omega) \in \Omega^1(M)$ as follows. We need to construct a section of T^*M, which assigns to each $x \in M$ a covector $\varphi^*(\omega)(x) \in T^*_xM$. Here we make use of the fact that T_xM and T^*_xM are dual vector spaces, so that we can identify such a covector with a linear function on T_xM. To construct such a function, we take a vector $v \in T_xM$ and push it (linearly) to a vector $\varphi_*(v) \in T_yN$. Now we use the covector $\omega(y) \in T^*yN$, which is a linear function on T_yN. This is exactly the construction given in the definition of the pullback.
To illustrate this definition we write the pullback in coordinates. Let (x^a) be coordinates on M and (y^α) coordinates on N, as in the previous section. Using these coordinates a covector field $\omega \in \Omega^1(N)$ takes the form $\omega_\alpha dy^\alpha$, while a vector $v \in T_xM$ can be written as $v = v^a\partial_a$. The definition of the pullback then reads

$$\langle v, \varphi^*(\omega)(x)\rangle = \langle \varphi_*(v), \omega(\varphi(x))\rangle = \varphi_*(v)^\alpha\omega_\alpha(y(x)) = v^a\frac{\partial y^\alpha}{\partial x^a}\omega_\alpha(y(x))\,, \tag{20.2}$$

so that $\varphi^*(\omega)$ can be written in coordinates in the form

$$\varphi^*(\omega)(x) = \omega_\alpha(y(x))\frac{\partial y^\alpha}{\partial x^a}dx^a\,. \tag{20.3}$$

We now have pullbacks of 0-forms (real functions) and 1-forms (covector fields) on N. One may already guess that this procedure can be extended to arbitrary k-forms on N. For this purpose, recall that an element of $\Lambda^kT^*_yN$ can be viewed as an alternating multilinear form on T_yN, i.e., a function from $T_yN \times \ldots \times T_yN$ to $\mathbb{R}$ which is linear in each argument and totally antisymmetric with respect to permutations of its arguments. With this in mind we can define the pullback of a differential form as follows.

Definition 20.3 (Pullback of a differential form)**.** Let M and N be manifolds and $\varphi : M \to N$ a smooth map. The *pullback* of a k-form $\omega \in \Omega^k(N)$ to M along φ is the k-form $\varphi^*(\omega) \in \Omega^1(M)$ such that for all $x \in M$ and $v_1, \ldots, v_k \in T_xM$ holds

$$\varphi^*(\omega)(x)(v_1, \ldots, v_k) = \omega(\varphi(x))(\varphi_*(v_1), \ldots, \varphi_*(v_k))\,. \tag{20.4}$$

Again one easily checks that this definition indeed yields a k-form on M. Also the coordinate expression can be easily derived. Following the same procedure as above one easily sees that

$$\varphi^*(\omega_{\alpha_1\cdots\alpha_k}dy^{\alpha_1} \wedge \ldots \wedge dy^{\alpha_k}) = \omega_{\alpha_1\cdots\alpha_k}\frac{\partial y^{\alpha_1}}{\partial x^{a_1}}\cdots\frac{\partial y^{\alpha_k}}{\partial x^{a_k}}dx^{a_1} \wedge \ldots \wedge dx^{a_k}\,. \tag{20.5}$$

A bit less obvious are the following very useful properties of the pullback of differential forms.

Theorem 20.1. *Let M and N be manifolds and $\varphi : M \to N$ a smooth map. For any differential forms $\alpha \in \Omega^p(N), \beta \in \Omega^q(N)$ on N the pullback satisfies*

$$\varphi^*(\alpha) \wedge \varphi^*(\beta) = \varphi^*(\alpha \wedge \beta) \quad \text{and} \quad d(\varphi^*(\alpha)) = \varphi^*(d\alpha)\,. \tag{20.6}$$

The proof is rather lengthy, but simple, so we will not discuss it here. We finally generalize the pullback even further. In a similar way as an element of $\Lambda^k T_y^* N$ can be regarded as an alternating multilinear form on $T_y N$, an element of $\bigotimes^k T_y^* N$ corresponds to a (general) multilinear form on $T_y N$. This allows us to extend the pullback to covariant tensor fields, i.e., tensor fields of type $(0, s)$. In fact, the definition is identical to the case of a differential form.

Definition 20.4 (Pullback of a covariant tensor field)**.** Let M and N be manifolds and $\varphi : M \to N$ a smooth map. The *pullback* of a covariant tensor field $A \in T_s^0 N$ to M along φ is the covariant tensor field $\varphi^*(A) \in T_s^0 M$ such that for all $x \in M$ and $v_1, \ldots, v_k \in T_x M$ holds

$$\varphi^*(A)(x)(v_1, \ldots, v_k) = A(\varphi(x))(\varphi_*(v_1), \ldots, \varphi_*(v_k)) \,. \tag{20.7}$$

It should be clear now that this is indeed a tensor field of type $(0, s)$ on M and that its coordinate expression is given by

$$\varphi^*(A_{\alpha_1 \cdots \alpha_k} dy^{\alpha_1} \otimes \ldots \otimes dy^{\alpha_k}) = A_{\alpha_1 \cdots \alpha_k} \frac{\partial y^{\alpha_1}}{\partial x^{a_1}} \cdots \frac{\partial y^{\alpha_k}}{\partial x^{a_k}} dx^{a_1} \otimes \ldots \otimes dx^{a_k} \,. \tag{20.8}$$

21 Diffeomorphisms and coordinate transformations

We have seen in the previous sections that the ways we can transfer objects along an arbitrary smooth map $\varphi : M \to N$ are limited since φ is in general neither injective nor surjective. We can remove these limitations by taking φ to be a diffeomorphism, i.e., a bijective map whose inverse is again smooth. In this case the differential $\varphi_* : TM \to TN$ becomes a vector bundle isomorphism, and we can make use of various derived vector bundle isomorphisms to transfer single tensors and tensor fields freely between both manifolds. This will be done in this section. We start be defining the pullback of a vector field.

Definition 21.1 (Pullback of a vector field)**.** Let M and N be manifolds and $\varphi : M \to N$ a diffeomorphism. The *pullback* of a vector field $X \in \mathrm{Vect}(N)$ to M along φ is the vector field $\varphi^*(X) \in \mathrm{Vect}(M)$ such that $\varphi^*(X)(x) = \varphi_*^{-1}(X(\varphi(x)))$ for each $x \in M$.

In the definition we have explicitly used the inverse of φ_*, which should remind us that this construction is valid only if φ is a diffeomorphism. In coordinates (x^a) on M and (y^a) on N (where we now use the same type of letters for the indices, because diffeomorphic manifolds necessarily have the same dimension) we find that

$$\varphi^*(X) = X^a \frac{\partial x^b}{\partial y^a} \partial_b \,, \tag{21.1}$$

which follows from the rule for the derivative of inverse functions on $\mathbb{R}^n$. Since we can now pull back both vector and covector fields, we can also pull back arbitrary tensor fields. The definition is as follows.

Definition 21.2 (Pullback of a tensor field)**.** Let M and N be manifolds and $\varphi : M \to N$ a diffeomorphism. The *pullback* of tensor fields on N to tensor fields on M is defined as the linear function $\varphi^* : \Gamma(T^r_sN) \to \Gamma(T^r_sM)$ that for any r vector fields $X_1, \ldots, X_r \in \mathrm{Vect}(N)$ and s 1-forms $\omega_1, \ldots, \omega_s \in \Omega^1(N)$ holds

$$\varphi^*(X_1 \otimes \ldots \otimes X_r \otimes \omega_1 \otimes \ldots \otimes \omega_s) = \varphi^*(X_1) \otimes \ldots \otimes \varphi^*(X_r) \otimes \varphi^*(\omega_1) \otimes \ldots \otimes \varphi^*(\omega_s)\,. \quad (21.2)$$

In coordinates we find for a tensor field $A \in \Gamma(T^r_sN)$ the pullback

$$\begin{aligned}
&\varphi^*(A^{a_1\cdots a_r}{}_{b_1\cdots b_s}\partial'_{a_1} \otimes \ldots \otimes \partial'_{a_r} \otimes dy^{b_1} \otimes \ldots \otimes dy^{b_s}) \\
&\quad = A^{a_1\cdots a_r}{}_{b_1\cdots b_s}\frac{\partial x^{c_1}}{\partial y^{a_1}} \cdots \frac{\partial x^{c_r}}{\partial y^{a_r}}\frac{\partial y^{b_1}}{\partial x^{d_1}} \cdots \frac{\partial y^{b_s}}{\partial x^{d_s}}\partial_{c_1} \otimes \ldots \otimes \partial_{c_r} \otimes dx^{d_1} \otimes \ldots \otimes dx^{d_s}\,, \quad (21.3)
\end{aligned}$$

where we wrote (∂'_a) for the coordinate basis of T_yN.
We finally remark that all formulas derived in this lecture also hold for the special case $M = N$ and $\varphi = \mathrm{id}_M$. This may seem trivial, but also in this case we are allowed to use two different sets of coordinates (x^a) and (y^a). The formulas above then simply describe how the coordinate expressions for tensor fields transform under a change of coordinates.

A Dictionary

English	Estonian
homomorphism	homomorfism
isomorphism	isomorfism
pushforward	edasitõuge
pullback	tagasitõmme