Differential geometry for physicists - Lecture 5

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13 Differential forms

In the last lecture we already discussed tensor bundles and tensor fields. We now come to a useful class of tensor fields, whose component expressions are completely antisymmetric in all indices. These tensor fields are called *differential forms* and play a role for calculating derivatives and integrals. We start with a few basic definitions.

Definition 13.1 (Exterior power bundle)**.** For a vector bundle E over a manifold M and $k \in \mathbb{N}$ the k'th *exterior power bundle* $\Lambda^k E$ is the union

$$\Lambda^k E = \bigcup_{x \in M} \Lambda^k E_x \,, \tag{13.1}$$

where $\Lambda^k E_x$ is the k'th exterior power space of the fiber vector space E_x over $x \in M$.

Recall from linear algebra that the exterior power $\Lambda^k V$ of a vector space of dimension n with basis $(e_i, i = 1, \ldots, n)$ is spanned by the vectors

$$e_{i_1} \wedge \ldots \wedge e_{i_k} = \frac{1}{k!} \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma) e_{i_{\sigma(1)}} \otimes \ldots \otimes e_{i_{\sigma(k)}} \,, \tag{13.2}$$

where the sum is taken over all permutations σ (elements of the symmetric group S_k permuting k objects) and $\operatorname{sgn}(\sigma)$ is the signature of σ. It follows that there are $\binom{n}{k}$ linearly independent vectors of this type, which constitute a basis of $\Lambda^k V$. The following statement is thus straightforward to prove.

Theorem 13.1. *The exterior power bundle $\Lambda^k E$ of a vector bundle E of rank n is a vector bundle of rank*

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \,. \tag{13.3}$$

In this lecture we are in particular interested in exterior powers of the cotangent bundle and their sections, which have their own name:

Definition 13.2 (Differential form)**.** A *differential form* (or more precisely a *k-form*) on a manifold M is a section of the bundle $\Lambda^k T^* M$ for $k \in \mathbb{N}$. The space of all k-forms on M is denoted $\Omega^k(M)$.

Given coordinates (x^a) on M, we can use the coordinate basis (dx^a) of T_x^*M to construct a basis of $\Lambda^k T_x^*M$ with basis elements of the form $dx^{a_1} \wedge \ldots \wedge dx^{a_k}$. A differential form $\omega \in \Omega^k(M)$ can thus be expressed in the form

$$\omega = \omega_{a_1 \cdots a_k} dx^{a_1} \wedge \ldots \wedge dx^{a_k}\,, \tag{13.4}$$

where the components are totally antisymmetric, $\omega_{a_1 \cdots a_k} = \omega_{[a_1 \cdots a_k]}$. It thus becomes clear that a k-form is simply a totally antisymmetric tensor field of type $(0, k)$. Here we used the bracket notation for (anti)symmetrizing over indices:

$$A_{[a_1 \cdots a_k]} = \frac{1}{k!} \sum_{\sigma \in S_k} \mathrm{sgn}(\sigma) A_{a_{\sigma(1)} \cdots a_{\sigma(k)}} \tag{13.5a}$$

$$A_{(a_1 \cdots a_k)} = \frac{1}{k!} \sum_{\sigma \in S_k} A_{a_{\sigma(1)} \cdots a_{\sigma(k)}} \tag{13.5b}$$

There are some special cases. For $k = 0$ we have $\Lambda^0 T_x^*M \cong \mathbb{R}$, so that a 0-form is simply a real function and $\Omega^0(M) \cong C^\infty(M, \mathbb{R})$. We also encountered $\Lambda^1 T_x^*M \cong T_x^*M$, so that a 1-form is the same as a covector field. This justifies the notation $\Omega^1(M)$ for the space of covector fields introduced in the last lecture.
In the following we will study a few operations on differential forms and their properties.

14 The exterior product

Recall from linear algebra that given a vector space V, the exterior algebra defines a wedge product

$$\begin{array}{cccc} \wedge \ : & \Lambda^p V \times \Lambda^q V & \to & \Lambda^{p+q} V \\ & (u, v) & \mapsto & u \wedge v \end{array}\,, \tag{14.1}$$

which acts on basis vectors in the obvious way,

$$(e_{i_1} \wedge \ldots \wedge e_{i_p}) \wedge (e_{j_1} \wedge \ldots \wedge e_{j_q}) = e_{i_1} \wedge \ldots \wedge e_{i_p} \wedge e_{j_1} \wedge \ldots \wedge e_{j_q}\,, \tag{14.2}$$

and is linear in both u and v. Pointwise application of the wedge product to differential forms allows us to define the following:

Definition 14.1 (Exterior product)**.** Let M be a manifold and $\alpha \in \Omega^p(M)$ and $\beta \in \Omega^q(M)$. Their *exterior product* is the differential form $\alpha \wedge \beta \in \Omega^{p+q}(M)$ such that for all $x \in M$

$$(\alpha \wedge \beta)(x) = \alpha(x) \wedge \beta(x)\,. \tag{14.3}$$

Using coordinates (x^a), we have

$$\begin{aligned} \alpha \wedge \beta &= (\alpha_{a_1 \cdots a_p} dx^{a_1} \wedge \ldots \wedge dx^{a_p}) \wedge (\beta_{b_1 \cdots b_q} dx^{b_1} \wedge \ldots \wedge dx^{b_q}) \\ &= \alpha_{[a_1 \cdots a_p} \beta_{b_1 \cdots b_q]} dx^{a_1} \wedge \ldots \wedge dx^{a_p} \wedge dx^{b_1} \wedge \ldots \wedge dx^{b_q}\,. \end{aligned} \tag{14.4}$$

The antisymmetrization comes from the fact that the wedge product of the basis elements (dx^a) is totally antisymmetric.
The following properties of the exterior product follow directly from the properties of the wedge product.

Theorem 14.1. *For $\alpha \in \Omega^p(M)$, $\beta \in \Omega^q(M)$ and $\gamma \in \Omega^r(M)$, the exterior product satisfies:*

- *Graded anticommutativity:*
$$\alpha \wedge \beta = (-1)^{pq} \beta \wedge \alpha\,. \tag{14.5}$$

- *Associativity:*
$$\alpha \wedge (\beta \wedge \gamma) = (\alpha \wedge \beta) \wedge \gamma = \alpha \wedge \beta \wedge \gamma\,. \tag{14.6}$$

- $\mathbb{R}$*-linearity in each factor.*

A special case is given if $p = 0$ or $q = 0$. In this case one of the terms in the wedge product is a real function $f \in C^\infty(M, \mathbb{R})$, and the exterior product reduces to the ordinary product

$$f \wedge \alpha = \alpha \wedge f = f\alpha. \tag{14.7}$$

15 The exterior derivative

We have seen in the previous lecture that the total derivative df of a function $f \in \Omega^0(M) \cong C^\infty(M, \mathbb{R})$ is a covector field, and hence a 1-form. The total derivative can thus be viewed as a function $d : \Omega^0(M) \to \Omega^1(M)$, which is a special case of the following construction.

Definition 15.1 (Exterior derivative)**.** For a manifold M, the *exterior derivative* $d : \Omega^k(M) \to \Omega^{k+1}(M)$ for all $k \in \mathbb{N}$ is the unique linear function such that:

- df is the total derivative for any $f \in \Omega^0(M) \cong C^\infty(M, \mathbb{R})$.
- $d(d\omega) = 0$ for any $\omega \in \Omega^k(M)$.
- $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta$ for $\alpha \in \Omega^p(M)$ and $\beta \in \Omega^q(M)$, where $p, q \in \mathbb{N}$.

In coordinates (x^a) we can write a k-form as $\omega = \omega_{a_1 \cdots a_k} dx^{a_1} \wedge \ldots \wedge dx^{a_k}$ and use the definition above to derive the formula

$$\begin{aligned}
d\omega &= d(\omega_{a_1 \cdots a_k} dx^{a_1} \wedge \ldots \wedge dx^{a_k}) \\
&= d(\omega_{a_1 \cdots a_k}) \wedge dx^{a_1} \wedge \ldots \wedge dx^{a_k} \\
&\quad + \omega_{a_1 \cdots a_k} \sum_{i=1}^{k} (-1)^{i-1} dx^{a_1} \wedge \ldots \wedge dx^{a_{i-1}} \wedge d(dx^{a_i}) \wedge dx^{a_{i+1}} \wedge \ldots \wedge dx^{a_k} \\
&= \partial_{[b} \omega_{a_1 \cdots a_k]} dx^b \wedge dx^{a_1} \wedge \ldots \wedge dx^{a_k}\,,
\end{aligned} \tag{15.1}$$

where the antisymmetrization in the last line again comes from the total antisymmetry of the wedge product.

16 The interior product

Also the pairing $\langle X, \omega \rangle$ between a vector field $X \in \mathrm{Vect}(M)$ and a covector field $\omega \in \Omega^1(M)$ introduced in the previous lecture is a special case of a more general construction, which we discuss in this section and which is defined as follows.

Definition 16.1 (Interior product)**.** For a manifold M, the *interior product* ι : $\mathrm{Vect}(M) \times \Omega^{k+1}(M) \to \Omega^k(M)$ is the unique function such that for any $X \in \mathrm{Vect}(M)$:

- $\iota_X \alpha = \langle X, \alpha \rangle$ for $\alpha \in \Omega^1(M)$.
- $\iota_X(\lambda\alpha + \mu\beta) = \lambda \iota_X \alpha + \mu \iota_X \beta$ for $\lambda, \mu \in \mathbb{R}$ and $\alpha, \beta \in \Omega^{k+1}(M)$.
- $\iota_X(\alpha \wedge \beta) = (\iota_X \alpha) \wedge \beta + (-1)^p \alpha \wedge (\iota_X \beta)$ for $\alpha \in \Omega^p(M)$ and $\beta \in \Omega^q(M)$, where $p, q \in \mathbb{N}$.

For a vector field $X = X^a \partial_a$ and a differential form $\omega = \omega_{a_1 \cdots a_k} dx^{a_1} \wedge \ldots \wedge dx^{a_k}$ expressed in coordinates (x^a) we can directly use the properties given in the definition above to read off the coordinate formula

$$\begin{aligned}
\iota_X \omega &= \iota_{X^b \partial_b}(\omega_{a_1 \cdots a_k} dx^{a_1} \wedge \ldots \wedge dx^{a_k}) \\
&= X^b \omega_{a_1 \cdots a_k} \sum_{i=1}^{k} (-1)^{i-1} \langle \partial_b, dx^{a_i} \rangle dx^{a_1} \wedge \ldots \wedge dx^{a_{i-1}} \wedge dx^{a_{i+1}} \wedge \ldots \wedge dx^{a_k} \\
&= \omega_{a_1 \cdots a_k} \sum_{i=1}^{k} (-1)^{i-1} X^{a_i} dx^{a_1} \wedge \ldots \wedge dx^{a_{i-1}} \wedge dx^{a_{i+1}} \wedge \ldots \wedge dx^{a_k} \\
&= k X^{a_1} \omega_{a_1 \cdots a_k} dx^{a_2} \wedge \ldots \wedge dx^{a_k} \,,
\end{aligned} \tag{16.1}$$

where the last line follows from the fact that we took the components $\omega_{a_1 \cdots a_k}$ to be totally antisymmetric. This antisymmetry also plays a role in the following statement.

Theorem 16.1. *For $X, Y \in \mathrm{Vect}(M)$ and $\omega \in \Omega^k(M)$ the interior product satisfies $\iota_X(\iota_Y \omega) = -\iota_Y(\iota_X \omega)$.*

We will not prove this here, and instead present another theorem, which can be helpful in practical calculations.

Theorem 16.2. *Given a k-form $\omega \in \Omega^k(M)$ and $k+1$ vector fields $X_0, \ldots X_k \in \mathrm{Vect}(M)$, the exterior derivative, interior product and Lie bracket are related by*

$$\begin{aligned}
\iota_{X_k} \cdots \iota_{X_0} d\omega = &\sum_{i=0}^{k} (-1)^i X_i \left(\iota_{X_k} \cdots \iota_{X_{i+1}} \iota_{X_{i-1}} \cdots \iota_{X_0} \omega \right) \\
&+ \sum_{i=0}^{k-1} \sum_{j=i+1}^{k} (-1)^{i+j} \iota_{X_k} \cdots \iota_{X_{j+1}} \iota_{X_{j-1}} \cdots \iota_{X_{i+1}} \iota_{X_{i-1}} \cdots \iota_{X_0} \iota_{[X_i, X_j]} \omega \,.
\end{aligned} \tag{16.2}$$

For a 1-form $\omega \in \Omega^1(M)$ this formula reduces to

$$\begin{aligned}
\iota_Y \iota_X d\omega &= X(\iota_Y \omega) - Y(\iota_X \omega) - \iota_{[X,Y]} \omega \\
&= X(\langle Y, \omega \rangle) - Y(\langle X, \omega \rangle) - \langle [X, Y], \omega \rangle \,.
\end{aligned} \tag{16.3}$$

17 Volume forms

The last concept we introduce in this lecture is based on what we have learned about differential forms, and will finally lead us to integrals, and which we define as follows.

Definition 17.1 (Volume form). A *volume form* on a manifold M of dimension n is a nowhere vanishing n-form, i.e., a differential form $\omega \in \Omega^n(M)$ such that $\omega(x) \neq 0$ for all $x \in M$.

Using coordinates (x^a), a volume form can always be written in the form $\omega = w\, dx^1 \wedge \ldots \wedge dx^n$ with $w(x) \neq 0$ everywhere. Note that although at first sight it looks like w is a real function on M, this is not the case - the value of w in this definition depends on the choice of coordinates (x^a), while the value of a real function $f \in C^\infty(M, \mathbb{R})$ depends only on a point on M, but not on the choice of coordinates used for its description. However, functions can be used to compare volume forms. If ω is a volume form and $f \in C^\infty(M, \mathbb{R})$ is nowhere vanishing, then obviously also $f\omega$ is a volume form. In fact, every volume form can be expressed by any other volume form and a function:

Theorem 17.1. *Let ω and ω' be volume forms on a manifold M. Then there exists a unique nowhere vanishing function $f \in C^\infty(M, \mathbb{R})$ such that $\omega' = f\omega$.*

Not every manifold allows for a volume form. In fact, volume forms come together with some additional structure, which we will define next.

Definition 17.2 (Orientable manifold). A manifold is called *orientable* if it possesses an atlas such that the determinants of the Jacobian matrices of all transition functions are positive.

With this definition we can now state:

Theorem 17.2. *A manifold possesses a volume form if and only if it is orientable.*

We will revisit this topic in the next lecture.

A Dictionary

English	Estonian
differential form	diferentsiaalvorm
exterior power	välisaste
exterior product	väliskorrutis
exterior derivative	välisdiferentsiaal
interior product	sisekorrutis (?)
volume form	ruumala vorm
orientable manifold	orienteeritav muutkond