Differential geometry for physicists - Lecture 4

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10 The cotangent bundle

We now come to a concept which is somehow dual to the tangent bundle. While elements of the tangent bundle can be interpreted as velocities along trajectories, elements of the cotangent bundle measure how a function changes along a manifold. We start with the definition of the cotangent space.

Definition 10.1 (Cotangent space)**.** Let M be a manifold and $x \in M$. Let $I_x \subset C^\infty(M, \mathbb{R})$ be the ideal of real functions f on M for which $f(x) = 0$ and $I_x^2 \subset I_x$ the ideal generated by functions fg with $f, g \in I_x$. Both I_x and I_x^2 are vector spaces, and I_x^2 is a subspace of I_x. The *cotangent space* T_x^*M at x is the quotient vector space I_x/I_x^2.

We also illustrate this definition using local coordinates (x^a) around a point $x_0 \in M$. Using Hadamard's lemma we can write any smooth function f in a certain neighborhood around x_0 in the form

$$f(x) = f(x_0) + (x^a - x_0^a)\tilde{f}_a(x)\,, \tag{10.1}$$

where $\tilde{f}_a$ are smooth functions defined in a neighborhood around x_0. The ideal I_{x_0} contains those functions f for which $f(x_0) = 0$. For the product $h = fg$ of two functions $f, g \in I_{x_0}$ we thus have

$$h(x) = f(x)g(x) = (x^a - x_0^a)(x^b - x_0^b)\tilde{f}_a(x)\tilde{g}_b(x) = (x^a - x_0^a)\tilde{h}_a(x)\,, \tag{10.2}$$

from which we see that $I_{x_0}^2$ contains those functions $h \in I_{x_0}$ for which $\tilde{h}_a(x_0) = 0$. The equivalence class $[f]_{x_0} \in I_{x_0}/I_{x_0}^2$ of $f \in I_{x_0}$ thus contains all functions $g \in I_{x_0}$ such that $\tilde{f}_a(x_0) = \tilde{g}_a(x_0)$, and is thus fully characterized by $\tilde{f}_a(x_0)$. This shows that $T_{x_0}^*M$ is a vector space of dimension $\dim M$. Our choice of coordinates induces a basis of $T_{x_0}^*M$, which we denote (dx^a), and in which we can write $[f]_{x_0} \in T_{x_0}^*M$ as

$$[f]_{x_0} = \tilde{f}_a(x_0)dx^a\,. \tag{10.3}$$

From a geometric point of view, we see that $\tilde{f}_a(x_0)$ simply describes the change of $f(x)$ along the direction x^a at $x = x_0$, given by

$$\tilde{f}_a(x_0) = \left.\frac{\partial}{\partial x^a}f(x)\right|_{x=x_0}\,. \tag{10.4}$$

This shows that the cotangent space is somehow "dual" to the tangent space (which we will make precise in the next section): while the tangent space $T_{x_0}M$ consists of all possible ways to differentiate functions at x_0 (or all possible "differential operators" at x_0, so that two differential operators are the same if they yield the same result, no matter what function we supply as input), the cotangent space consists of all equivalence classes of functions that we can feed a differential operator as input data (where equivalence is defined such that two functions belong to the same equivalence class if differentiating them yields the same result, no matter which differential operator we apply to them).
We can proceed similarly to the construction of the tangent bundle and assemble the cotangent spaces to form the cotangent bundle.

Definition 10.2 (Cotangent bundle)**.** The *cotangent bundle* of a manifold M is the disjoint union

$$T^*M = \biguplus_{x \in M} T_x^*M \,. \tag{10.5}$$

The canonical projection of the tangent bundle is the function $\tilde{\pi} : T^*M \to M$ such that $\tilde{\pi}(p) = x$ for $p \in T_x^*M$.

Also here we take the disjoint union, in full analogy to the construction of the tangent bundle. One may already guess the following theorems.

Theorem 10.1. *The cotangent bundle T^*M of a manifold M of dimension n is a manifold of dimension $2n$.*

Theorem 10.2. *The structure $(T^*M, M, \tilde{\pi})$ is a vector bundle of rank $n = \dim M$.*

We will not prove these theorems here, because the proof proceeds in exactly the same way as shown in the previous section for the tangent bundle: using coordinates (x^a), one obtains a basis (dx^a) of the tangent space, and can express any tangent space element as $p = p_a dx^a$. This yields coordinates (x^a, p_a) on the cotangent bundle, which enter the proof in the same way as the coordinates (x^a, v^a) on the tangent bundle.
Sections of the cotangent bundle are of similar importance as sections of the tangent bundle, and also deserve their own name.

Definition 10.3 (Covector field)**.** A *covector field* (or *1-form*) on a manifold M is a section of the cotangent bundle T^*M. The space of all covector fields on M is denoted $\Gamma(T^*M)$ or $\Omega^1(M)$.

The term 1-form and the notation $\Omega^1(M)$ will become clear in the next lecture, when we discuss general p-forms, with $0 \leq p \leq \dim M$. As it was also the case with vector fields, we can use coordinates (x^a) to write a covector field in the form $\omega = \omega_a dx^a$, where the component functions ω_a are smooth. Every smooth function on a manifold defines a covector field as follows.

Definition 10.4 (Total differential)**.** Let M be a manifold and $f \in C^\infty(M,\mathbb{R})$ a function on M. Its *total differential* $df(x)$ at a point $x \in M$ is the equivalence class $[f - f(x)]_x \in T_x^*M$ of $f - f(x) \in I_x$ modulo I_x^2. It defines a covector field df.

For the components ω_a of $\omega = df$ in a coordinate basis follows that $\omega_a = \partial_a f$.

11 Relation between tangent and cotangent bundle

In the last section we already had some hints that there is a duality between the tangent and cotangent bundles. We now make this precise and give the following theorem.

Theorem 11.1. *The tangent space T_xM and the cotangent space T_x^*M at any point x on a manifold M are dual vector spaces.*

Proof. To show that T_xM is the dual vector space of T_x^*M, we need to show that there is an isomorphism $\theta : T_xM \to (T_x^*M)^*$, which we construct as follows. Recall that the elements of $T_x^*M = I_x/I_x^2$ are equivalence classes $[f]_x = f + I_x^2$ of functions $f \in I_x$. For such an equivalence class $[f]_x \in T_x^*M$ and a derivation $v \in T_xM$ we define

$$\theta(v) : T_x^*M \to \mathbb{R}, [f]_x \mapsto v(f)\,. \tag{11.1}$$

We still need to check that this is well-defined and does not depend on the choice of the representative f. Since derivations are linear functions, this is equivalent to showing that v vanishes on I_x^2. Since the elements of I_x^2 are products of functions $f, g \in I_x$, we have

$$v(fg) = v(f)g(x) + f(x)v(g) = 0\,, \tag{11.2}$$

since $f(x) = g(x) = 0$. Further, we see that $\theta(v)$ is linear, since

$$\theta(v)(\lambda[f]_x+\mu[g]_x) = \theta(v)([\lambda f+\mu g]_x) = v(\lambda f+\mu g) = \lambda v(f)+\mu v(g) = \lambda\theta(v)([f]_x)+\mu\theta(v)([g]_x)\,. \tag{11.3}$$

To see that θ is an isomorphism of the vector spaces T_xM and $(T_x^*M)^*$, we need to show that it is linear and possesses an inverse. Linearity follows from

$$\theta(\lambda v + \mu w)([f]_x) = \lambda v(f) + \mu w(f) = \lambda\theta(v)([f]_x) + \mu\theta(w)([f]_x)\,. \tag{11.4}$$

We finally show the existence of an inverse $\vartheta : (T_x^*M)^* \to T_xM$ by explicit construction. Let $\alpha \in (T_x^*M)^*$ and define

$$\vartheta(\alpha) : C^\infty(M,\mathbb{R}) \to \mathbb{R}, f \mapsto \alpha([f - f(x)]_x) \tag{11.5}$$

To see that $\vartheta(\alpha)$ is a derivation, we check its linearity

$$\begin{aligned}\vartheta(\alpha)(\lambda f + \mu g) &= \alpha([\lambda(f - f(x)) + \mu(g - g(x))]_x) = \alpha(\lambda[f - f(x)]_x + \mu[g - g(x)]_x)\\ &= \lambda\alpha([f - f(x)]_x) + \mu\alpha([g - g(x)]_x) = \lambda\vartheta(\alpha)(f) + \mu\vartheta(\alpha)(g)\end{aligned} \tag{11.6}$$

and product rule

$$\begin{aligned}\vartheta(\alpha)(fg) &= \alpha([fg-f(x)g(x)]_x) = \alpha([(f-f(x))(g-g(x))+f(x)(g-g(x))+(f-f(x))g(x)]_x)\\ &= f(x)\alpha([g - g(x)]_x) + \alpha([f - f(x)]_x)g(x) = f(x)\vartheta(\alpha)(g) + \vartheta(\alpha)(f)g(x)\,.\end{aligned} \tag{11.7}$$

We finally need to check that the functions θ and ϑ defined above are inverses of each other. We first check that

$$\theta(\vartheta(\alpha))([f]_x) = \vartheta(\alpha)(f) = \alpha([f - \underbrace{f(x)}_{=0}]_x) = \alpha([f]_x) \tag{11.8}$$

for $\alpha \in (T_x^*M)^*$ and $f \in I_x$. Conversely, for $v \in T_xM$ and $f \in C^\infty(M, \mathbb{R})$ we have

$$\vartheta(\theta(v))(f) = \theta(v)([f - f(x)]_x) = v(f - f(x))\,. \tag{11.9}$$

To see that the latter equals $v(f)$, we need to show that a derivation v vanishes on a constant function c. This follows from the linearity of v together with the product rule, since

$$v(c)f = v(cf) - cv(f) = cv(f) - cv(f) = 0 \tag{11.10}$$

for all $f \in C^\infty(M, \mathbb{R})$. We have thus shown that θ and ϑ are indeed inverses of each other, so that $T_xM \cong (T_x^*M)^*$. Since T_x^*M is a finite-dimensional real vector space of dimension $\dim M$, which we have shown using Hadamard's lemma, it follows that also $(T_x^*M)^*$ and thus T_xM are real vector spaces of dimension $\dim M$. Finally, since the double dual V^{**} of a finite-dimensional vector space V is again isomorphic to V, it follows that also $T_x^*M \cong (T_xM)^*$. □

This rather lengthy proof was necessary since we provided an own definition for both tangent and cotangent spaces. In the literature one often finds another approach, which simply defines the cotangent space as the dual of the tangent space. This approach is of course valid. However, the approach we used here gave as a deeper understanding of the structure of these spaces and an interpretation for their elements in terms of functions on the manifold, which will be useful during the remainder of the lecture course. Instead of explicitly writing the isomorphisms θ and ϑ constructed above, we will simply write

$$\langle v, p\rangle = \theta(v)(p) \tag{11.11}$$

for the canonical pairing between $v \in T_xM$ and $p \in T_x^*M$, and also

$$\langle X, \omega\rangle(x) = \langle X(x), \omega(x)\rangle \tag{11.12}$$

for $X \in \Gamma(TM)$ and $\omega \in \Gamma(T^*M)$. To clarify this notation, we make use of it now for showing that also the coordinate bases (∂_a) and (dx^a) are related.

Theorem 11.2. *Given coordinates (x^a) on a manifold M, the coordinate basis (∂_a) of T_xM and (dx^a) of T_x^*M are dual bases, i.e., $\langle \partial_a, dx^b\rangle = \delta_a^b$.*

Proof. Recall that we defined the coordinate basis of T_xM such that for a tangent vector $v \in T_xM$ and a function $f \in C^\infty(M, \mathbb{R})$ holds $v(f) = v^a\partial_a f$, while we expressed a cotangent vector $[f]_x \in T_x^*M$ as $\partial_a f dx^a$. It thus follows directly that

$$v^a\partial_b f\langle \partial_a, dx^b\rangle = \langle v^a\partial_a, \partial_b f dx^b\rangle = \langle v, [f]_x\rangle = v(f) = v^a\partial_a f = v^a\partial_b f\delta_a^b\,, \tag{11.13}$$

so that $\langle \partial_a, dx^b\rangle = \delta_a^b$. □

12 Tensor bundles

The tangent and cotangent bundles we introduced so far are the building blocks of another structure, called *tensor bundles*, which we will frequently encounter during the remainder of the course and extensively use in physics. In fact, physical quantities are usually modeled by *tensor fields* on a spacetime manifold, i.e., sections of a tensor bundle. In this section we will explain this notion. But before we arrive at this notion, we will slightly generalize the "duality" between the tangent and cotangent bundles from the last section to general vector bundles, and then discuss their role as building blocks of a tensor bundle.

Definition 12.1 (Dual bundle)**.** Let M be a manifold and E a vector bundle over M of rank k. Its *dual bundle* E^* is the union

$$E^* = \bigcup_{x \in M} E_x^* \,, \tag{12.1}$$

where E_x^* is the dual vector space of the fiber E_x.

Our previous experience with vector bundles allows us to guess that the following will hold.

Theorem 12.1. *The dual bundle E^* as defined above is a vector bundle of rank k over M.*

Proof. Instead of giving a full proof, it should suffice to just sketch the basic ideas. Recall that in the definition of a vector bundle we encounter at some point a vector space isomorphism $\theta_x : E_x \to \mathbb{R}^k$ for each $x \in M$. From this we derive another vector space isomorphism $\vartheta_x : \mathbb{R} \to E_x^*$ by defining

$$\vartheta_x(a)(v) = a \cdot \theta_x(v) \tag{12.2}$$

for $a \in \mathbb{R}^k$ and $v \in E_x$, where $\cdot$ is the Euclidean scalar product of $\mathbb{R}^k$. Its inverse ϑ_x^{-1} takes the same role in showing that E^* is a vector bundle of rank k as θ_x takes in the context of E. □

It should not be surprising that the tangent and cotangent bundles are a typical example, which we state as follows.

Theorem 12.2. *The cotangent bundle T^*M of a manifold M is the dual bundle of the tangent bundle TM (and vice versa).*

We will not prove this here. Instead, we will continue by introducing another concept, which allows us to put different vector bundles together and will finally lead us to tensors.

Definition 12.2 (Tensor product bundle)**.** Let M be a manifold and E and F vector bundles over M of rank k_E and k_F, respectively. Their *tensor product bundle* $E \otimes F$ is the union

$$E \otimes F = \bigcup_{x \in M} E_x \otimes F_x \,, \tag{12.3}$$

where $E_x \otimes F_x$ is the tensor product space of the fibers E_x and F_x over $x \in M$.

We briefly review the tensor product of vector spaces. If $e_i, i = 1, \ldots, k_E$ and $f_j, j = 1, \ldots, k_F$ are bases of E_x and F_x, then $e_i \otimes f_j$ is a basis of $E_x \otimes F_x$. It is a vector space of dimension $k_E k_F$, which leads us to another educated guess.

Theorem 12.3. *The tensor product bundle $E \otimes F$ as defined above is a vector bundle of rank $k_E k_F$ over M.*

Again we will omit the proof here. The most important vector bundles we have encountered so far are the tangent and cotangent bundles. We thus use them as factors for a particular class of tensor product bundles, which we define as follows.

Definition 12.3 (Tensor bundle)**.** Let M be a manifold. The *tensor bundle* of type (r, s) for $r, s \in \mathbb{N}$ is the tensor product bundle

$$T^r_s M = \underbrace{TM \otimes \ldots \otimes TM}_{r \text{ times}} \otimes \underbrace{T^*M \otimes \ldots \otimes T^*M}_{s \text{ times}} . \tag{12.4}$$

The following fact about the tensor bundle now directly follows from our more general theorem on tensor product bundles.

Theorem 12.4. *The tensor bundle $T^r_s M$ as defined above is a vector bundle of rank $(\dim M)^{r+s}$ over M.*

To illustrate the definition, we introduce coordinates (x^a) on M. For any point $x \in M$ we then have the coordinate bases (∂_a) of $T_x M$ and (dx^a) of $T^*_x M$. The corresponding coordinate basis of $T^r_{s\,x} M$ is then given by the elements

$$\partial_{a_1} \otimes \ldots \otimes \partial_{a_r} \otimes dx^{b_1} \otimes \ldots \otimes dx^{b_s} , \tag{12.5}$$

where each index runs from 1 to $\dim M$, so that the basis has $(\dim M)^{r+s}$ elements. Any element $V \in T^r_{s\,x} M$ takes the form

$$V = V^{a_1 \cdots a_r}{}_{b_1 \cdots b_s} \partial_{a_1} \otimes \ldots \otimes \partial_{a_r} \otimes dx^{b_1} \otimes \ldots \otimes dx^{b_s} , \tag{12.6}$$

with r upper and s lower indices.
One can see from the definition that we already encountered two examples of tensor bundles, namely the tangent bundle $TM = T^1_0 M$ and the cotangent bundle $T^*M = T^0_1 M$. Another bundle, which we have not yet encountered directly, is the trivial line bundle $M \times \mathbb{R} = T^0_0 M$. To see that we have encountered it indirectly, we need to discuss sections of tensor bundles.

Definition 12.4 (Tensor field)**.** A *tensor field* of type (r, s) on a manifold M is a section of the tensor bundle $T^r_s M$. The set of all tensor fields of type (r, s) on M is denoted $\Gamma(T^r_s M)$.

Now it is clear that vector fields are tensor fields of type $(1, 0)$, while covector fields are tensor fields of type $(0, 1)$. But what about real functions? Recall that a real function $f \in C^\infty(M, \mathbb{R})$ is a smooth map $f : M \to \mathbb{R}$. However, each real function uniquely

determines a smooth section of the trivial line bundle $M \times \mathbb{R} = T_0^0 M$. In other words, there is a canonical isomorphism such that $C^\infty(M, \mathbb{R}) \cong \Gamma(T_0^0 M)$. We can thus interpret a real function as a tensor field of type $(0, 0)$ (and see further examples in the future why this makes sense). In physics, a tensor field of type $(0, 0)$ is also called a *scalar field.*
Using coordinates (x^a) on M, we can write any tensor fields $T \in \Gamma(T_s^r M)$ in the form

$$T(x) = T^{a_1 \cdots a_r}{}_{b_1 \cdots b_s}(x) \partial_{a_1} \otimes \ldots \otimes \partial_{a_r} \otimes dx^{b_1} \otimes \ldots \otimes dx^{b_s} \,, \tag{12.7}$$

where the components $T^{a_1 \cdots a_r}{}_{b_1 \cdots b_s}(x)$ are smooth functions of x.
There are different ways to obtain tensor fields from simpler ones. A rather simple construction works as follows.

Definition 12.5 (Tensor field product)**.** Let M be a manifold and $T \in \Gamma(T_s^r M)$ and $U \in \Gamma(T_u^t M)$ be tensor fields. Their *tensor product* is a tensor field $T \otimes U \in \Gamma(T_{s+u}^{r+t} M)$ such that for each $x \in M$,

$$(T \otimes U)(x) = T(x) \otimes U(x) \,. \tag{12.8}$$

This definition can most easily be understood using coordinates (x^a) on M. Let $T \in \Gamma(T_s^r M)$ and $U \in \Gamma(T_u^t M)$, and write $V = T \otimes U$. The tensor product is given by

$$\begin{aligned} T \otimes U = &\left(T^{a_1 \cdots a_r}{}_{b_1 \cdots b_s} \partial_{a_1} \otimes \ldots \otimes \partial_{a_r} \otimes dx^{b_1} \otimes \ldots \otimes dx^{b_s}\right) \\ &\otimes \left(U^{c_1 \cdots c_t}{}_{d_1 \cdots d_u} \partial_{c_1} \otimes \ldots \otimes \partial_{c_t} \otimes dx^{d_1} \otimes \ldots \otimes dx^{d_u}\right) \\ = &\, T^{a_1 \cdots a_r}{}_{b_1 \cdots b_s} U^{c_1 \cdots c_t}{}_{d_1 \cdots d_u} \\ &\partial_{a_1} \otimes \ldots \otimes \partial_{a_r} \otimes dx^{b_1} \otimes \ldots \otimes dx^{b_s} \otimes \partial_{c_1} \otimes \ldots \otimes \partial_{c_t} \otimes dx^{d_1} \otimes \ldots \otimes dx^{d_u} \end{aligned} \tag{12.9}$$

and yields the components

$$V^{a_1 \cdots a_r}{}_{b_1 \cdots b_s}{}^{c_1 \cdots c_t}{}_{d_1 \cdots d_u} = T^{a_1 \cdots a_r}{}_{b_1 \cdots b_s} U^{c_1 \cdots c_t}{}_{d_1 \cdots d_u} \,. \tag{12.10}$$

One might be worried that the basis elements ∂_a and dx^a appear now in "mixed order", in contrast to the definition of the tensor bundle. This is not a problem, since the tensor product bundles $TM \otimes T^*M$ and $T^*M \otimes TM$ are canonically isomorphic, so one can simply define a new tensor field $\tilde{V}$ such that

$$\tilde{V}^{a_1 \cdots a_r c_1 \cdots c_t}{}_{b_1 \cdots b_s d_1 \cdots d_u} = V^{a_1 \cdots a_r}{}_{b_1 \cdots b_s}{}^{c_1 \cdots c_t}{}_{d_1 \cdots d_u} \,. \tag{12.11}$$

However, this does *not* mean that changing the order of indices does not change the tensor field - V and $\tilde{V}$ carry the same information, but encoded differently. As another simple example, the tensor fields

$$A_{ab} dx^a \otimes dx^b \neq A_{ba} dx^a \otimes dx^b = A_{ab} dx^b \otimes dx^a \tag{12.12}$$

are (for general A_{ab}) not the same!
After showing a way how to construct higher tensor fields from simpler ones, we also show a way how to obtain simpler tensor fields.

Definition 12.6 (Tensor field contraction)**.** Let M be a manifold and $\Gamma(T^r_sM)$ the space of tensors of type (r,s) on M with $r,s \geq 1$. The *contraction* of the k'th and l'th tensor component, where $1 \leq k \leq r$ and $1 \leq l \leq s$, is the unique linear function

$$\operatorname{tr}^k_l : \Gamma(T^r_sM) \to \Gamma(T^{r-1}_{s-1}M)\,, \tag{12.13}$$

such that

$$\operatorname{tr}^k_l(A \otimes B \otimes C \otimes D \otimes E \otimes F) = \langle B, E\rangle A \otimes C \otimes D \otimes F \tag{12.14}$$

for all $A \in \Gamma(T^{k-1}_0M)$, $B \in \Gamma(T^1_0M)$, $C \in \Gamma(T^{r-k}_0M)$, $D \in \Gamma(T^0_{l-1}M)$, $E \in \Gamma(T^0_1M)$, $F \in \Gamma(T^0_{s-l}M)$.

Also this construction is most easily illustrated using coordinates. Let $T \in \Gamma(T^r_sM)$ a tensor field of type (r,s) on M. Its contraction of the k'th and l'th component then simply takes the form

$$\begin{aligned}
\operatorname{tr}^k_l T &= T^{a_1\cdots a_r}{}_{b_1\cdots b_s} \operatorname{tr}^k_l \left(\partial_{a_1} \otimes \ldots \otimes \partial_{a_r} \otimes dx^{b_1} \otimes \ldots \otimes dx^{b_s}\right)\\
&= T^{a_1\cdots a_r}{}_{b_1\cdots b_s} \langle \partial_{a_k}, dx^{b_l}\rangle\\
&\quad \left(\partial_{a_1} \otimes \ldots \otimes \partial_{a_{k-1}} \otimes \partial_{a_{k+1}} \otimes \ldots \otimes \partial_{a_r} \otimes dx^{b_1} \otimes \ldots \otimes dx^{b_{l-1}} \otimes dx^{b_{l+1}} \otimes \ldots \otimes dx^{b_s}\right)\\
&= T^{a_1\cdots a_{k-1} c a_{k+1}\cdots a_r}{}_{b_1\cdots b_{l-1} c b_{l+1}\cdots b_s}\\
&\quad \left(\partial_{a_1} \otimes \ldots \otimes \partial_{a_{k-1}} \otimes \partial_{a_{k+1}} \otimes \ldots \otimes \partial_{a_r} \otimes dx^{b_1} \otimes \ldots \otimes dx^{b_{l-1}} \otimes dx^{b_{l+1}} \otimes \ldots \otimes dx^{b_s}\right)\,.
\end{aligned} \tag{12.15}$$

In other words, the components of $\operatorname{tr}^k_l T$ are obtained simply by summation over the k'th upper and l'th lower indices. To illustrate this with another example, we apply it to the following theorem.

Theorem 12.5. *Let M be a manifold, $f \in C^\infty(M,\mathbb{R})$ a function on M and $X \in \operatorname{Vect}(M)$ a vector field on M. Then $Xf = \operatorname{tr}^1_1(X \otimes df) = \langle X, df\rangle$.*

Proof. It is clear from the definition of a tensor contraction that $\operatorname{tr}^1_1(X \otimes df) = \langle X, df\rangle$. To see that this also equals Xf, recall that for every $x \in M$ we obtain $(Xf)(x)$ by applying the derivation $v = X(x) \in T_xM$ to f. Further, df is defined for all x as the equivalence class $df(x) = [f - f(x)]_x \in T^*_xM = I_x/I^2_x$. Finally, the pairing $\langle v, [f - f(x)]_x\rangle$ is given by $v(f)$, which completes the proof. □

There is an even faster way to see this using coordinates, where one easily reads off

$$Xf = X^a\partial_a f = \langle X^a\partial_a, \partial_b f\, dx^b\rangle = \langle X, df\rangle\,. \tag{12.16}$$

A Dictionary

English	Estonian
cotangent space	
cotangent bundle	
covector field	
dual vector space	kaasruum
dual vector bundle	
tensor product	
tensor product bundle	
tensor bundle	
tensor field	
tensor contraction	