Differential geometry for physicists - Lecture 3

Manuel Hohmann

3. March 2015

7 Curves and functions on a manifold

In this lecture we will come to more applied aspects of differential geometry. We start with two very basic and often encountered objects.

Definition 7.1 (Curve)**.** A (smooth) *curve* on a manifold M is a (smooth) map $\gamma : \mathbb{R} \to M$.

If one fixes a chart (U, ϕ) of M with coordinates (x^a), one usually denotes the components of $(\phi \circ \gamma)(t)$ by $\gamma^a(t)$. We will also make use of this notation. Smoothness of the curve γ then simply means that the components γ^a must be smooth functions.

Definition 7.2 (Real function)**.** A (smooth) *real function* on a manifold M is a (smooth) map $f : M \to \mathbb{R}$.

Given a chart (U, ϕ) with coordinates (x^a) one often encounters a slightly dangerous abuse of notation. Instead of $f \circ \phi^{-1}$ for the coordinate expression of f one simply writes f. However, this should not lead to confusion if the chart is clear from the context, so we will use the same convention here. Also here it is now clear how the smoothness of f can be understood.
If we denote the set of all smooth maps from a manifold M to a manifold N by $C^\infty(M, N)$, then the set of smooth curves on M is given by $C^\infty(\mathbb{R}, M)$, while the set of smooth real functions on M is given by $C^\infty(M, \mathbb{R})$. Note that these sets are kind of dual to each other.

8 The tangent bundle

Every manifold is naturally equipped with a number of structures. One of the most basic and important structures is the tangent bundle. Geometrically it can be seen as the space of all vectors tangent to a manifold. In physics it appears most naturally in the context of mechanics: if the space of all possible positions of a point mass is modeled as a manifold, then its velocity is an element of the tangent bundle. The space of all tangent vectors at a given point is called the tangent space, and in can be defined in a number of different, but equivalent ways. Here we use a particularly simple definition in terms of derivations, and provide its geometric interpretation a bit later.

Definition 8.1 (Tangent space)**.** Let M be a manifold and $x \in M$. A *derivation* at x is a linear function $D : C^\infty(M, \mathbb{R}) \to \mathbb{R}$ such that

$$D(fg) = D(f)g(x) + f(x)D(g) \tag{8.1}$$

for all $f, g \in C^\infty(M, \mathbb{R})$. The set of all derivations at x is called the *tangent space* at x and denoted T_xM. It is a vector space, where addition and scalar multiplication are defined as

$$(D_1 + D_2)(f) = D_1(f) + D_2(f) \quad \text{and} \quad (\lambda D)(f) = \lambda D(f)\,. \tag{8.2}$$

Since this definition is rather abstract, we will illustrate it using local coordinates (x^a), where we make use of the conventions mentioned above and don't explicitly write the chart (U, ϕ). We can thus regard a function $f \in C^\infty(M, \mathbb{R})$ as a function of $n = \dim M$ coordinates with domain $U \subset \mathbb{R}^n$. Every derivation $D \in T_xM$ in a point $x_0 \in U$ can be uniquely written in the form

$$D : f \mapsto D(f) = \sum_{a=1}^{\dim M} v^a \left.\frac{\partial}{\partial x^a} f(x)\right|_{x=x_0} = v^a \left.\partial_a f(x)\right|_{x=x_0} \tag{8.3}$$

with $v \in \mathbb{R}^n$, and every $v \in \mathbb{R}^n$ conversely defines a derivation. While the latter is clear and can be seen from a direct calculation, the former is less obvious and follows from Hadamard's lemma. We will not discuss this further here, and simply accept it as a fact for now (it will be proven later). To summarize, the partial derivatives ∂_a at x form a basis of the tangent space T_xM, called the coordinate basis of this choice of coordinates.

Here we have introduced two more useful and convenient notation conventions. For the partial derivatives $\partial/\partial x^a$ with respect to the coordinates x^a we simply write ∂_a. Note that the upper index turns into a lower one here - we will discuss this further when we come to tensor bundles. Further, if we have a term where the same index appears exactly once in upper and in lower position, it implies that the sum over this index is taken, and that the index runs over its complete range, in this case over all coordinates. Using these notations, we can also write a tangent space element simply in the form $v^a\partial_a$. In other words, (∂_a) is a basis of the tangent space, called the coordinate basis corresponding to the coordinates (x^a).

So far we have discussed the tangent spaces at each point separately. We will now put them together:

Definition 8.2 (Tangent bundle)**.** The *tangent bundle* of a manifold M is the disjoint union

$$TM = \biguplus_{x \in M} T_xM\,. \tag{8.4}$$

The canonical projection of the tangent bundle is the function $\pi : TM \to M$ such that $\pi(v) = x$ for $v \in T_xM$.

It is important to note that we take the *disjoint* union of all tangent spaces, i.e., we consider elements of TM to be different if they are taken from the tangent spaces T_xM and T_yM at different points $x \neq y$. For example, the function $D : C^\infty(M, \mathbb{R}) \to \mathbb{R}, f \mapsto 0$ is obviously a derivation both at x and y, and it would be contained only once in TM if we would naively take the union of all tangent spaces as defined at the beginning of this section. However, by taking the disjoint union, the elements of TM are actually pairs (x, D) of a point $x \in M$ and a derivation $D \in T_xM$, such that $(x, D) \neq (y, D)$. The projection π is then simply the function $\pi : (x, D) \mapsto x$.
The name already suggests that the tangent bundle is a fiber bundle - and even more, a vector bundle. We will show this in two steps.

Theorem 8.1. *The tangent bundle TM of a manifold M of dimension n is a manifold of dimension $2n$.*

Proof. We need to construct an atlas of TM. Let $x \in M$ and $(U, \phi) \in \mathcal{A}$ be a chart in an atlas $\mathcal{A}$ of M such that $x \in U$. Define $\tilde{U} = \pi^{-1}(U) \subset TM$ and let $\tilde{\phi} : \tilde{U} \to \mathbb{R}^{2n}$ the function that assigns to $(x, D) \in \tilde{U}$ the pair consisting of $\phi(x)$ and $v \in \mathbb{R}^n$ such that $D(f) = v^a \partial_a f$. Apply this procedure to every chart in $\mathcal{A}$. One easily checks that this yields an atlas $\tilde{\mathcal{A}}$ of TM. □

Theorem 8.2. *The structure (TM, M, π) is a vector bundle of rank $n = \dim M$.*

Proof. For $x \in M$, let (U, ϕ) be a chart of M such that $x \in U$. Define a function $\hat{\phi} : \pi^{-1}(U) \to U \times \mathbb{R}^n$ such that $\hat{\phi}(x, D) = (x, v)$ and $v \in \mathbb{R}^n$ as in the previous proof. One can easily show that $\hat{\phi}$ has the properties listed in the definition of a vector bundle. □

We now come to more practical aspects of the tangent bundle, which are closer to physics. One of the most important aspects is the tangent vector of a curve, which can be interpreted as the velocity of a point mass along its trajectory and which is defined as follows.

Definition 8.3 (Tangent vector of a curve)**.** Let $\gamma \in C^\infty(\mathbb{R}, M)$ be a curve on a manifold M. Its *tangent vector* at $t \in \mathbb{R}$ is the derivation $\dot{\gamma}(t) \in T_{\gamma(t)}M$ defined by

$$\dot{\gamma}(t)(f) = (f \circ \gamma)'(t) \tag{8.5}$$

for $f \in C^\infty(M, \mathbb{R})$.

Also here we are interested in a coordinate description, so we will work in local coordinates (x^a). As mentioned before, we can use these coordinates to write a curve $\gamma \in C^\infty(\mathbb{R}, M)$ using its components $\gamma^a(t)$, and that a function $f \in C^\infty(M, \mathbb{R})$ depends on the coordinates x^a. If we set these coordinates equal to $\gamma^a(t)$, we obtain the composition $f \circ \gamma : \mathbb{R} \to \mathbb{R}$. By the chain rule we then have

$$(f \circ \gamma)'(t_0) = \left.\frac{\partial \gamma^a}{\partial t}\right|_{t=t_0} \left.\frac{\partial f}{\partial x^a}\right|_{x^a=\gamma^a(t_0)} = \dot{\gamma}^a(t_0)\, \partial_a f|_{x^a=\gamma^a(t_0)}\,, \tag{8.6}$$

so that the coordinate expression of the tangent vector $\dot{\gamma}(t_0) \in T_{\gamma(t_0)}M$ is simply $\dot{\gamma}^a(t_0)\partial_a$.

9 Vector fields

If we consider a fluid instead of a point mass, we have a velocity at each point of the fluid, so we need to assign a tangent space element to every point. We already encountered this type of assignment and called it a section. Sections of the tangent bundle are so important that they deserve their own name.

Definition 9.1 (Vector field)**.** A *vector field* on a manifold M is a section of the tangent bundle TM. The space of all vector fields on M is denoted $\Gamma(TM)$ or $\mathrm{Vect}(M)$.

Let $X \in \mathrm{Vect}(M)$. If we use local coordinates (x^a), we can write $X(x)$ for $x \in M$ using the coordinate basis of T_xM as $X^a(x)\partial_a$, and the whole vector field X as $X^a\partial_a$. The condition that X is a section, and thus a smooth map, then simply means that the component functions X^a must be smooth.
Since a vector field assigns to any point of a manifold a derivation at that point, we can define the following construction.

Definition 9.2 (Action of a vector field on a function)**.** Let M be a manifold, $X \in \mathrm{Vect}(M)$ a vector field on M and $f \in C^\infty(M, \mathbb{R})$ a real function on M. For each $x \in M$, the vector field X defines a derivation $X(x) \in T_xM$. Via these derivations X *acts* on f, i.e., it defines a real function $Xf \in C^\infty(M, \mathbb{R})$ given by

$$(Xf)(x) = X(x)(f) \tag{9.1}$$

for all $x \in M$.

To see that Xf is indeed a smooth function, one can evaluate it using local coordinates (x^a). In these coordinates the vector field X takes the form $X^a\partial_a$, and $Xf = X^a\partial_a f$, which should be read in the obvious way:

$$(Xf)(x) = X^a(x)\partial_a f(x) = X^a(x)\frac{\partial f}{\partial x^a}(x)\,. \tag{9.2}$$

Some more properties follow from the definition:

- Xf is $\mathbb{R}$-linear in the first argument:

$$(\lambda X + \mu Y)f = \lambda(Xf) + \mu(Yf) \quad \text{for} \quad \lambda, \mu \in \mathbb{R}\,. \tag{9.3}$$

- Xf is $\mathbb{R}$-linear in the second argument:

$$X(\lambda f + \mu g) = \lambda(Xf) + \mu(Xg) \quad \text{for} \quad \lambda, \mu \in \mathbb{R}\,. \tag{9.4}$$

- Xf satisfies the Leibniz rule for the second argument:

$$X(fg) = (Xf)g + f(Xg)\,. \tag{9.5}$$

This action allows us to give a bit more structure to the set $\mathrm{Vect}(M)$ as follows.

Definition 9.3 (Commutator of vector fields)**.** Let M be a manifold and $X, Y \in \mathrm{Vect}(M)$ vector fields. Their *commutator* is the unique vector field $[X, Y] \in \mathrm{Vect}(M)$ such that for all $f \in C^\infty(M, \mathbb{R})$,

$$[X, Y]f = X(Yf) - Y(Xf)\,. \tag{9.6}$$

Of course one must check that such a unique vector field $[X, Y]$ really exists, i.e., that the definition above assigns to each point $x \in M$ a derivation at x. It is clear from the definition above that $[X, Y](x) : C^\infty(M, \mathbb{R}) \to \mathbb{R}$ is a linear function. To see that it is a derivation, one calculates

$$\begin{aligned}
[X, Y](fg) &= X(Y(fg)) - Y(X(fg)) \\
&= X((Yf)g + f(Yg)) - Y((Xf)g - f(Xg)) \\
&= (X(Yf))g + (Yf)(Xg) + (Xf)(Yg) + f(X(Yg)) \\
&\quad - (Y(Xf))g - (Xf)(Yg) - (Yf)(Xg) - f(Y(Xg)) \\
&= ([X, Y]f)g + f([X, Y]g)\,.
\end{aligned} \tag{9.7}$$

If we express the vector fields $X = X^a\partial_a$ and $Y = Y^a\partial_a$ in a coordinate basis, we see that

$$[X, Y]f = X^a\partial_a(Y^b\partial_b f) - Y^a\partial_a(X^b\partial_b f) = (X^a\partial_a Y^b - Y^a\partial_a X^b)\partial_b f = [X, Y]^b\partial_b f\,. \tag{9.8}$$

This gives us an explicit formula for the components $[X, Y]^a$ in these coordinates. This formula has the same form in all coordinate systems, since we have made no reference to particular coordinates in the definition of $[X, Y]$. Another property of the commutator is now easy to show.

Theorem 9.1. *The set* $\mathrm{Vect}(M)$ *of vector fields on a manifold* M *carries the structure of a real Lie algebra, with the Lie bracket given by the commutator.*

Proof. It follows from the linearity of Xf in the first argument that $\mathrm{Vect}(M)$ is a real vector space. The same property implies that the commutator $[X, Y]$ is linear in both arguments. Further, one can see immediately from the definition that it is antisymmetric. Finally, we check the Jacobi identity

$$\begin{aligned}
0 &\stackrel{?}{=} [X, [Y, Z]]f + [Y, [Z, X]]f + [Z, [X, Y]]f \\
&= X(Y(Zf)) - X(Z(Yf)) - Y(Z(Xf)) + Z(Y(Xf)) + Y(Z(Xf)) - Y(X(Zf)) \\
&\quad - Z(X(Yf)) + X(Z(Yf)) + Z(X(Yf)) - Z(Y(Xf)) - X(Y(Zf)) + Y(X(Zf))\,.
\end{aligned} \tag{9.9}$$

□

A Dictionary

English	Estonian
curve	joon
real function	reaalne funktsioon
derivation	derivatsioon
tangent space	puutujaruum
tangent bundle	puutujakihtkond
tangent vector	puutujavektor
vector field	vektoriväli
commutator	kommutaator