Differential geometry for physicists - Lecture 2

Manuel Hohmann

17. February 2015

4 Fiber bundles

In the last lecture we have introduced the direct product of manifolds. We now discuss an important concept, called a *fiber bundle*, which can be viewed as a local version of a product manifold. Recall that in the case of the direct product $M \times N$ of two manifolds we have projections pr_M and pr_N onto each factor. One can show that the pre-image of a point $p \in M$ under pr_M is again a manifold which is diffeomorphic to N. We write: $\mathrm{pr}_M^{-1}(p) \cong N$. Of course the construction of the direct product is symmetric in M and N, so that also $\mathrm{pr}_N^{-1}(q) \cong M$ for $q \in N$.
For a fiber bundle, only one half of this is true. It consists of a manifold E called the *total space*, another manifold B called the *base space* and a map $\pi : E \to B$ called the *projection* or *bundle map*, such that for any $p \in B$ the pre-image $\pi^{-1}(p)$ is diffeomorphic to a manifold F called the *fiber*. In addition, we need a condition which guarantees that the total space of the fiber bundle "locally looks like" a direct product. We define:

Definition 4.1 (Fiber bundle). A (smooth) *fiber bundle* (E, B, π, F) consists of (smooth) manifolds E, B, F and a (smooth) surjective map π, such that for any $p \in B$ there exists an open set $U \subset B$ containing p and a diffeomorphism $\phi : \pi^{-1}(U) \to U \times F$ such that the diagram

$$\begin{tikzcd} \pi^{-1}(U) \arrow[r, "\phi"'] \arrow[d, "\pi"] & U \times F \arrow[dl, "\mathrm{pr}_U"] \\ U & \end{tikzcd} \tag{4.1}$$

commutes. The pair (U, ϕ) is called a *local trivialization*.

It should be remarked that also here we have used only smooth objects (manifolds and maps) in order to define a smooth fiber bundle. There exist also more general versions of fiber bundles. However, the most common and useful ones in physics are smooth.
We do not need to explicitly demand that $\pi^{-1}(p) \cong F$ for any $p \in B$, because this follows from the definition given above. To see this, note that $\mathrm{pr}_U^{-1}(p) \cong F$, as for any direct product. Since ϕ is a diffeomorphism, it follows that also $\pi^{-1}(p) = \phi^{-1}(\mathrm{pr}_U^{-1}(p)) \cong F$.
Another common notation for a fiber bundle is the "function notation" $\pi : E \to B$, when the fiber manifold F is clear from the context. Sometimes, with a slight abuse of terminology, also the total space E is simply called a fiber bundle, when the base manifold and the projection are known.

Example 4.1 (Trivial fiber bundle). The most simple class of fiber bundles are *trivial fiber bundles*. Given manifolds M, N, one can construct two different trivial fiber bundles, which are given by $(M \times N, M, \mathrm{pr}_M, N)$ and $(M \times N, N, \mathrm{pr}_N, M)$. It is easy to check that these are indeed fiber bundles.

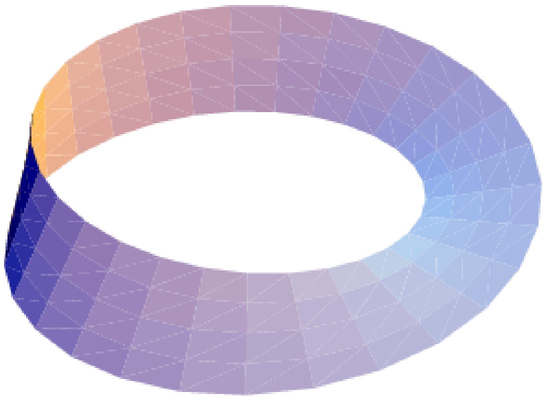

Figure 1: Möbius strip

Example 4.2 (Möbius strip). Let $\tilde{U}_1 = (0, 2\pi) \times (-1, 1)$, $\tilde{U}_2 = (-\pi, \pi) \times (-1, 1)$ and the functions

$$\begin{array}{cccc} \tilde{\phi}_i & : & \tilde{U}_i & \to & \mathbb{R}^3 \\ & & (t, s) & \mapsto & \left(\left(R + Ws\cos\frac{t}{2}\right)\cos t, \left(R + Ws\cos\frac{t}{2}\right)\sin t, Ws\sin\frac{t}{2}\right) \end{array}, \tag{4.2}$$

$i = 1, 2$, with constants $0 < W < R$. Let further $M = \tilde{\phi}_1(\tilde{U}_1) \cup \tilde{\phi}_2(\tilde{U}_2)$. It is easy to show that M carries the structure of a two-dimensional manifold, and that an atlas is given by the charts $(U_i = \tilde{\phi}_i(\tilde{U}_i), \phi_i = \tilde{\phi}_i^{-1})$. This manifold is called the *Möbius strip*. Now consider the function

$$\begin{array}{cccc} \tilde{\pi} & : & \{x \in \mathbb{R}^3 | x_1^2 + x_2^2 > 0\} & \to & \mathbb{R}^2 \\ & & (x_1, x_2, x_3) & \mapsto & \left(\frac{x_1}{\sqrt{x_1^2+x_2^2}}, \frac{x_2}{\sqrt{x_1^2+x_2^2}}\right) \end{array} \tag{4.3}$$

Looking at the compositions

$$\begin{array}{cccc} \tilde{\pi} \circ \tilde{\phi}_i & : & \tilde{U}_i & \to & \mathbb{R}^2 \\ & & (t, s) & \mapsto & (\cos t, \sin t) \end{array} \tag{4.4}$$

one can see that the restriction of $\tilde{\pi}$ to M defines a smooth map $\pi : M \to S^1 = \{x \in \mathbb{R}^2 | x_1^2 + x_2^2 = 1\}$ and that the pre-images $\pi^{-1}(p)$ are diffeomorphic to $(-1, 1)$. One can show that $(M, S^1, \pi, (-1, 1))$ is a non-trivial fiber bundle.

5 Vector bundles

Often we encounter fiber bundles whose fibers are not just manifolds, but also carry additional structure. The most common structure is that of a (real or complex) vector space.

In this case the fiber bundle is called a *vector bundle*. To keep things simple for now, we will restrict ourselves to real vector bundles, which are defined as follows.

Definition 5.1 (Vector bundle)**.** A (real, smooth) *vector bundle* of rank $k \in \mathbb{N}$ is a fiber bundle $(E, B, \pi, \mathbb{R}^k)$ such that for all $p \in B$ the pre-image $\pi^{-1}(p)$ is a real vector space of dimension k and such that the restrictions of the local trivializations $\phi : \pi^{-1}(U) \to U \times \mathbb{R}^k$ to a fiber $\pi^{-1}(p)$ for $p \in U$ are vector space isomorphisms from $\pi^{-1}(p)$ to $\{p\} \times \mathbb{R}^k$.

One may ask why we want the local trivializations to restrict to vector space isomorphisms. This is a typical example for a very common situation that we have two different structures, here that of a manifold and that of a vector space, which we want to be *compatible*. In this case it guarantees that on every fiber $\pi^{-1}(p)$ for $p \in B$, which is both a manifold diffeomorphic to $\mathbb{R}^k$ and a vector space isomorphic to $\mathbb{R}^k$, both

- the scalar multiplication $\cdot : \mathbb{R} \times \pi^{-1}(p) \to \pi^{-1}(p)$
- and the addition $+ : \pi^{-1}(p) \times \pi^{-1}(p) \to \pi^{-1}(p)$

are smooth maps. Further, it guarantees that if $p, p' \in B$ are "close to each other", then:

- The zero elements of the vector spaces $\pi^{-1}(p)$ and $\pi^{-1}(p')$ are "close to each other".
- If $v \in \pi^{-1}(p)$ and $v' \in \pi^{-1}(p')$ are "close to each other", then also λv and $\lambda v'$ are "close to each other" for any $\lambda \in \mathbb{R}$.
- If $v \in \pi^{-1}(p)$ is "close to" $v' \in \pi^{-1}(p')$ and $w \in \pi^{-1}(p)$ is "close to" $w' \in \pi^{-1}(p')$, then also $v + w$ and $v' + w'$ are "close to each other".

Of course, we need to define what we mean by being "close to each other". This will be made precise in the next section, when we discuss sections of fiber (and in particular vector) bundles.

Example 5.1 (Möbius strip as a vector bundle). In the last section we discussed the Möbius strip as a fiber bundle $(M, S^1, \pi, (-1, 1))$. However, the open interval $(-1, 1)$ and the real line $\mathbb{R}$ are diffeomorphic, one-dimensional manifolds, so that one can also view the Möbius strip as a fiber bundle $(M, S^1, \pi, \mathbb{R})$, which one may call an "infinite Möbius strip". This can be seen most easily by changing the charts from our previous definition such that $\tilde{U}_1 = (0, 2\pi) \times \mathbb{R}$, $\tilde{U}_2 = (-\pi, \pi) \times \mathbb{R}$ and the functions

$$\begin{array}{cccc} \tilde{\phi}_i & : & \tilde{U}_i & \to & \mathbb{R}^3 \\ & & (t, s) & \mapsto & \left(\left(R + \frac{Ws}{\sqrt{1+s^2}} \cos \frac{t}{2}\right) \cos t, \left(R + \frac{Ws}{\sqrt{1+s^2}} \cos \frac{t}{2}\right) \sin t, \frac{Ws}{\sqrt{1+s^2}} \sin \frac{t}{2}\right) \end{array} . \tag{5.1}$$

On each fiber $\pi^{-1}(p) \cong \mathbb{R}$ one has the usual structure of the one-dimensional vector space $\mathbb{R}$.

6 Sections

When dealing with fiber bundles we often work with maps $f : B \to E$ which assign to each point $p \in B$ on the base manifold a point $f(p) \in \pi^{-1}(p) \subset E$ on the fiber over p. These maps are called *sections*, and are defined as follows.

Definition 6.1 (Section)**.** A (smooth) *section* of a fiber bundle (E, B, π, F) is a (smooth) map $f : B \to E$ such that $\pi \circ f = \mathrm{id}_B$.

What we have defined here is also called a *global* section, since its domain is the whole base manifold B. Not every bundle admits global sections - there are bundles for which no global sections exist. However, it is always possible to find *local* sections, i.e., maps $f : U \to E$ defined on an open set $U \subset B$ such that $\pi \circ f = \mathrm{id}_B$.
The set of all sections of a fiber bundle is often denoted $\Gamma(E, B, \pi, F)$, or simply $\Gamma(E)$ if it is clear which are the other ingredients of the fiber bundle.
Vector bundles always admit global sections, the most simple one given in the following example.

> *Example* 6.1 (Zero section). Every vector bundle $(E, B, \pi, \mathbb{R}^k)$ has at least one section, called the *zero section*, which assigns to each $p \in B$ the zero element of the vector space $\pi^{-1}(p)$.

It is not difficult to show that the zero section is indeed a section. We will prove a more general statement here, from which also this property of the zero section follows.

Theorem 6.1. *The set of all sections of a vector bundle is a vector space, where scalar multiplication and addition are defined pointwise.*

Proof. Let f, g be sections of a vector bundle $(E, B, \pi, \mathbb{R}^k)$ and $\lambda, \mu \in \mathbb{R}$. We have to check that also the function $h = \lambda f + \mu g$ defined by

$$\begin{array}{rcccc} h & : & B & \to & E \\ & & p & \mapsto & h(p) = \lambda f(p) + \mu g(p) \end{array} \tag{6.1}$$

is a smooth section, i.e., a smooth map such that $\pi \circ h = \mathrm{id}_B$. We first have to check that this function is well-defined. Since both f and g are sections, they satisfy $\pi \circ f = \pi \circ g = \mathrm{id}_B$. For any $p \in B$ we thus have $f(p) \in \pi^{-1}(p)$ and $g(p) \in \pi^{-1}(p)$. Since $\pi^{-1}(p)$ carries the structure of a vector space, there is a well-defined element $\lambda f(p) + \mu g(p) = h(p) \in \pi^{-1}(p)$, so that the function h is indeed well-defined. This also shows that $\pi \circ h = \mathrm{id}_B$.
We finally show that h is a smooth map. To see this, let (U, ϕ) be a local trivialization around some point $p \in B$. The functions $\phi \circ f : U \to U \times \mathbb{R}^k$ and $\phi \circ g : U \to U \times \mathbb{R}^k$ are smooth maps, since ϕ is a diffeomorphism. Now $U \times \mathbb{R}^k$ is a product manifold, so that its elements are pairs (p, x) with $p = \mathrm{pr}_U(p, x) \in U$ and $x = \mathrm{pr}_{\mathbb{R}^k}(p, x) \in \mathbb{R}^k$. Since f and g are sections, we have by definition for every $p \in B$:

$$(\phi \circ f)(p) = ((\underbrace{\mathrm{pr}_U \circ \phi}_{=\pi} \circ f)(p), (\mathrm{pr}_{\mathbb{R}^k} \circ \phi \circ f)(p)) = (p, (\mathrm{pr}_{\mathbb{R}^k} \circ \phi \circ f)(p)) \in \{p\} \times \mathbb{R}^k \tag{6.2}$$

and the same for g. The function $\mathrm{pr}_{\mathbb{R}^k} \circ \phi \circ f$ is a smooth map, since projections are smooth. We define a function

$$\tilde{h} : U \to U \times \mathbb{R}^k, p \mapsto (p, \lambda(\mathrm{pr}_{\mathbb{R}^k} \circ \phi \circ f)(p) + \mu(\mathrm{pr}_{\mathbb{R}^k} \circ \phi \circ g)(p)) \,. \tag{6.3}$$

This is a smooth map, since sums and multiples of smooth functions on $\mathbb{R}^k$ are smooth. Using the fact that

$$(\phi \circ h)(p) = \phi(\lambda f(p) + \mu g(p)) = \lambda \phi(f(p)) + \mu \phi(g(p)) \in \{p\} \times \mathbb{R}^k \,, \tag{6.4}$$

since ϕ restricts to an isomorphism of vector spaces on every fiber, it is now easy to see that $h = \phi^{-1} \circ \tilde{h}$ is smooth on U. Finally, we can find such a trivialization (U, ϕ) for all $p \in B$, and thus h is a smooth map. □

Now this is the precise notion of what we meant by being "close to each other" in the previous section. It means that if f, g are smooth sections of a vector bundle ("$f(p)$ is close to $f(p')$" if "p is close to p'" and the same for g), then also $\lambda f + \mu g$ is a smooth section ("$\lambda f(p) + \mu g(p)$ is close to $\lambda f(p') + \mu g(p')$") for any $\lambda, \mu \in \mathbb{R}$.

A Dictionary

English	Estonian
surjective map	pealekujutus
injective map	üksühene kujutus
fiber bundle	kihtkond
total space	kihtkonna ruum
base space	kihtkonna baas
fiber	kiht
local trivialization	lokaalne trivialisatsioon
vector bundle	vektorkihtkond
section	lõige
Möbius strip	Möbiuse leht