Differential geometry for physicists - Lecture 1

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1 Manifolds

The most important concept we will be dealing with in this lecture course is that of a *manifold*. A manifold can be viewed as a set with an additional structure, called an *atlas*. For our purposes it will be most useful to adopt the following definition of an atlas:

Definition 1.1 (Atlas)**.** Let M be a set. An *atlas* $\mathcal{A}$ of dimension n on M is a collection of pairs (U_i, ϕ_i), such that the following properties hold:

- Every pair (U_i, ϕ_i), called a *chart*, consists of a set $U_i \subset M$ and an injective function $\phi_i : U_i \to \mathbb{R}^n$, such that the image $\phi_i(U_i) \subset \mathbb{R}^n$ is open.
- The sets U_i cover M:
$$\bigcup_i U_i = M\,. \tag{1.1}$$
- Any two charts (U_i, ϕ_i) and (U_j, ϕ_j) are *compatible* in the sense that the *transition function*
$$\phi_{ij} : \phi_i(U_i \cap U_j) \to \phi_j(U_i \cap U_j) \tag{1.2}$$
defined by $\phi_{ij} = \phi_j \circ \phi_i^{-1}$ is smooth.

We need to make a few remarks on this definition. First, what we have defined is also called a *real* atlas, since the target space of all functions ϕ_i is the real vector space $\mathbb{R}^n$. Second, we have defined a *smooth* or C^∞-atlas by demanding that all transition functions are smooth, i.e., that they are continuous and infinitely often continuously differentiable. A less strict definition would have been that of a *topological* atlas, where the transition functions only need to be continuous, or a C^k-atlas for $k \in \mathbb{N}$, where they need to be k times continuously differentiable. However, in physics it is often convenient to assume that everything is smooth, and so we will stick to this assumption. We further define:

Definition 1.2 (Maximal atlas)**.** An atlas $\mathcal{A}$ on a set M is called *maximal* if there exists no further chart (U, ϕ) on M which is compatible with all charts in $\mathcal{A}$ and which is not already contained in $\mathcal{A}$.

Note that any atlas $\mathcal{A}$ defines a maximal atlas $\bar{\mathcal{A}}$, which contains all charts which are compatible with all charts of $\mathcal{A}$. Finally, we define:

Definition 1.3 (Manifold)**.** A *manifold* is a set M (its *space*) together with a maximal atlas $\mathcal{A}$ on M.

Again, it should be remarked that what we have defined here is a *real, smooth* (or C^∞) manifold, and that any other kind of atlas can be used to define a different class of manifolds. Instead of calling the pair $(M, \mathcal{A})$ a manifold, it is also common to call M itself a manifold and to take $\mathcal{A}$ as implicitly defined. We will make use of this convention and explicitly write the atlas $\mathcal{A}$ only if it is needed.

One may ask why we want a *maximal* atlas in this definition. The answer becomes clear if we ask the question when two manifolds are the same. There are actually two equivalent ways to define this. We can say that two (non-maximal) atlases $\mathcal{A}, \mathcal{A}'$ define the same manifold structure on M if they are compatible. We could thus define a manifold as a set M together with an equivalence class of compatible atlases. However, two atlases are compatible if and only if their completions to maximal atlases agree, $\bar{\mathcal{A}} = \bar{\mathcal{A}}'$. Therefore, an equivalence class of compatible atlases is essentially the same as a maximal atlas.

A special role is given to charts in the application of differential geometry to physics. In this context, a chart together with an assignment of names to the components of $\mathbb{R}^n$ is also called a set of (local) *coordinates*, while a transition function is also called a *change of coordinates*. There are different ways to label coordinates. One possibility is to give an explicit name to each coordinate, such as (x, y, z) or (r, θ, φ) for a chart of a three-dimensional manifold. Another common possibility is to write coordinates as indexed quantities, such as $(x^a, a = 1, \ldots, 3)$, where here the coordinates are named (x^1, x^2, x^3). It is conventional to use upper indices for coordinates - these must not be confused with powers!

A similar notation is used for transition functions. Let the coordinates of the chart (U_i, ϕ_i) be denoted by (x^a) and those of (U_j, ϕ_j) by (x'^a), where $a = 1, \ldots, n$. The transition function ϕ_{ij} is then commonly written as $x'(x)$, and specified in terms of the coordinate functions

$$x'^1(x^1, \ldots, x^n), \ldots, x'^n(x^1, \ldots, x^n)\,. \tag{1.3}$$

The requirement that a transition function must be smooth is then expressed by the requirement that all component functions must be continuous, that they must be infinitely often partially differentiable and that all partial derivatives are continuous.

One should be careful here, because in the physics literature one often finds coordinates corresponding to charts which cover *almost*, but not all of M. An example is the description of the two-dimensional sphere S^2 by latitude $-\pi/2 < \theta < \pi/2$ and longitude $0 < \varphi < 2\pi$, which does not include the poles and the zero meridian. It is also conventional to "cure" this problem by redefining the coordinate range in the form $-\pi/2 \leq \theta \leq \pi/2$ and $0 \leq \varphi < 2\pi$, but this is not even a chart anymore, since it does not define a function onto an open subset of $\mathbb{R}^2$! The missing / added points here are called "coordinate singularities". One can work with this description, but one must pay attention to all the possible illnesses that occur at coordinate singularities and know how to deal with them (namely, by using proper charts). The following example gives such charts for the sphere S^2.

Example 1.1. Imagine a sphere S^2 of radius 1 embedded in Euclidean space $\mathbb{R}^3$. Let this sphere be intersected by a plane through its center. Call the intersection of the plane and the sphere the equator, and define the two poles of the sphere in the usual geographical sense. Fix one of these poles p_1 and pick a second, different point p on the sphere. Connect these two points by straight line. The straight extension of this line intersects the plane exactly once. Keeping the pole p_1 fixed and moving the second point p, we obtain a function that assigns to every point $p \in S^2 \setminus \{p_1\}$ a point $\phi_1(p)$ of the plane. In the "usual" latitude / longitude description of the sphere this function takes the form

$$\phi_1(\theta, \varphi) = \left(\frac{\cos\theta \cos\varphi}{1 - \sin\theta}, \frac{\cos\theta \sin\varphi}{1 - \sin\theta} \right) . \tag{1.4}$$

Do the same with the other pole p_2 to construct a second function $\phi_2 : S^2 \setminus \{p_2\} \to \mathbb{R}^2$. This function can be written as

$$\phi_2(\theta, \varphi) = \left(\frac{\cos\theta \cos\varphi}{1 + \sin\theta}, \frac{\cos\theta \sin\varphi}{1 + \sin\theta} \right) . \tag{1.5}$$

It is now easy to check that we have constructed two charts of the sphere which constitute a smooth atlas. Using Cartesian coordinates (x, y) for the first chart and (x', y') for the second chart on $\mathbb{R}^2$, where we choose the origin of the plane to coincide with the center of the sphere, we also have two sets of coordinates on (almost all of) the sphere. In these coordinates we can write the transition function $\phi_{21} = \phi_2 \circ \phi_1^{-1} : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}^2 \setminus \{0\}$ as

$$x' = \frac{x}{x^2 + y^2} , \quad y' = \frac{y}{x^2 + y^2} . \tag{1.6}$$

Note that in each chart one point of the sphere is missing, so that one needs to take the other chart to make statements regarding this point.

In this lecture course we will use coordinates whenever it is necessary, which is the case for explicit calculations of examples (and which is also the most important application of coordinates in physics). Sometimes we will introduce a particular set of coordinates, sometimes we will simply assume that some set of coordinates is given, which we do not specify any further. But most of the time, whenever it is possible, we will avoid the use of coordinates.

2 Maps

The second very important we need is that of a *map* between manifolds. We define:

Definition 2.1 (Map)**.** Let M, N be manifolds. A *map* from M to N is a function $f : M \to N$ such that for each point $p \in M$ exist charts (U, ϕ) of M and (V, χ) on N such that:

- $p \in U$ and $f(p) \in V$.
- The function $\chi \circ f \circ \phi^{-1} : \phi(U \cap f^{-1}(V)) \to \chi(f(U) \cap V)$ is smooth.

Note that again we have restricted ourselves to the most useful case in physics, which is that of a *smooth* map between smooth manifolds. We will not consider more general maps between manifolds.

Given coordinates $(x^a, a = 1, \ldots, m)$ on M and $(y^b, b = 1, \ldots, n)$ on N, where m and n are the respective dimensions of M and N, a map is often conventionally expressed as $y(x)$, which is to be interpreted in the same way as the transition functions discussed in the previous section. Again, for a smooth map the component functions must be continuous, infinitely often partially differentiable and all their partial derivatives must be continuous. We finally introduce a particularly useful type of map:

Definition 2.2 (Diffeomorphism)**.** A (smooth) map $f : M \to N$ which is bijective and whose inverse $f^{-1} : N \to M$ is again a (smooth) map, is called a *diffeomorphism.* If such a diffeomorphism exists, the manifolds M, N are called *diffeomorphic.*

Every manifold M is equipped with at least one diffeomorphism, which is the identical map $\mathrm{id}_M : M \to M, p \mapsto p$. In fact, there exist many more diffeomorphisms on a manifold, as we will see later.

3 Product manifold and projections

Given manifolds M and N with atlases $\mathcal{A}_M$ and $\mathcal{A}_N$, one can easily construct another manifold as follows:

Definition 3.1 (Product manifold)**.** Let M and N be manifolds of dimensions m and n with atlases $\mathcal{A}_M$ and $\mathcal{A}_N$. On the Cartesian product $M \times N = \{(p, q)|p \in M, q \in N\}$ define an atlas $\mathcal{A}_{M\times N}$ of dimension $m + n$ with charts (W_{ij}, ψ_{ij}) as follows:

- The sets W_{ij} are given by $W_{ij} = U_i \times V_j$.
- The functions $\psi_{ij} : W_{ij} \to \mathbb{R}^{m+n}$ are given by $\psi_{ij}(p, q) = (\phi_i(p), \chi_j(q))$.

The completion of this atlas to a maximal atlas then turns $M \times N$ into a manifold, called the *product manifold* (or *direct product*).

One can easily check that this is indeed an atlas. The product manifold comes with a set of useful maps:

Definition 3.2 (Projection map)**.** Let M and N be manifolds and $M \times N$ their direct product. The maps $\mathrm{pr}_M : M \times N \to M, (p, q) \mapsto p$ and $\mathrm{pr}_N : M \times N \to N, (p, q) \mapsto q$ are called the *projections* onto the first and second factor, respectively.

Again, it is easy to check that the projections are indeed smooth maps.
We also take a brief look at the coordinates one can use on a product manifold. Given coordinates (x^a) on M and (y^b) on N, corresponding to charts (U, ϕ) and (V, χ), the corresponding coordinates for the product chart (W, ψ) as constructed above are simply (x^a, y^b).

Example 3.1. Let $M = \mathbb{R}$ the line and $N = S^1$ the circle. Their direct product is the cylinder $\mathbb{R} \times S^1$.

Example 3.2. Let $M_i = S^1, i = 1, \dots n$. The direct product $M_1 \times \dots \times M_n$ is the n-dimensional torus T^n.

A Dictionary

English	Estonian
open set	lahtine hulk
closed set	kinnine hulk
subset	alamhulk
vector space	vektorruum
continuous function	pidev funktsioon
differentiable function	diferentseeruv funktsioon
smooth function	sile funktsioon
(local) chart	(lokaalne) kaart
transition function	üleminekufunktsioon
atlas	atlas
manifold	muutkond
topological manifold	topoloogiline muutkond
smooth manifold	sile muutkond
change of coordinates	koordinaatteisendus
map	kujutus
diffeomorphism	difeomorfism
diffeomorphic manifolds	difeomorfsed muutkonnad
direct product	otsekorrutis
projection	projektsioon