Differential geometry for physicists - Assignment 16

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1. U(1) **gauge theory**
   Let $\pi : P \to M$ be a principal G-bundle with $G = \mathrm{U}(1)$ the complex numbers with modulus 1 and $F = \mathbb{C}$ with the canonical left action $\rho(g, z) = gz$.

   (a) Show that given coordinates (x^μ) on a trivializing neighborhood $U \subset M$ and the canonical coordinates on U(1) and F, a gauge field $\Omega : M \to C$ can be expressed by the coordinate expression $A_\mu(x) dx^\mu$, while a matter field can be expressed by a complex function $\phi(x)$.

   (b) Show that in these coordinates the Koszul connection of the associated bundle connection on $P \times_\rho F$ is given by $\nabla\phi = (\partial_\mu\phi - iA_\mu\phi)dx^\mu$.

   (c) Let $g \in T^0_2 M$ be a non-degenerate tensor field (a metric) and $m > 0$. Show that

   $$\mathcal{L} = g^{\mu\nu}(\nabla_\mu\phi)^*(\nabla_\nu\phi) - m\phi^*\phi$$

   is invariant under gauge transformations of the form

   $$A'_\mu(x) = A_\mu(x) + \partial_\mu\Lambda(x)\,, \quad \phi'(x) = \phi(x)e^{i\Lambda(x)}\,.$$