Differential geometry for physicists - Assignment 14

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19. May 2015

1. **Cross product**
   Consider the Lie group $G = \mathrm{SO}(3)$ as given in the example in the lecture. Let $M = \mathbb{R}^3 \times \mathbb{R}^3$ with left action $\rho_M(g,(x,y)) = (gx, gy)$ and $N = \mathbb{R}^3$ with left action $\rho_N(g,x) = gx$, where gx denotes the multiplication of a matrix and a vector. Prove that the *cross product* $\times : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$ is an equivariant map.

2. **Hopf fibration**
   Consider the groups $G = \mathrm{SU}(2)$ and $H = \mathrm{SO}(2)$.

   (a) Show that
   $$\mathrm{SU}(2) = \left\{ \left(\begin{array}{cc} a & b \\ -\bar{b} & \bar{a} \end{array} \right) \middle| a, b \in \mathbb{C}, a\bar{a} + b\bar{b} = 1 \right\} \cong S^3 \,.$$

   (b) Show that
   $$\mathrm{SO}(2) = \left\{ \left(\begin{array}{cc} c & d \\ -d & c \end{array} \right) \middle| c, d \in \mathbb{R}, x^2 + y^2 = 1 \right\} \cong S^1 \,.$$

   (c) Use the action $\rho : \mathrm{SU}(2) \times \mathbb{R}^3 \to \mathbb{R}^3$ defined by
   $$\rho(g,x) = x' \,, \quad g \left(\begin{array}{cc} ix_1 & -x_2 + ix_3 \\ x_2 + ix_3 & -ix_1 \end{array} \right) g^{-1} = \left(\begin{array}{cc} ix'_1 & -x'_2 + ix'_3 \\ x'_2 + ix'_3 & -ix'_1 \end{array} \right)$$
   to show that $\mathrm{SU}(2)/\mathrm{SO}(2)$ is diffeomorphic to S^2.

   (d) Calculate the fundamental vector field $\tilde{X}$ of
   $$X = \left(\begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right) \in \mathfrak{so}(2)$$
   on $\mathrm{SU}(2)$.

3. **Tangent bundle**
   Let M be a manifold of dimension $\dim M = n$ and $\mathrm{GL}(M)$ its general linear frame bundle. Consider the left action of $G = GL(n)$ on $F = \mathbb{R}^n$.

   (a) Show that the associated bundle $\mathrm{GL}(M) \times_\rho \mathbb{R}^n$ is diffeomorphic to the tangent bundle TM.

   (b) Show that there is a one-to-one correspondence between vector fields $X \in \mathrm{Vect}(M)$ on M and equivariant maps $\phi \in C^\infty_{\mathrm{GL}(n)}(\mathrm{GL}(M), \mathbb{R}^n)$.