Differential geometry for physicists - Assignment 12

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1. **Commutator of Lie derivatives**
   Consider the commutator rule for Lie derivatives:

   $$\mathcal{L}_{[X,Y]}T = \mathcal{L}_X\mathcal{L}_Y T - \mathcal{L}_Y\mathcal{L}_X T\,.$$

   (a) Prove this rule in the cases that T is a real function $f \in C^\infty(M,\mathbb{R})$, a vector field $X \in \mathrm{Vect}(M)$ and a one-form $\omega \in \Omega^1(M)$. Perform these proofs both in coordinates (x^a) on M and without coordinates, using only the expressions for the Lie derivatives of functions, vector fields and differential forms given in the lecture.

   (b) Prove that if this rule holds for the Lie derivatives of tensor fields S and T, then it holds also for $S \otimes T$. Choose whether you use coordinates or not in this proof.

   (c) What can you conclude from the result of both proofs together?

2. **Cartan's magic formula**
   Consider the Lie derivative of differential forms given by

   $$\mathcal{L}_X\omega = \iota_X(d\omega) + d(\iota_X\omega)\,.$$

   (a) Prove this formula for one-forms ω.

   (b) Prove that if this formula holds for the Lie derivatives of differential forms ω and σ, then it holds also for $\omega \wedge \sigma$.

   (c) What can you conclude from the result of both proofs together?