Differential geometry for physicists - Assignment 11

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1. **Double cover of the rotation group**
   Consider the special unitary group SU(2) of complex 2×2 matrices with determinant 1 such that $AA^\dagger = \mathbb{1}$.

   (a) Show that every element $g \in \mathrm{SU}(2)$ can uniquely be written in the form

   $$g = \begin{pmatrix} g_1 + ig_2 & g_3 + ig_4 \\ -g_3 + ig_4 & g_1 - ig_2 \end{pmatrix}$$

   with $g_i \in \mathbb{R}$ and $g_1^2 + g_2^2 + g_3^2 + g_4^2 = 1$, so that SU(2) is diffeomorphic to S^3.

   (b) Show that for $g \in \mathrm{SU}(2)$ and $x \in \mathbb{R}^3$ there exists $x' \in \mathbb{R}^3$ such that

   $$g \begin{pmatrix} ix_1 & -x_2 + ix_3 \\ x_2 + ix_3 & -ix_1 \end{pmatrix} g^{-1} = \begin{pmatrix} ix'_1 & -x'_2 + ix'_3 \\ x'_2 + ix'_3 & -ix'_1 \end{pmatrix} .$$

   (c) Determine the 3×3 matrix $\varphi(g)$ for which holds

   $$\varphi(g) \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x'_1 \\ x'_2 \\ x'_3 \end{pmatrix} ,$$

   where g, x, x' are as in the previous part of this exercise.

   (d) Show that $\varphi(g) \in \mathrm{SO}(3)$.

   (e) Show that $\varphi : \mathrm{SU}(2) \to \mathrm{SO}(3)$ is a Lie group homomorphism. What is $\varphi^{-1}(\mathbb{1}) \subset \mathrm{SU}(2)$ (the *kernel* of φ)?

2. **Orbit and stabilizer**
   Let $\phi : G \times M \to M$ be the left action of a Lie group G on a manifold M and $x, y \in M$ such that they have the same orbit $\mathcal{O}_x = \mathcal{O}_y$ under this action. Show that their stabilizers G_x and G_y are isomorphic and construct an isomorphism $\varphi : G_x \to G_y$.