Differential geometry for physicists - Assignment 10

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1. **First order Lagrangian of a point mass on a metric manifold with potential** Consider the example given in the previous lecture. Let $M = \mathbb{R}$ and Q a manifold of dimension n. Let $E = \mathbb{R} \times Q$ be the trivial fiber bundle with projection $\pi : \mathbb{R} \times Q \to \mathbb{R}$ onto the first factor. Sections of this bundle are uniquely expressed by maps $\gamma \in C^\infty(\mathbb{R}, Q)$, i.e., by curves on Q. We use the one-dimensional Euclidean coordinate t on $\mathbb{R}$ and arbitrary coordinates (q^a) on Q, so that we have coordinates (t, q^a) on $\mathbb{R} \times Q$. From these coordinates we derive the coordinates $(t, q^a_{(0)}, q^a_{(1)}, \ldots)$ on $J^\infty(E) \cong \mathbb{R} \times TQ$. For brevity, we write $q^a_{(0)} = q^a$, $q^a_{(1)} = \dot{q}^a$, $q^a_{(2)} = \ddot{q}^a$ etc.

   Let further $g \in \Gamma(T^0_2 Q)$ be a non-degenerate, positive definite, symmetric tensor field of type $(0, 2)$ (the *metric*) and $V \in C^\infty(Q, \mathbb{R})$ (the *potential*). Consider the Lagrangian given by

$$L(t, q, \dot{q}, \ldots) = \left(\frac{1}{2} g_{ab}(q) \dot{q}^a \dot{q}^b - V(q)\right) dt \in \Omega^{1,0}(J^\infty(E))\,. \tag{0.1}$$

   (a) Calculate the vertical derivative $d_V L$.

   (b) Calculate the projection $\mathcal{E}L = \varrho(d_V L)$ onto $\mathcal{F}^1(J^\infty(E))$.

   (c) Show that $\mathcal{E}L$ is of the form $\omega_a \theta^a \wedge dt$, where $\theta^a = \theta^a_{(0)}$.

   (d) Let $\gamma \in C^\infty(\mathbb{R}, Q)$ be a curve, which determines a section $\sigma = (\mathrm{id}_{\mathbb{R}}, \gamma) : \mathbb{R} \to \mathbb{R} \times Q$, and write it in coordinates in the form $t \mapsto q^a(t)$. Derive the Euler-Lagrange equations for such curves.