Differential geometry for physicists - Assignment 6

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1. **Pullback to the sphere**
Consider a chart of the sphere S^2 corresponding to the usual latitude / longitude coordinates $\theta \in (-\pi/2, \pi/2)$ and $\phi \in (0, 2\pi)$. Use this chart to define the map

$$\varphi : S^2 \to \mathbb{R}^3, (\theta, \phi) \mapsto (\cos\theta\cos\phi, \cos\theta\sin\phi, \sin\theta)\,.$$

   (a) Let
   $$v = v^\theta \partial_\theta + v^\phi \partial_\phi \in T_{(\theta,\phi)}S^2$$
   be a tangent vector and
   $$w = w^x \partial_x + w^y \partial_y + w^z \partial_z = \varphi_*(v) \in T_{\varphi(\theta,\phi)}\mathbb{R}^3\,.$$
   Calculate the components of w as functions of the components of v.

   (b) Calculate the pullback $\varphi^*(g)$ of the tensor field
   $$g = dx \otimes dx + dy \otimes dy + dz \otimes dz\,.$$

   (c) Calculate the pullback $\varphi^*(\omega)$ of the 1-form
   $$\omega = x\,dx + y\,dy + z\,dz\,.$$

2. **Pullback of functions and 1-forms**
Let M and N be manifolds and $\varphi : M \to N$ a smooth map. Show (without using coordinates) that the pullback of a real function $f \in C^\infty(N, \mathbb{R})$ and its total derivative $df \in \Omega^1(N)$ satisfy $\varphi^*(df) = d(\varphi^*(f))$. Use the fact that a function $g \in C^\infty(M, \mathbb{R})$ and a vector $v \in T_xM$ satisfy $v(g) = \langle v, dg(x)\rangle$.

3. **Pullback of a volume form along a diffeomorphism**
Let M and N be manifolds of dimension n and $\varphi : M \to N$ a diffeomorphism. Use coordinates (x^a) on M and (y^a) on N and let
$$\omega = w\,dy^1 \wedge \ldots \wedge dy^n \in \Omega^n(N)$$
be a volume form on N. Show that
$$\varphi^*(\omega) = w\,\det\left(\frac{\partial y^a}{\partial x^b}\right) dx^1 \wedge \ldots \wedge dx^n \in \Omega^n(M)\,.$$