Differential geometry for physicists - Assignment 5

Manuel Hohmann

24. March 2015

1. **Differential forms on a sphere**
   Consider a chart of the sphere S^2 corresponding to the usual latitude / longitude coordinates $\theta \in (-\pi/2, \pi/2)$ and $\varphi \in (0, 2\pi)$. Within this chart, we define the functions

   $$f(\theta, \varphi) = \sin\theta\,, \quad g(\theta, \varphi) = \cos\theta\cos\varphi\,, \quad h(\theta, \varphi) = \cos\theta\sin\varphi\,,$$

   which can be uniquely smoothly extended to the complete sphere.

   (a) Calculate the 1-forms df, dg and dh.

   (b) Calculate the 2-forms $df \wedge dg$, $dg \wedge dh$ and $dh \wedge df$.

   (c) Show that there exists a 2-form ω such that $df \wedge dg = -h\omega$, $dg \wedge dh = -f\omega$ and $dh \wedge df = -g\omega$, and write ω using the coordinates defined above.

2. **Exterior derivative, interior product, Lie bracket**
   Let (x^a) be coordinates on a manifold M. Consider a 1-form $\omega = \omega_a dx^a$ and vector fields $X = X^a\partial_a$ and $Y = Y^a\partial_a$. Write the formula

   $$\iota_Y\iota_X d\omega = X(\iota_Y\omega) - Y(\iota_X\omega) - \iota_{[X,Y]}\omega$$

   using these coordinates. What do you see from this result?