Differential geometry for physicists - Assignment 4

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1. **Covector and tensor fields on a sphere**
   Consider a chart of the sphere S^2 corresponding to the usual latitude / longitude coordinates $\theta \in (-\pi/2, \pi/2)$ and $\varphi \in (0, 2\pi)$. Within this chart, we define the function and vector fields

   $$X(\theta, \varphi) = \cos\theta \sin\varphi \partial_\varphi \,, \quad Y(\theta, \varphi) = \cos\theta \cos\varphi \partial_\theta \,, \quad f(\theta, \varphi) = \cos\theta \cos\varphi \,,$$

   which can be uniquely smoothly extended to the complete sphere.

   (a) Calculate the total differential df.

   (b) Calculate the tensor fields $X \otimes df$, $Y \otimes df$, $X \otimes Y$ and $Y \otimes X$.

   (c) Calculate the contractions $\operatorname{tr}_1^1(X \otimes df)$ and $\operatorname{tr}_1^1(Y \otimes df)$. Compare this result with your result from the previous assignment.

2. **Schwarzschild metric**
   Consider the tensor field

   $$g = -\left(1 - \frac{r_0}{r}\right) dt \otimes dt + \left(1 - \frac{r_0}{r}\right)^{-1} dr \otimes dr + r^2 \left(d\theta \otimes d\theta + \sin^2\theta d\phi \otimes d\phi\right) \in \Gamma(T_2^0 M)$$

   in coordinates (t, r, θ, ϕ) on $M = \mathbb{R}^4 \backslash \{0\}$. This tensor field is called the *Schwarzschild metric*. Further define

   $$g(X, Y) = \operatorname{tr}_1^1 \operatorname{tr}_2^2(X \otimes Y \otimes g) = X^a Y^b g_{ab}$$

   for $X, Y \in \operatorname{Vect}(M)$. Consider further the vector field $X = X^t \partial_t + X^r \partial_r$.

   (a) Calculate $g(X, X)$.

   (b) Determine a solution for the components $X^t(r)$ and $X^r(r)$ (which do not depend on t, θ, ϕ) such that $g(X, X) = 0$ everywhere on M.

   (c) Keep $X^r(r) = 0$ everywhere and $X^t(r) > 0$. In which regions is $g(X, X) < 0$?

   (This tensor field describes the geometry of a black hole. In the second exercise you determine the directions of light propagation, and in the third exercise the places where an observer can rest without moving, i.e., the places where he can still overcome the gravity of the black hole.)