Differential geometry for physicists - Assignment 3

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1. **Tangent bundle lift**
   Let M be a manifold and $\gamma \in C^\infty(\mathbb{R}, M)$ a curve on M. Define a curve $\dot\gamma : \mathbb{R} \to TM, t \mapsto \dot\gamma(t)$ (the tangent bundle lift of γ), which assigns to $t \in \mathbb{R}$ the tangent vector $\dot\gamma(t) \in T_{\gamma(t)}M$. Show that $\dot\gamma \in C^\infty(\mathbb{R}, TM)$, i.e., show that $\dot\gamma$ is a smooth curve on the tangent bundle.

2. **Vector fields and functions on a sphere**
   Consider a chart of the sphere S^2 corresponding to the usual latitude / longitude coordinates $\theta \in (-\pi/2, \pi/2)$ and $\varphi \in (0, 2\pi)$. Within this chart, we define the function and vector fields

   $$X(\theta,\varphi) = \cos\theta \sin\varphi \partial_\varphi\,, \quad Y(\theta,\varphi) = \cos\theta\cos\varphi\partial_\theta\,, \quad f(\theta,\varphi) = \cos\theta\cos\varphi\,,$$

   which can be uniquely smoothly extended to the complete sphere.

   (a) Calculate the commutator $[X, Y]$.

   (b) Calculate Xf, Yf and $[X, Y]f$.

3. **Point on a rolling wheel**
   Consider the motion of a rolling wheel of radius R in $\mathbb{R}^2$. The wheel is rotating with angular velocity ω, such that its center follows the curve given by $(x, y) = (R\omega t, 0)$. At $t = 0$, when the center of the wheel crosses the y-axis, mark a point along the y-axis with distance d over the center of the wheel, i.e., the point $(x, y) = (0, d)$. Let this point be fixed to the wheel, so that it keeps its distance to the center of the wheel and follows its rotation.

   (a) Describe the motion of the point by a curve γ. Is this a smooth curve $\gamma \in C^\infty(\mathbb{R}, \mathbb{R}^2)$?

   (b) Calculate $\dot\gamma(t)$.

   (c) Are there values of t for which $\dot\gamma(t) = 0$? How does this depend on the values of d, R, ω?