Selected Topics in the Theories of Gravity

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1 Tensor densities

We know that tensors are quantities A whose components change under a coordinate transformation $x^\mu \to x'^\mu$ according to the transformation law

$$A'^{\mu_1\cdots\mu_r}{}_{\nu_1\ldots\nu_s} = A^{\rho_1\cdots\rho_r}{}_{\sigma_1\ldots\sigma_s}\frac{\partial x'^{\mu_1}}{\partial x^{\rho_1}}\cdots\frac{\partial x'^{\mu_r}}{\partial x^{\rho_r}}\frac{\partial x^{\sigma_1}}{\partial x'^{\nu_1}}\cdots\frac{\partial x^{\sigma_s}}{\partial x'^{\nu_s}}\,. \tag{1.1}$$

We further know that the covariant derivative

$$\begin{aligned}\nabla_\rho A^{\mu_1\cdots\mu_r}{}_{\nu_1\ldots\nu_s} &= \partial_\rho A^{\mu_1\cdots\mu_r}{}_{\nu_1\ldots\nu_s}\\ &\quad + \Gamma^{\mu_1}{}_{\rho\sigma}A^{\sigma\cdots\mu_r}{}_{\nu_1\ldots\nu_s} + \ldots + \Gamma^{\mu_r}{}_{\rho\sigma}A^{\mu_1\cdots\sigma}{}_{\nu_1\ldots\nu_s}\\ &\quad - \Gamma^{\sigma}{}_{\rho\nu_1}A^{\mu_1\cdots\mu_r}{}_{\sigma\ldots\nu_s} - \ldots - \Gamma^{\sigma}{}_{\rho\nu_s}A^{\mu_1\cdots\mu_r}{}_{\nu_1\ldots\sigma}\end{aligned} \tag{1.2}$$

of a tensor A is again a tensor, provided that the connection coefficients satisfy the transformation law

$$\Gamma'^{\mu}{}_{\nu\sigma} = \Gamma^{\rho}{}_{\pi\tau}\frac{\partial x'^\mu}{\partial x^\rho}\frac{\partial x^\pi}{\partial x'^\nu}\frac{\partial x^\tau}{\partial x'^\sigma} + \frac{\partial x'^\mu}{\partial x^\kappa}\frac{\partial^2 x^\kappa}{\partial x'^\nu\partial x'^\sigma}\,. \tag{1.3}$$

However, sometimes we find quantities which are not tensors, but obey a different simple transformation law, such es $\sqrt{-g}$, which transforms according to

$$\sqrt{-g'} = \sqrt{-\det\left(g'^{\mu\nu}\right)} = \sqrt{-\det\left(g^{\rho\sigma}\frac{\partial x'^\mu}{\partial x_\rho}\frac{\partial x'^\nu}{\partial x_\sigma}\right)} = \sqrt{-g}\left(\det\frac{\partial x}{\partial x'}\right)\,. \tag{1.4}$$

We thus generalize the concept of tensors to tensor densities. A tensor density $\mathfrak{A}$ of weight $w \in \mathbb{R}$ is a quantity which transforms under coordinate changes according to

$$\mathfrak{A}'^{\mu_1\cdots\mu_r}{}_{\nu_1\ldots\nu_s} = \mathfrak{A}^{\rho_1\cdots\rho_r}{}_{\sigma_1\ldots\sigma_s}\frac{\partial x'^{\mu_1}}{\partial x^{\rho_1}}\cdots\frac{\partial x'^{\mu_r}}{\partial x^{\rho_r}}\frac{\partial x^{\sigma_1}}{\partial x'^{\nu_1}}\cdots\frac{\partial x^{\sigma_s}}{\partial x'^{\nu_s}}\left(\det\frac{\partial x}{\partial x'}\right)^w\,. \tag{1.5}$$

The covariant derivative of a tensor density of weight w is again a tensor density of weight w if it is given by

$$\begin{aligned}\nabla_\rho \mathfrak{A}^{\mu_1\cdots\mu_r}{}_{\nu_1\ldots\nu_s} &= \partial_\rho \mathfrak{A}^{\mu_1\cdots\mu_r}{}_{\nu_1\ldots\nu_s} - w\Gamma^{\sigma}{}_{\sigma\rho}\mathfrak{A}^{\mu_1\cdots\mu_r}{}_{\nu_1\ldots\nu_s}\\ &\quad + \Gamma^{\mu_1}{}_{\rho\sigma}\mathfrak{A}^{\sigma\cdots\mu_r}{}_{\nu_1\ldots\nu_s} + \ldots + \Gamma^{\mu_r}{}_{\rho\sigma}\mathfrak{A}^{\mu_1\cdots\sigma}{}_{\nu_1\ldots\nu_s}\\ &\quad - \Gamma^{\sigma}{}_{\rho\nu_1}\mathfrak{A}^{\mu_1\cdots\mu_r}{}_{\sigma\ldots\nu_s} - \ldots - \Gamma^{\sigma}{}_{\rho\nu_s}\mathfrak{A}^{\mu_1\cdots\mu_r}{}_{\nu_1\ldots\sigma}\,.\end{aligned} \tag{1.6}$$

Note that this holds for any covariant derivative, not only for the Levi-Civita connection. Some properties of tensor densities:

- A tensor density of weight 0 is a tensor.
- The product of two tensor densities $\mathfrak{A}, \mathfrak{B}$ of weights w and w' is a tensor density of weight $w + w'$.
- The Leibnitz rule $\nabla_\mu(\mathfrak{A}\mathfrak{B}) = (\nabla_\mu\mathfrak{A})\mathfrak{B} + \mathfrak{A}(\nabla_\mu\mathfrak{B})$ holds also for tensor densities.

A useful example for a tensor density of weight 1 is $\sqrt{-g}$.

2 Palatini method of variation

In the first lecture we have seen how to derive the Einstein equations from the Einstein-Hilbert action

$$S_G[g] = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\,(R[g] - 2\Lambda) \tag{2.1}$$

by variation with respect to the metric. In this calculation we took $R[g]$ to be the Ricci tensor which is calculated from the Levi-Civita connection $\Gamma^{\mu}{}_{\nu\sigma}$ of $g_{\mu\nu}$. We now follow a different approach, in which $\Gamma^{\mu}{}_{\nu\sigma}$ is not given by the Levi-Civita connection, but an arbitrary torsion-free (i.e., symmetric in its lower indices) connection. The Riemann tensor

$$R^{\mu}{}_{\nu\rho\sigma}[\Gamma] = \partial_{\rho}\Gamma^{\mu}{}_{\nu\sigma} - \partial_{\sigma}\Gamma^{\mu}{}_{\nu\rho} + \Gamma^{\mu}{}_{\rho\tau}\Gamma^{\tau}{}_{\nu\sigma} - \Gamma^{\mu}{}_{\sigma\tau}\Gamma^{\tau}{}_{\nu\rho} \tag{2.2}$$

then depends only on the connection and not on the metric. The same holds for the Ricci tensor

$$R_{\mu\nu}[\Gamma] = R^{\rho}{}_{\mu\rho\nu}[\Gamma]\,. \tag{2.3}$$

We then write the Einstein-Hilbert action in the form

$$S_G[g,\Gamma] = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\,(g^{\mu\nu}R_{\mu\nu}[\Gamma] - 2\Lambda) \tag{2.4}$$

with independent variables $g_{\mu\nu}$ and $\Gamma^{\mu}{}_{\nu\sigma}$. Consequently we must vary this action with respect to both variables independently. We have already seen in the first lecture that variation of the terms $\sqrt{-g}$ and $g^{\mu\nu}$ yields the expression

$$\delta_g S_G = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\left[\frac{1}{2}g^{\mu\nu}(g^{\rho\sigma}R_{\rho\sigma}[\Gamma] - 2\Lambda) - g^{\mu\rho}g^{\nu\sigma}R_{\rho\sigma}[\Gamma]\right]\delta g_{\mu\nu}\,, \tag{2.5}$$

where we now simply replaced $R_{\mu\nu}[g]$ with $R_{\mu\nu}[\Gamma]$. From this expression, together with some matter action, we read off the Einstein equations

$$R_{\mu\nu}[\Gamma] - \frac{1}{2}g_{\mu\nu}g^{\rho\sigma}R_{\rho\sigma}[\Gamma] + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}\,. \tag{2.6}$$

We now come to the variation of the action (2.4) with respect to the connection coefficients, which takes the form

$$\delta_\Gamma S_G = \frac{1}{16\pi G}\int d^4x\sqrt{-g}g^{\mu\nu}\delta_\Gamma R_{\mu\nu}[\Gamma]\,. \tag{2.7}$$

Recall also from the first lecture that the variation of the Ricci tensor with respect to the connection yields

$$\delta_\Gamma R_{\mu\nu}[\Gamma] = \nabla_{\rho}\delta\Gamma^{\rho}{}_{\mu\nu} - \nabla_{\nu}\delta\Gamma^{\rho}{}_{\mu\rho}\,. \tag{2.8}$$

We further introduce the tensor density

$$\mathfrak{g}^{\mu\nu} = \sqrt{-g}g^{\mu\nu}\,, \tag{2.9}$$

so that the variation of the action reads

$$\delta_\Gamma S_G = \frac{1}{16\pi G}\int d^4x\,\mathfrak{g}^{\mu\nu}\left(\nabla_{\rho}\delta\Gamma^{\rho}{}_{\mu\nu} - \nabla_{\nu}\delta\Gamma^{\rho}{}_{\mu\rho}\right)\,. \tag{2.10}$$

We now apply the Leibniz rule for tensor densities and reorder indices to obtain

$$\delta_\Gamma S_G = \frac{1}{16\pi G}\int d^4x\left[\nabla_{\rho}\left(\mathfrak{g}^{\mu\nu}\delta\Gamma^{\rho}{}_{\mu\nu} - \mathfrak{g}^{\mu\rho}\delta\Gamma^{\nu}{}_{\mu\nu}\right) - \left(\nabla_{\rho}\mathfrak{g}^{\mu\nu}\delta\Gamma^{\rho}{}_{\mu\nu} - \nabla_{\nu}\mathfrak{g}^{\mu\nu}\delta\Gamma^{\rho}{}_{\mu\rho}\right)\right]\,. \tag{2.11}$$

Here the first term is the covariant divergence of a vector density $\mathfrak{A}^\mu$ of weight 1. For this expression we find

$$\nabla_\mu \mathfrak{A}^\mu = \partial_\mu \mathfrak{A}^\mu - \Gamma^\nu{}_{\nu\mu} \mathfrak{A}^\mu + \Gamma^\mu{}_{\mu\nu} \mathfrak{A}^\nu = \partial_\mu \mathfrak{A}^\mu \,. \tag{2.12}$$

Since this is only a partial derivative, this term does not contribute to the integral. The remaining terms take the form

$$\delta_\Gamma S_G = \frac{1}{16\pi G} \int d^4x \left(\nabla_\sigma \mathfrak{g}^{\mu\sigma} \delta^\nu_\rho - \nabla_\rho \mathfrak{g}^{\mu\nu} \right) \delta\Gamma^\rho{}_{\mu\nu} \,. \tag{2.13}$$

This must vanish for arbitrary variations $\delta\Gamma^\rho{}_{\mu\nu}$. Taking into account the symmetry in the lower two indices we can thus read off the equation

$$\frac{1}{2} \nabla_\sigma \mathfrak{g}^{\mu\sigma} \delta^\nu_\rho + \frac{1}{2} \nabla_\sigma \mathfrak{g}^{\nu\sigma} \delta^\mu_\rho - \nabla_\rho \mathfrak{g}^{\mu\nu} = 0 \tag{2.14}$$

By contracting the indices ρ and ν we obtain the equation

$$\nabla_\nu \mathfrak{g}^{\mu\nu} = 0 \,. \tag{2.15}$$

Inserting this again we then find the equation

$$\nabla_\rho \mathfrak{g}^{\mu\nu} = 0 \,. \tag{2.16}$$

This means that $\mathfrak{g}^{\mu\nu}$ must be covariantly constant. From this finally follows

$$\nabla_\rho g_{\mu\nu} = 0 \,, \tag{2.17}$$

so that ∇ must indeed be the unique metric-compatible torsion-free connection, which is the Levi-Civita connection.