Teleparallel axions and cosmology

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Manuel Hohmann, Christian Pfeifer

Laboratory of Theoretical Physics - Institute of Physics - University of Tartu
Center of Excellence “The Dark Side of the Universe”

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Presentation for the virtual axion institute

Outline

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Cosmological field equations - vector branch

- Tetrad field equations for the vector branch $\overset{\mathcal{V}}{\theta}{}^a{}_\mu$:

$$-9c_v\left(H^2 + \frac{k}{A^2}\right) - \mathcal{Z}\dot{\phi}^2 - 2\kappa^2\mathcal{V} = 2\kappa^2\rho\,, \tag{1a}$$

$$3c_v\left(2\dot{H} + 3H^2 + \frac{k}{A^2}\right) - \mathcal{Z}\dot{\phi}^2 + 2\kappa^2\mathcal{V} = 2\kappa^2 p\,. \tag{1b}$$

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- Scalar field equation for the vector branch $\overset{\mathcal{V}}{\theta}{}^a{}_\mu$:

$$2\mathcal{Z}\left(\ddot{\phi}+3H\dot{\phi}\right)+\mathcal{Z}_\phi\dot{\phi}^2+2\kappa^2\mathcal{V}_\phi=0\,. \tag{2}$$

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⇒ Pseudo-scalar field becomes minimally coupled.

Cosmological field equations - axial branch

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$$-9c_v H^2 + 4c_a \frac{k}{A^2} - \mathcal{Z}\dot{\phi}^2 - 2\kappa^2\mathcal{V} = 2\kappa^2\rho\,, \tag{3a}$$

$$3c_v\left(2\dot{H} + 3H^2\right) - \frac{4}{3}c_a\frac{k}{A^2} + 2\frac{bu\dot{\phi}}{A} - \mathcal{Z}\dot{\phi}^2 + 2\kappa^2\mathcal{V} = 2\kappa^2 p\,, \tag{3b}$$

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$\Rightarrow$ Pseudo-scalar field obtains additional non-minimal coupling to gravity.

Cosmological axions as effective fluid

- Restrict constant parameters to general relativity values:

$$c_a = \frac{3}{2}\,, \quad c_v = -\frac{2}{3}\,, \quad c_t = \frac{2}{3}\,. \tag{5}$$

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⇒ Effective fluid form of cosmological field equations:

$$3\left(H^2 + \frac{k}{A^2}\right) = \kappa^2(\rho + \rho_\phi)\,, \tag{6a}$$

$$-\left(2\dot{H} + 3H^2 + \frac{k}{A^2}\right) = \kappa^2(p + p_\phi)\,. \tag{6b}$$

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- Effective energy density and pressure of axion field:

$$\rho_\phi = \frac{1}{2\kappa^2}\mathcal{Z}\dot{\phi}^2 + \mathcal{V}\,, \quad p_\phi = \frac{1}{2\kappa^2}\mathcal{Z}\dot{\phi}^2 - \mathcal{V} - \begin{cases} \frac{bu\dot{\phi}}{\kappa^2 A} & \text{for the axial tetrad } \overset{\mathcal{A}}{\theta}{}^a{}_\mu\,, \\ 0 & \text{for the vector tetrad } \overset{\mathcal{V}}{\theta}{}^a{}_\mu\,. \end{cases} \tag{7}$$

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⇒ Additional parity-odd pressure contribution for axial tetrad only.

Dynamical system analysis

- Introduce phase space coordinates (α, β) for vacuum case $\rho = p = 0$:

$$\dot{\phi} = \sqrt{2\kappa^2 \frac{\mathcal{V}}{\mathcal{Z}}} \frac{\alpha}{\sqrt{1-\alpha^2}}\,, \quad H = \sqrt{\frac{\kappa^2}{3-3\alpha^2}\mathcal{V}}\cos\beta\,, \quad A = u\left(\sqrt{\frac{\kappa^2}{3-3\alpha^2}\mathcal{V}}\sin\beta\right)^{-1}\,. \tag{8}$$

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$\Rightarrow$ Autonomous dynamical system:

$$\frac{\dot{\alpha}}{\sqrt{\Lambda}} = \left(\frac{b}{\sqrt{2}} \sin\beta - \sqrt{3}\alpha\right) \sqrt{1-\alpha^2} \cos\beta, \tag{9a}$$

$$\frac{\dot{\beta}}{\sqrt{\Lambda}} = \left(\frac{3\alpha^2 - 1}{\sqrt{3}} - \frac{b}{\sqrt{2}}\alpha \sin\beta\right) \frac{\sin\beta}{\sqrt{1-\alpha^2}}. \tag{9b}$$

Cosmological phase space

$\diamond, \blacklozenge$ Big Bang / Big Crunch singularities:

$$\alpha = \pm 1\,. \tag{10}$$

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$\star$ Saddle points:

$$\alpha = \frac{b \pm \sqrt{b^2 + 8}}{2\sqrt{6}}\,\mathrm{sgn}\,u\,, \quad \cos\beta = 0\,. \tag{12}$$

Figure 1: minimal coupling $b = 0$.

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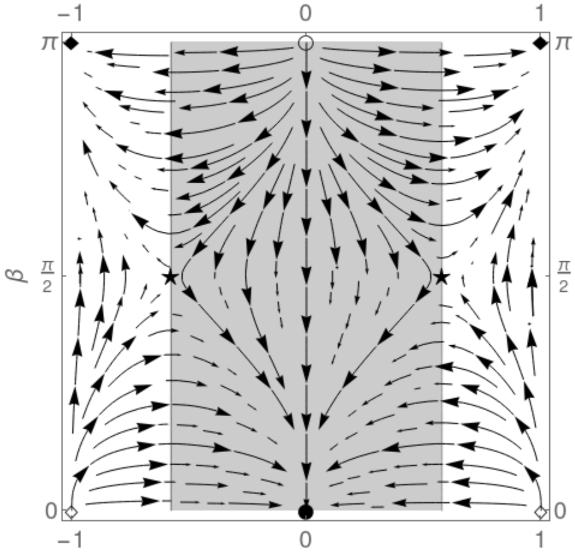

Cosmological phase space

Figure 2: weak coupling $0 < b < \sqrt{8/3}$.

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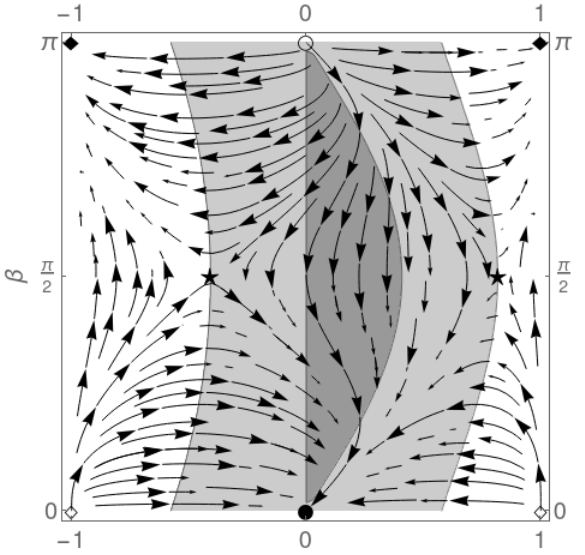

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Figure 3: critical coupling $b = \sqrt{8/3}$.

- ◇, ◆ Big Bang / Big Crunch singularities:

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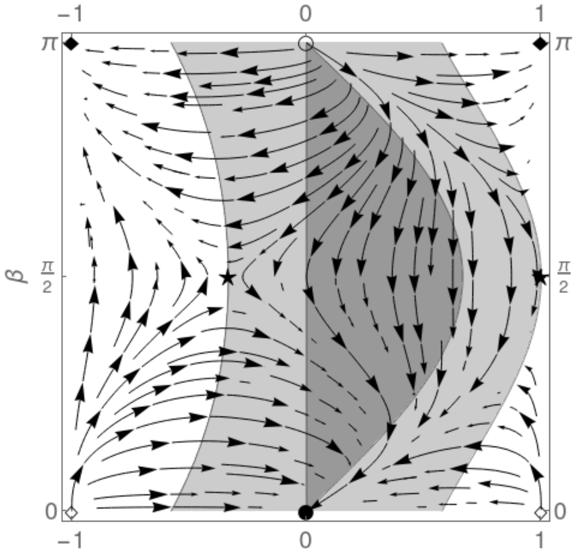

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Figure 4: strong coupling $b > \sqrt{8/3}$.

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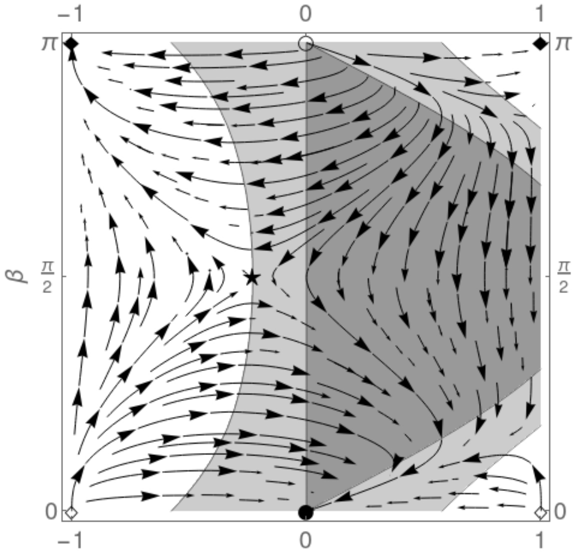

Outline

Multiple axions

- Replace single axion by a multiplet $\phi = (\phi^A, A = 1, \ldots, n)$:
  - For each axion, include pair b_A and $\tilde{b}_A$ of coupling parameters.
  - Parameter functions $\mathcal{V}$ and $\mathcal{Z}$ depend on all pseudo-scalar fields ϕ^A.
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⇒ Generalized action for multi-axion teleparallel gravity:

$$S_{\mathrm{g}}[\theta, \omega, \phi] = \frac{1}{2\kappa^2} \int \mathrm{d}^4x\, \theta \Big[c_v T_{\mathrm{vec}} + c_a T_{\mathrm{axi}} + c_t T_{\mathrm{ten}} + b_A \phi^A P + \tilde{b}_A \phi^A \tilde{P} + \mathcal{Z}_{AB}(\phi) g^{\mu\nu} \partial_\mu \phi^A \partial_\nu \phi^B + 2\kappa^2 \mathcal{V}(\phi) \Big] . \quad (13)$$

Dynamical couplings

- Replace constant parameters in the action by dynamical coupling functions of ϕ:
  - Axion couplings b and $\tilde{b}$ replaced by functions $\mathcal{B}(\phi)$ and $\tilde{\mathcal{B}}(\phi)$.
  - Even terms governed by $c_{a,t,v}$ receive non-minimal coupling through $\mathcal{C}_{a,t,v}(\phi)$.

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⇒ Generalized action with dynamical, non-minimal couplings:

$$S_{\mathrm{g}}[\theta,\omega,\phi] = \int \mathrm{d}^4x\,\theta\,\frac{1}{2\kappa^2}\Big[\mathcal{C}_v(\phi)\,T_{\mathrm{vec}} + \mathcal{C}_a(\phi)\,T_{\mathrm{axi}} + \mathcal{C}_t(\phi)\,T_{\mathrm{ten}} + \mathcal{B}(\phi)\phi P + \tilde{\mathcal{B}}(\phi)\phi\tilde{P} + \mathcal{Z}(\phi)g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + 2\kappa^2\mathcal{V}(\phi)\Big]\,. \quad (14)$$

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⇒ Further generalization to multiple axion fields:

$$S_{\mathrm{g}}[\theta,\omega,\phi] = \frac{1}{2\kappa^2}\int d^4x\,\theta\Big[\mathcal{C}_v(\phi)\,T_{\mathrm{vec}} + \mathcal{C}_a(\phi)\,T_{\mathrm{axi}} + \mathcal{C}_t(\phi)\,T_{\mathrm{ten}} + \mathcal{B}_A(\phi)\phi^A P + \tilde{\mathcal{B}}_A(\phi)\phi^A\tilde{P} + \mathcal{Z}_{AB}(\phi)g^{\mu\nu}\partial_\mu\phi^A\partial_\nu\phi^B + 2\kappa^2\mathcal{V}(\phi)\Big]\,. \quad (15)$$

Symmetric teleparallel axions

- Symmetric teleparallel gravity: [Nester, Yo '98; Beltrán Jiménez, Heisenberg, Koivisto '17/18]
  - Consider metric $g_{\mu\nu}$ and independent connection $\Gamma^{\mu}{}_{\nu\rho}$ as dynamical variables.
  - $\Gamma^{\mu}{}_{\nu\rho}$ required to have vanishing torsion, $T^{\mu}{}_{\nu\rho} = 0$, and curvature, $R^{\mu}{}_{\nu\rho\sigma} = 0$.
  - Gravitational interaction mediated by non-vanishing non-metricity $Q_{\rho\mu\nu} = \nabla_{\rho} g_{\mu\nu}$.

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- Scalar and pseudo-scalar invariants:
  - Five scalar invariants quadratic in non-metricity:

$$\mathcal{Q}_1 = Q^{\rho\mu\nu} Q_{\rho\mu\nu}\,, \;\; \mathcal{Q}_2 = Q^{\mu\nu\rho} Q_{\rho\mu\nu}\,, \;\; \mathcal{Q}_3 = Q^{\rho\mu}{}_{\mu} Q_{\rho\nu}{}^{\nu}\,, \;\; \mathcal{Q}_4 = Q^{\mu}{}_{\mu\rho} Q_{\nu}{}^{\nu\rho}\,, \;\; \mathcal{Q}_5 = Q^{\mu}{}_{\mu\rho} Q^{\rho\nu}{}_{\nu}\,. \tag{16}$$

  - One pseudo-scalar invariant:

$$\hat{\mathcal{Q}} = \epsilon^{\mu\nu\rho\sigma} Q_{\mu\nu\lambda} Q_{\rho\sigma}{}^{\lambda}\,. \tag{17}$$

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$\Rightarrow$ Action for symmetric teleparallel gravity with axion field:

$$S_{\mathrm{g}}[g, \Gamma, \phi] = \int d^4x \sqrt{-g} \frac{1}{2\kappa^2} \left[\sum_{i=1}^{5} c_i \mathcal{Q}_i + b\phi\hat{\mathcal{Q}} + \mathcal{Z}(\phi) g^{\mu\nu} \partial_{\mu}\phi \partial_{\nu}\phi + 2\kappa^2 \mathcal{V}(\phi) \right]. \quad (18)$$

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- May also be generalized to multiple axions and dynamical couplings.

General teleparallel axions

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- Further scalar and pseudo-scalar invariants combining torsion and non-metricity:
  - Three additional scalar invariants:

$$Q_{\mu\nu\rho}T^{\rho\mu\nu}\,, \quad Q^{\mu}{}_{\mu\rho}T_{\nu}{}^{\nu\rho}\,, \quad Q_{\rho\mu}{}^{\mu}T_{\nu}{}^{\nu\rho}\,. \tag{19}$$

  - Three additional pseudo-scalar invariants:

$$\epsilon^{\mu\nu\rho\sigma}Q_{\mu\nu}{}^{\tau}T_{(\tau\rho)\sigma}\,, \quad \epsilon^{\mu\nu\rho\sigma}Q^{\tau}{}_{\tau\mu}T_{\nu\rho\sigma}\,, \quad \epsilon^{\mu\nu\rho\sigma}Q_{\mu\tau}{}^{\tau}T_{\nu\rho\sigma}\,, \tag{20}$$

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  - Three additional pseudo-scalar invariants:

$$\epsilon^{\mu\nu\rho\sigma}Q_{\mu\nu}{}^{\tau}T_{(\tau\rho)\sigma}\,, \quad \epsilon^{\mu\nu\rho\sigma}Q^{\tau}{}_{\tau\mu}T_{\nu\rho\sigma}\,, \quad \epsilon^{\mu\nu\rho\sigma}Q_{\mu\tau}{}^{\tau}T_{\nu\rho\sigma}\,, \tag{20}$$

⇒ General teleparallel gravity allows 6 different terms to couple to axions.

Outline

Summary

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   - Dynamical fields are tetrad $\theta^a{}_\mu$ and flat, metric-compatible spin connection $\omega^a{}_{b\mu}$.
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4. Extensions and alternatives:
   - Possible to use non-metricity instead of or in addition to torsion.
   - Possible generalization with multiple axion fields or dynamical couplings.