Classification of cosmological tetrads and teleparallel geometries

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Outline

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Definition of metric-affine geometry

- Metric tensor $g_{\mu\nu}$:
  - Defines length of and angle between tangent vectors.
  - Defines length of curves and proper time.
  - Defines causality (spacelike and timelike directions).

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  - Defines parallel transport along arbitrary curves.
  - Defines autoparallel curves via parallel transport of tangent vector.

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  - Defines autoparallel curves via parallel transport of tangent vector.

! In general the connection is defined independently of the metric.

Properties of metric-affine geometry

- Three characteristic quantities:
  - Curvature:

$$R^{\mu}{}_{\nu\rho\sigma} = \partial_{\rho}\Gamma^{\mu}{}_{\nu\sigma} - \partial_{\sigma}\Gamma^{\mu}{}_{\nu\rho} + \Gamma^{\mu}{}_{\tau\rho}\Gamma^{\tau}{}_{\nu\sigma} - \Gamma^{\mu}{}_{\tau\sigma}\Gamma^{\tau}{}_{\nu\rho}\,. \tag{1}$$

  - Torsion:

$$T^{\mu}{}_{\nu\rho} = \Gamma^{\mu}{}_{\rho\nu} - \Gamma^{\mu}{}_{\nu\rho}\,. \tag{2}$$

  - Nonmetricity:

$$Q_{\mu\nu\rho} = \nabla_{\mu}g_{\nu\rho} = \partial_{\mu}g_{\nu\rho} - \Gamma^{\sigma}{}_{\nu\mu}g_{\sigma\rho} - \Gamma^{\sigma}{}_{\rho\mu}g_{\nu\sigma}\,. \tag{3}$$

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- Some special classes of connections used in gravity theory:
  - Levi-Civita connection: $T = Q = 0$.
  - Metric teleparallelism: $R = Q = 0$.
  - Symmetric teleparallelism: $R = T = 0$.

Decomposition of the connection

- Affine connection can be decomposed:

$$\Gamma^{\mu}{}_{\nu\rho} = \mathring{\Gamma}^{\mu}{}_{\nu\rho} + K^{\mu}{}_{\nu\rho} + L^{\mu}{}_{\nu\rho} \,. \tag{4}$$

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- Parts of the decomposition:
  - Levi-Civita connection of the metric:

$$\overset{\circ}{\Gamma}{}^{\mu}{}_{\nu\rho} = \frac{1}{2}g^{\mu\sigma}\left(\partial_{\nu}g_{\sigma\rho} + \partial_{\rho}g_{\nu\sigma} - \partial_{\sigma}g_{\nu\rho}\right)\,. \tag{5}$$

  - Contortion:

$$K^{\mu}{}_{\nu\rho} = \frac{1}{2}\left(T_{\nu}{}^{\mu}{}_{\rho} + T_{\rho}{}^{\mu}{}_{\nu} - T^{\mu}{}_{\nu\rho}\right)\,. \tag{6}$$

  - Disformation:

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- All three components depend on the metric.

Tetrad and spin connection formulation

- Metric teleparallelism conventionally formulated using:
  - Tetrad / coframe: $\theta^a = \theta^a{}_\mu \mathrm{d}x^\mu$ with inverse $e_a = e_a{}^\mu \partial_\mu$.
  - Spin connection: $\omega^a{}_b = \omega^a{}_{b\mu} \mathrm{d}x^\mu$.

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- Induced metric-affine geometry:
  - Metric:

$$g_{\mu\nu} = \eta_{ab} \theta^a{}_\mu \theta^b{}_\nu \,. \tag{8}$$

  - Affine connection:

$$\Gamma^\mu{}_{\nu\rho} = e_a{}^\mu \left(\partial_\rho \theta^a{}_\nu + \omega^a{}_{b\rho} \theta^b{}_\nu \right) \,. \tag{9}$$

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- Conditions on the spin connection:
  - Flatness $R = 0$:

$$\partial_\mu \omega^a{}_{b\nu} - \partial_\nu \omega^a{}_{b\mu} + \omega^a{}_{c\mu} \omega^c{}_{b\nu} - \omega^a{}_{c\nu} \omega^c{}_{b\mu} = 0 \,. \tag{10}$$

  - Metric compatibility $Q = 0$:

$$\eta_{ac} \omega^c{}_{b\mu} + \eta_{bc} \omega^c{}_{a\mu} = 0 \,. \tag{11}$$

Local Lorentz invariance

- Local Lorentz transformation of the tetrad only:

$$\theta^a{}_\mu \mapsto \theta'^a{}_\mu = \Lambda^a{}_b \theta^b{}_\mu \,. \tag{12}$$

  ✓ Metric is invariant: $g'_{\mu\nu} = g_{\mu\nu}$.
  ↯ Connection is not invariant: $\Gamma'^\mu{}_{\nu\rho} \neq \Gamma^\mu{}_{\nu\rho}$.

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⇒ Metric-affine geometry equivalently described by:
- Metric $g_{\mu\nu}$ and affine connection $\Gamma^\mu{}_{\nu\rho}$.
- Equivalence class of tetrad $\theta^a{}_\mu$ and spin connection $\omega^a{}_{b\mu}$.
- Equivalence defined with respect to local Lorentz transformations.

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  - Equivalence defined with respect to local Lorentz transformations.

- Teleparallel geometry admits Weitzenböck gauge: $\omega^a{}_{b\mu} \equiv 0$.

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Symmetry transformations of metric-affine geometry

- Finite spacetime transformation:
  - Action $\varphi : G \times M \to M$ of symmetry group G with $x' = \varphi_u(x)$.
  - Transformations of fundamental geometric objects:
    - ⋆ Metric:

$$(\varphi_u^* g)_{\mu\nu}(x) = g_{\tau\omega}(x') \frac{\partial x'^\tau}{\partial x^\mu} \frac{\partial x'^\omega}{\partial x^\nu} \,. \tag{14}$$

    - ⋆ Connection coefficients:

$$(\varphi_u^* \Gamma)^\mu{}_{\nu\rho}(x) = \Gamma^\sigma{}_{\tau\omega}(x') \frac{\partial x^\mu}{\partial x'^\sigma} \frac{\partial x'^\tau}{\partial x^\nu} \frac{\partial x'^\omega}{\partial x^\rho} + \frac{\partial x^\mu}{\partial x'^\sigma} \frac{\partial^2 x'^\sigma}{\partial x^\nu \partial x^\rho} \,. \tag{15}$$

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- Infinitesimal spacetime transformation:
  - Generating vector fields X_ξ on M with $\xi \in \mathfrak{g}$.
  - Lie derivatives of fundamental geometric objects are tensor fields:
    - ⋆ Metric:
$$(\mathcal{L}_{X_\xi} g)_{\mu\nu} = X_\xi^\rho \partial_\rho g_{\mu\nu} + \partial_\mu X_\xi^\rho g_{\rho\nu} + \partial_\nu X_\xi^\rho g_{\mu\rho} \,. \tag{16}$$
    - ⋆ Connection coefficients:
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Symmetry of tetrad and spin connection

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    - Spin connection:

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$$(\mathcal{L}_{X_\xi}\theta)^a{}_\mu = X_\xi^\nu\partial_\nu\theta^a{}_\mu + \partial_\mu X_\xi^\nu\theta^a{}_\nu = -\boldsymbol{\lambda}_\xi{}^a{}_b\theta^b{}_\mu\,. \tag{20}$$

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- Symmetry requires $\mathbf{\Lambda}_u(x) \in \mathrm{SO}(1,3)$ and $\boldsymbol{\lambda}_\xi(x) \in \mathfrak{so}(1,3)$.

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- Consider two elements $u, v \in G$ of the symmetry group.

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⇒ Consistency condition if tetrad is symmetric w.r.t. u and v:

$$(\boldsymbol{\Lambda}_{uv}^{-1})^a{}_b \theta^b{}_\mu = (\varphi_{uv}^* \theta)^a{}_\mu = (\varphi_v^* \varphi_u^* \theta)^a{}_\mu = (\boldsymbol{\Lambda}_v^{-1})^a{}_b (\boldsymbol{\Lambda}_u^{-1})^b{}_c \theta^c{}_\mu \quad (22)$$

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$\Rightarrow$ $\mathbf{\Lambda}$ must be local group homomorphism:

$$\mathbf{\Lambda}_{uv} = \mathbf{\Lambda}_u\mathbf{\Lambda}_v\,. \quad (23)$$

$\Rightarrow$ $\boldsymbol{\lambda}$ must be local Lie algebra homomorphism:

$$\boldsymbol{\lambda}_{[\xi,\zeta]} = [\boldsymbol{\lambda}_\xi, \boldsymbol{\lambda}_\zeta]\,. \quad (24)$$

Lorentz transformations and Weitzenböck gauge

- Recall Weitzenböck gauge in metric teleparallelism: $\omega^a{}_{b\mu} \equiv 0$.

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$\Rightarrow$ Symmetry condition for finite transformations:

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\begin{aligned}
0 &\equiv (\varphi_u^*\omega)^a{}_{b\mu}(x) = \omega^c{}_{d\nu}(x')\frac{\partial x'^\nu}{\partial x^\mu} \\
&= (\mathbf{\Lambda}_u^{-1})^a{}_c(x)\left[(\mathbf{\Lambda}_u)^d{}_b(x)\omega^c{}_{d\mu}(x) + \partial_\mu(\mathbf{\Lambda}_u)^c{}_b(x)\right] \\
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&= \partial_\mu\boldsymbol{\lambda}_\xi^a{}_b + \omega^a{}_{c\mu}\boldsymbol{\lambda}_\xi^c{}_b - \omega^c{}_{b\mu}\boldsymbol{\lambda}_\xi^a{}_c \\
&\equiv \partial_\mu\boldsymbol{\lambda}_\xi^a{}_b\,.
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\tag{26}
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⇒ Homomorphisms $\mathbf{\Lambda}$ and $\boldsymbol{\lambda}$ are constant in Weitzenböck gauge.

Outline

Generators of cosmological symmetry

- Use spherical coordinates t, r, ϑ, φ.

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- Generating vector fields:
  - Rotations:

$$R_1 = \sin\varphi\partial_\vartheta + \frac{\cos\varphi}{\tan\vartheta}\partial_\varphi\,, \tag{27a}$$

$$R_2 = -\cos\varphi\partial_\vartheta + \frac{\sin\varphi}{\tan\vartheta}\partial_\varphi\,, \tag{27b}$$

$$R_3 = -\partial_\varphi\,, \tag{27c}$$

  - Translations:

$$T_1 = \chi\sin\vartheta\cos\varphi\partial_r + \frac{\chi}{r}\cos\vartheta\cos\varphi\partial_\vartheta - \frac{\chi\sin\varphi}{r\sin\vartheta}\partial_\varphi\,, \tag{28a}$$

$$T_2 = \chi\sin\vartheta\sin\varphi\partial_r + \frac{\chi}{r}\cos\vartheta\sin\varphi\partial_\vartheta + \frac{\chi\cos\varphi}{r\sin\vartheta}\partial_\varphi\,, \tag{28b}$$

$$T_3 = \chi\cos\vartheta\partial_r - \frac{\chi}{r}\sin\vartheta\partial_\vartheta\,. \tag{28c}$$

Generators of cosmological symmetry

- Use spherical coordinates t, r, ϑ, φ.
- Generating vector fields:
  - Rotations:

$$R_1 = \sin\varphi\partial_\vartheta + \frac{\cos\varphi}{\tan\vartheta}\partial_\varphi\,, \tag{27a}$$

$$R_2 = -\cos\varphi\partial_\vartheta + \frac{\sin\varphi}{\tan\vartheta}\partial_\varphi\,, \tag{27b}$$

$$R_3 = -\partial_\varphi\,, \tag{27c}$$

  - Translations:

$$T_1 = \chi\sin\vartheta\cos\varphi\partial_r + \frac{\chi}{r}\cos\vartheta\cos\varphi\partial_\vartheta - \frac{\chi\sin\varphi}{r\sin\vartheta}\partial_\varphi\,, \tag{28a}$$

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$$T_3 = \chi\cos\vartheta\partial_r - \frac{\chi}{r}\sin\vartheta\partial_\vartheta\,. \tag{28c}$$

- Here $\chi = \sqrt{1-(ur)^2}$, and u can be real or imaginary.

Symmetry group and algebra

- Commutation relations of symmetry generators:

$$[R_i, R_j] = \epsilon_{ijk} R_k \,, \tag{29a}$$

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- Symmetry group depends on u:
    - $u^2 > 0$: positive spatial curvature, symmetry group SO(4).
    - $u^2 < 0$: negative spatial curvature, symmetry group $SO_0(3, 1)$.
    - $u^2 = 0$: vanishing spatial curvature, symmetry group ISO(3).

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- Helpful to compare with Lorentz algebra:

$$[J_i, J_j] = \epsilon_{ijk} J_k \,, \tag{30a}$$
$$[K_i, K_j] = -\epsilon_{ijk} J_k \,, \tag{30b}$$
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Cosmologically symmetric Riemannian geometry

- Most general metric with cosmological symmetry:

$$g_{tt} = -\mathcal{N}^2 \,, \quad g_{rr} = \frac{\mathcal{A}^2}{\chi^2} \,, \quad g_{\vartheta\vartheta} = \mathcal{A}^2 r^2 \,, \quad g_{\varphi\varphi} = \mathcal{A}^2 r^2 \sin^2\vartheta \quad (31)$$

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- Totally antisymmetric tensor $\epsilon_{\mu\nu\rho\sigma}$:

$$\epsilon_{tr\vartheta\varphi} = \sqrt{-\det g} = \frac{\mathcal{N}\mathcal{A}^3 r^2 \sin\vartheta}{\chi}\,. \tag{32}$$

$3+1$ split of Riemannian geometry

- Canonical $3+1$ split of the metric:
  - Unit normal (co-)vector field:

$$N = n^{\sharp} = \frac{1}{\mathcal{N}}\partial_t\,, \quad n = N^{\flat} = -\mathcal{N}\mathrm{d}t\,. \tag{33}$$

  - Spatial metric (gives projection onto spatial slices):

$$h = g + n \otimes n = \mathcal{A}^2\left[\frac{\mathrm{d}r \otimes \mathrm{d}r}{\chi^2} + r^2(\mathrm{d}\vartheta \otimes \mathrm{d}\vartheta + \sin^2\vartheta\mathrm{d}\varphi \otimes \mathrm{d}\varphi)\right]\,. \tag{34}$$

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- Induced spatial volume form via

$$\varepsilon_{\mu\nu\rho} = n^\sigma\epsilon_{\sigma\mu\nu\rho}\,, \quad \epsilon_{\mu\nu\rho\sigma} = 4\varepsilon_{[\mu\nu\rho}n_{\sigma]}\,. \tag{35}$$

so that

$$\varepsilon_{r\vartheta\varphi} = \frac{\mathcal{A}^3 r^2 \sin\vartheta}{\chi}\,, \quad \varepsilon_{tij} = 0\,. \tag{36}$$

Curvature of Levi-Civita connection

- Riemann tensor:

$$\overset{\circ}{R}_{\mu\nu\rho\sigma} = 2\frac{\dot{\mathcal{A}}^2 + u^2\mathcal{N}^2}{\mathcal{A}^2\mathcal{N}^2}h_{\mu[\rho}h_{\sigma]\nu} + 4\frac{\ddot{\mathcal{A}}\mathcal{N} - \dot{\mathcal{A}}\dot{\mathcal{N}}}{\mathcal{A}\mathcal{N}^3}n_{[\mu}h_{\nu][\rho}n_{\sigma]}\,. \quad (37)$$

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$$\overset{\circ}{R} = 6\frac{u^2\mathcal{N}^3 + \dot{\mathcal{A}}^2\mathcal{N} - \mathcal{A}\dot{\mathcal{A}}\dot{\mathcal{N}} + \mathcal{A}\ddot{\mathcal{A}}\mathcal{N}}{\mathcal{A}^2\mathcal{N}^3}\,. \quad (39)$$

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- Similar expressions for relevant teleparallel quantities?

Cosmologically symmetric teleparallel geometry

- What do we know so far?
  - Most general cosmologically symmetric metric.
  - Most general cosmologically symmetric, flat, metric connection.
  - Examples of tetrads and spin connections for $u^2 \in \{-1, 0, 1\}$.
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- We will now answer these questions.

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- Read off spin connection in diagonal gauge.

Step 1: irreducible reps. of the symmetry group

- $u \neq 0$: use “unitary trick” and complexification.
  - Irreps labeled by (m, n) with $\{2m, 2n, m+n\} \subset \mathbb{N}$.
  - Dimension given by $(2m+1)(2n+1)$.
  - Irreps with dimension at most 4:

$$(0,0)\,, \quad (0,1)\,, \quad (1,0)\,, \quad \left(\frac{1}{2},\frac{1}{2}\right)\,. \tag{40}$$

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- $u = 0$: use induced representations of Euclidean group.
  - Irreps induced by representations of SO(3).
  - Irreps labeled by spin $l \in \mathbb{N}$.
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Step 2: four-dimensional reps. of the symmetry group

- $u \neq 0$: four inequivalent representations.
  1. Trivial representation: $(0,0) \oplus (0,0) \oplus (0,0) \oplus (0,0)$.
  2. Anti-self-dual two-form: $(0,0) \oplus (0,1)$.
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⇒ Easier to work with Lie algebra representation:

$$\begin{array}{c} \phi : \mathfrak{g} \to \mathfrak{gl}(4) \\ \xi \mapsto \phi_\xi \end{array} . \tag{42}$$

Step 3: preservation of the Minkowski metric

- Need to find basis transformation P such that for all $u \in G$:

$$\mathbf{\Lambda}_u = P^{-1}\mathbf{\Phi}_u P \in \mathrm{SO}(1,3)\,. \tag{43}$$

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✓ Transformations exist for all four-dimensional representations.

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- Solve symmetry conditions in Weitzenböck gauge:
  - Solve three conditions for R_i to get spherical symmetry:
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Outline

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- Find Lorentz transformation to Weitzenböck gauge.

Step 1: most general connection

- Solve symmetry condition for cosmological generators:

$$0 = (\mathcal{L}_{X_\xi}\Gamma)^{\mu}{}_{\nu\rho} = \nabla_\rho\nabla_\nu X_\xi^\mu - X_\xi^\sigma R^\mu{}_{\nu\rho\sigma} - \nabla_\rho(X_\xi^\sigma T^\mu{}_{\nu\sigma})\,. \tag{49}$$

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\Gamma^r{}_{rr} = \frac{u^2 r}{\chi^2}\,, \quad \Gamma^t{}_{rr} = \frac{\mathcal{K}_2}{\chi^2}\,, \quad \Gamma^t{}_{\vartheta\vartheta} = \mathcal{K}_2 r^2\,, \quad \Gamma^t{}_{\varphi\varphi} = \mathcal{K}_2 r^2\sin^2\vartheta\,,\\
\Gamma^r{}_{\varphi\vartheta} = -\Gamma^r{}_{\vartheta\varphi} = \mathcal{K}_5 r^2\chi\sin\vartheta\,, \quad \Gamma^\vartheta{}_{r\varphi} = -\Gamma^\vartheta{}_{\varphi r} = \frac{\mathcal{K}_5\sin\vartheta}{\chi}\,,\\
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⇒ Depends on five free functions $\mathcal{K}_1(t), \ldots, \mathcal{K}_5(t)$ of time.

Step 2: impose metric compatibility

- Calculate nonmetricity:

$$Q_{\rho\mu\nu} = 2\mathcal{Q}_1 n_\rho n_\mu n_\nu + 2\mathcal{Q}_2 n_\rho h_{\mu\nu} + 2\mathcal{Q}_3 h_{\rho(\mu} n_{\nu)}\,, \tag{51}$$

where

$$\mathcal{Q}_1 = \frac{\dot{\mathcal{N}}}{\mathcal{N}^2} - \frac{\mathcal{K}_1}{\mathcal{N}}\,, \quad \mathcal{Q}_2 = \frac{1}{\mathcal{N}}\left(\mathcal{K}_4 - \frac{\dot{\mathcal{A}}}{\mathcal{A}}\right)\,, \quad \mathcal{Q}_3 = \frac{\mathcal{K}_3}{\mathcal{N}} - \frac{\mathcal{K}_2\mathcal{N}}{\mathcal{A}^2}\,. \tag{52}$$

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⇒ Metric-affine geometry satisfies $Q_{\rho\mu\nu} = 0$ if and only if

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⇒ Metricity determines $\mathcal{K}_1$, $\mathcal{K}_4$ and ratio between $\mathcal{K}_2$ and $\mathcal{K}_3$.

Step 3: impose flatness

- Calculate curvature of most general connection:

$$
\begin{aligned}
R_{\mu\nu\rho\sigma} = {} & 2\frac{\mathcal{K}_3(\mathcal{K}_4 - \mathcal{K}_1) + \dot{\mathcal{K}}_3}{\mathcal{N}^2} n_\nu n_{[\rho} h_{\sigma]\mu} - 2\frac{\mathcal{K}_3\mathcal{K}_5}{\mathcal{N}\mathcal{A}} n_\nu \varepsilon_{\mu\rho\sigma} \\
& + 2\frac{\mathcal{K}_2(\mathcal{K}_4 - \mathcal{K}_1) - \dot{\mathcal{K}}_2}{\mathcal{A}^2} n_\mu n_{[\rho} h_{\sigma]\nu} - 2\frac{\dot{\mathcal{K}}_5}{\mathcal{N}\mathcal{A}} \varepsilon_{\mu\nu[\rho} n_{\sigma]} \\
& + 2\frac{\mathcal{K}_2\mathcal{K}_5\mathcal{N}}{\mathcal{A}^3} n_\mu \varepsilon_{\nu\rho\sigma} + 2\frac{u^2 + \mathcal{K}_2\mathcal{K}_3 - \mathcal{K}_5^2}{\mathcal{A}^2} h_{\mu[\rho} h_{\sigma]\nu}\,.
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⇒ Several coupled conditions, whose solution depends on u.

Step 4: flat, metric, cosmological connection

- Two of three metricity conditions solved immediately:

$$\mathcal{K}_1 = \frac{\dot{\mathcal{N}}}{\mathcal{N}}, \quad \mathcal{K}_4 = \frac{\dot{\mathcal{A}}}{\mathcal{A}}. \tag{56}$$

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     - 2.2 "Vector" solution: $\mathcal{K}_5 = 0$ and $\mathcal{K}_2\mathcal{N}/\mathcal{A} = \mathcal{K}_3\mathcal{A}/\mathcal{N} = \pm iu$.

$\Rightarrow$ Solutions become complex depending on sign of u^2.

$\Rightarrow$ Smooth limit $u \to 0$ of solutions.

Step 5: diagonal tetrad and spin connection

- Start from diagonal tetrad:

$$\theta'^0 = \mathcal{N}\mathrm{d}t\,, \quad \theta'^1 = \frac{\mathcal{A}}{\chi}\mathrm{d}r\,, \quad \theta'^2 = \mathcal{A}r\mathrm{d}\vartheta\,, \quad \theta'^3 = \mathcal{A}r\sin\vartheta\mathrm{d}\varphi\,. \tag{57}$$

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$$0 = \nabla_\mu\theta'^a{}_\nu = \partial_\mu\theta'^a{}_\nu + \omega'^a{}_{b\mu}\theta'^b{}_\nu - \Gamma^\rho{}_{\nu\mu}\theta'^a{}_\rho\,. \tag{58}$$

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✓ Find same spin connections as using first approach.

⇝ Perform Lorentz transformation to Weitzenböck gauge.

✓ Obtain same non-diagonal tetrads as using first approach.

Outline

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- Proceed as before to determine cosmological tetrad.

Step 1: decomposition of the connection

- Recall decomposition of the connection:

$$\Gamma^{\mu}{}_{\nu\rho} = \mathring{\Gamma}^{\mu}{}_{\nu\rho} + K^{\mu}{}_{\nu\rho} + L^{\mu}{}_{\nu\rho}\,. \tag{59}$$

  - Levi-Civita connection $\mathring{\Gamma}^{\mu}{}_{\nu\rho}$ of the metric.
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- Contortion expressed in terms of torsion and metric:

$$K^{\mu}{}_{\nu\rho} = \frac{1}{2}\left(T_{\nu}{}^{\mu}{}_{\rho} + T_{\rho}{}^{\mu}{}_{\nu} - T^{\mu}{}_{\nu\rho}\right) \,. \tag{60}$$

Step 2: irreducible torsion decomposition

- Torsion decomposes into three irreducible parts:
  - Vector torsion:

$$\mathfrak{v}_\mu = T^\nu{}_{\nu\mu} \quad \Rightarrow \quad \mathfrak{V}^\mu{}_{\nu\rho} = \frac{2}{3}\delta^\mu_{[\nu}\mathfrak{v}_{\rho]}\,. \tag{61a}$$

  - Axial torsion:

$$\mathfrak{a}_\mu = \frac{1}{6}\epsilon_{\mu\nu\rho\sigma}T^{\nu\rho\sigma} \quad \Rightarrow \quad \mathfrak{A}_{\mu\nu\rho} = \epsilon_{\mu\nu\rho\sigma}\mathfrak{a}^\sigma\,. \tag{61b}$$

  - Tensor torsion:

$$\mathfrak{t}_{\mu\nu\rho} = T_{(\mu\nu)\rho} + \frac{1}{3}\left(T^\sigma{}_{\sigma(\mu}g_{\nu)\rho} - T^\sigma{}_{\sigma\rho}g_{\mu\nu}\right) \quad \Rightarrow \quad \mathfrak{T}^\mu{}_{\nu\rho} = \frac{4}{3}\mathfrak{t}^\mu{}_{[\nu\rho]}\,. \tag{61c}$$

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- Unique decomposition $T^\mu{}_{\nu\rho} = \mathfrak{V}^\mu{}_{\nu\rho} + \mathfrak{A}^\mu{}_{\nu\rho} + \mathfrak{T}^\mu{}_{\nu\rho}$ such that

$$\mathfrak{A}^\nu{}_{\nu\mu} = \mathfrak{T}^\nu{}_{\nu\mu} = 0\,, \quad \mathfrak{V}_{[\mu\nu\rho]} = \mathfrak{T}_{[\mu\nu\rho]} = 0\,. \tag{62}$$

Step 3: impose cosmological symmetry

- Most general torsion with cosmological symmetry:

$$T_{\mu\nu\rho} = 2\mathcal{T}_1 h_{\mu[\nu} n_{\rho]} + 2\mathcal{T}_2 \varepsilon_{\mu\nu\rho} \,. \tag{63}$$

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⇒ Torsion fully determined by scalar $\mathcal{T}_1(t)$ and pseudo-scalar $\mathcal{T}_2(t)$.

Step 4: impose flatness

- Calculate connection coefficients:

$$\mathcal{K}_1 = \frac{\dot{\mathcal{N}}}{\mathcal{N}}\,, \quad \mathcal{K}_2 = \frac{\mathcal{A}\dot{\mathcal{A}}}{\mathcal{N}^2} - \frac{\mathcal{A}^2\mathcal{T}_1}{\mathcal{N}}\,, \quad \mathcal{K}_3 = \frac{\dot{\mathcal{A}}}{\mathcal{A}} - \mathcal{N}\mathcal{T}_1\,, \quad \mathcal{K}_4 = \frac{\dot{\mathcal{A}}}{\mathcal{A}}\,, \quad \mathcal{K}_5 = \mathcal{A}\mathcal{T}_2\,. \tag{65}$$

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  1. "Pure vector" solution:

$$\mathcal{T}_1 = \frac{\dot{\mathcal{A}}}{\mathcal{A}\mathcal{N}} \pm \frac{iu}{\mathcal{A}}\,, \quad \mathcal{T}_2 = 0\,. \tag{66}$$

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⇝ Apply same procedure as before to obtain symmetric tetrads.

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Solution in Weitzenböck gauge

- Homomorphism of the symmetry algebra:

$$R_i \mapsto J_i\,, \quad T_i \mapsto iuK_i\,. \tag{68}$$

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⇒ Symmetric tetrad in Weitzenböck gauge:

$$\theta^0 = \mathcal{N}\chi \mathrm{d}t + iu\mathcal{A}\frac{r}{\chi}\mathrm{d}r\,, \tag{69a}$$

$$\theta^1 = \mathcal{A}\left[\sin\vartheta\cos\varphi\left(\mathrm{d}r + iu\frac{\mathcal{N}}{\mathcal{A}}r\mathrm{d}t\right) + r\cos\vartheta\cos\varphi\mathrm{d}\vartheta - r\sin\vartheta\sin\varphi\mathrm{d}\varphi\right] \tag{69b}$$

$$\theta^2 = \mathcal{A}\left[\sin\vartheta\sin\varphi\left(\mathrm{d}r + iu\frac{\mathcal{N}}{\mathcal{A}}r\mathrm{d}t\right) + r\cos\vartheta\sin\varphi\mathrm{d}\vartheta + r\sin\vartheta\cos\varphi\mathrm{d}\varphi\right] \tag{69c}$$

$$\theta^3 = \mathcal{A}\left[\cos\vartheta\left(\mathrm{d}r + iu\frac{\mathcal{N}}{\mathcal{A}}r\mathrm{d}t\right) - r\sin\vartheta\mathrm{d}\vartheta\right]\,, \tag{69d}$$

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⇒ Real for $u^2 \leq 0$, complex for $u^2 > 0$.

Transformation to diagonal gauge

- Diagonalizing Lorentz transformation:

$$\Lambda^a{}_b = \begin{pmatrix} \chi & -iur\sin\vartheta\cos\varphi & -iur\sin\vartheta\sin\varphi & -iur\cos\vartheta \\ -iur & \chi\sin\vartheta\cos\varphi & \chi\sin\vartheta\sin\varphi & \chi\cos\vartheta \\ 0 & \cos\vartheta\cos\varphi & \cos\vartheta\sin\varphi & -\sin\vartheta \\ 0 & -\sin\varphi & \cos\varphi & 0 \end{pmatrix} . \tag{70}$$

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- Spin connection in diagonal gauge:

$$\begin{gathered} \omega'^0{}_{1r} = \omega'^1{}_{0r} = -\frac{iu}{\chi}, \quad \omega'^0{}_{2\vartheta} = \omega'^2{}_{0\vartheta} = -iur, \\ \omega'^0{}_{3\varphi} = \omega'^3{}_{0\varphi} = -iur\sin\vartheta, \quad \omega'^1{}_{2\vartheta} = -\omega'^2{}_{1\vartheta} = -\chi, \\ \omega'^1{}_{3\varphi} = -\omega'^3{}_{1\varphi} = -\chi\sin\vartheta, \quad \omega'^2{}_{3\varphi} = -\omega'^3{}_{2\varphi} = -\cos\vartheta. \end{gathered} \tag{71}$$

Induced affine connection

- Parameter functions in the connection:

$$\mathcal{K}_1 = \frac{\dot{\mathcal{N}}}{\mathcal{N}}, \quad \mathcal{K}_2 = -iu\frac{\mathcal{A}}{\mathcal{N}}, \quad \mathcal{K}_3 = -iu\frac{\mathcal{N}}{\mathcal{A}}, \quad \mathcal{K}_4 = \frac{\dot{\mathcal{A}}}{\mathcal{A}}, \quad \mathcal{K}_5 = 0\,. \quad (72)$$

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$\Rightarrow$ Contortion:

$$K_{\mu\nu\rho} = 2\frac{\dot{\mathcal{A}} + iu\mathcal{N}}{\mathcal{A}\mathcal{N}}h_{\rho[\mu}n_{\nu]}\,. \tag{74}$$

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$$\theta^2 = \mathcal{A}\left[\frac{\sin\vartheta\sin\varphi}{\chi}\mathrm{d}r + r(\chi\cos\vartheta\sin\varphi - ur\cos\varphi)\mathrm{d}\vartheta + r\sin\vartheta(\chi\cos\varphi + ur\cos\vartheta\sin\varphi)\mathrm{d}\varphi\right]\,, \tag{76c}$$

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$$\Lambda^a{}_b = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \sin\vartheta\cos\varphi & \sin\vartheta\sin\varphi & \cos\vartheta \\ 0 & \chi\cos\vartheta\cos\varphi + ur\sin\varphi & \chi\cos\vartheta\sin\varphi - ur\cos\varphi & -\chi\sin\vartheta \\ 0 & ur\cos\vartheta\cos\varphi - \chi\sin\varphi & \chi\cos\varphi + ur\cos\vartheta\sin\varphi & -ur\sin\vartheta \end{pmatrix}. \tag{77}$$

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$$\Lambda^a{}_b = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \sin\vartheta\cos\varphi & \sin\vartheta\sin\varphi & \cos\vartheta \\ 0 & \chi\cos\vartheta\cos\varphi + ur\sin\varphi & \chi\cos\vartheta\sin\varphi - ur\cos\varphi & -\chi\sin\vartheta \\ 0 & ur\cos\vartheta\cos\varphi - \chi\sin\varphi & \chi\cos\varphi + ur\cos\vartheta\sin\varphi & -ur\sin\vartheta \end{pmatrix}. \tag{77}$$

- Spin connection in diagonal gauge:

$$\begin{aligned} \omega'^1{}_{2\vartheta} = -\omega'^2{}_{1\vartheta} = -\chi\,, \quad & \omega'^1{}_{2\varphi} = -\omega'^2{}_{1\varphi} = ur\sin\vartheta\,, \\ \omega'^1{}_{3\vartheta} = -\omega'^3{}_{1\vartheta} = -ur\,, \quad & \omega'^1{}_{3\varphi} = -\omega'^3{}_{1\varphi} = -\chi\sin\vartheta\,, \\ \omega'^2{}_{3r} = -\omega'^3{}_{2r} = \frac{u}{\chi}\,, \quad & \omega'^2{}_{3\varphi} = -\omega'^3{}_{2\varphi} = -\cos\vartheta\,. \end{aligned} \tag{78}$$

Induced affine connection and torsion

- Parameter functions in the connection:

$$\mathcal{K}_1 = \frac{\dot{\mathcal{N}}}{\mathcal{N}}, \quad \mathcal{K}_2 = \mathcal{K}_3 = 0, \quad \mathcal{K}_4 = \frac{\dot{\mathcal{A}}}{\mathcal{A}}, \quad \mathcal{K}_5 = -u. \tag{79}$$

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Outline

Coordinate independent torsion properties

- Torsion of general cosmological connection:

$$\mathfrak{v}_\mu = 3\frac{\mathcal{K}_4 - \mathcal{K}_3}{\mathcal{N}} n_\mu \,, \quad \mathfrak{a}_\mu = -2\frac{\mathcal{K}_5}{\mathcal{A}} n_\mu \,. \tag{82}$$

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⇒ Torsion of teleparallel cosmological connections:

	"vector"	"axial"
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✓ Not necessary to work with cumbersome coordinate expressions.

Cosmological dynamics

- Consider tetrad field equations of some teleparallel theory:

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  - Energy-momentum tensor takes perfect fluid form:

$$\Theta_{\mu\nu} = (\rho + p) n_\mu n_\nu + p g_{\mu\nu} = \rho n_\mu n_\nu + p h_{\mu\nu} \,. \tag{84}$$

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⇒ There exist only two branches of cosmological solutions.

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⇒ Axial torsion pseudo-vector reverses sign under reflection.

- ⇒ "Vector" branch of tetrads is parity-invariant, $\mathfrak{a}_\mu = 0$.
- ⇒ "Axial" branch of tetrads is not parity-invariant, $\mathfrak{a}_\mu = 2\frac{u}{\mathcal{A}} n_\mu \neq 0$.

Outline

Summary

- Metric-affine and teleparallel geometry:
    - Based on metric and independent connection.
    - Connection may exhibit curvature, torsion, nonmetricity.
    - Metric teleparallel geometry characterized by torsion only.
    - More conventional to work with tetrad and spin connection.

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  - Two branches of solutions depending on lapse and scale factor.
  - Simple formulas for torsion in terms of defining functions.
- Answered a few questions:
  - ✓ Determined most general cosmological teleparallel geometries.
  - ✓ Torsion & field equations expressed like in Riemannian geometry.
  - ✓ Effective way to work with perturbations in teleparallel cosmology.

Outlook

- Cosmological (background) dynamics:
  - Study theories which distinguish vector and axial torsion.
  - Different dynamics for different branches when $u \neq 0$?
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- Apply method to more general symmetries:
  - Spherical symmetry: black holes and quasinormal modes.
  - Planar symmetry: gravitational wave propagation.

Literature

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