HIDDEN COSMOLOGICAL DYNAMICS IN TELEPARALLEL GRAVITY

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Definitions

The *fundamental fields* on M are

- a *coframe field*

$$\theta^a = \theta^a{}_\mu \mathrm{d}x^\mu\,, \quad (1)$$

- a flat *spin connection*

$$\overset{\bullet}{\omega}{}^a{}_b = \overset{\bullet}{\omega}{}^a{}_{b\mu} \mathrm{d}x^\mu\,, \quad (2)$$

- N *scalar fields* ϕ^A,
- arbitrary *matter fields* χ^I.

These fields further define

- a *frame field* $e_a = e_a{}^\mu \partial_\mu$ with

$$\iota_{e_a}\theta^b = \delta^b_a\,, \quad (3)$$

a *metric*

$$g_{\mu\nu} = \eta_{ab}\theta^a{}_\mu \theta^b{}_\nu\,, \quad (4)$$

- a *volume form*

$$\theta \mathrm{d}^4x = \theta^0 \wedge \theta^1 \wedge \theta^2 \wedge \theta^3\,, \quad (5)$$

- the *Levi-Civita connection*

$$\overset{\circ}{\omega}{}_{ab} = -\frac{1}{2}(\iota_{e_b}\iota_{e_c}\mathrm{d}\theta_a + \iota_{e_c}\iota_{e_a}\mathrm{d}\theta_b - \iota_{e_a}\iota_{e_b}\mathrm{d}\theta_c)\theta^c\,, \quad (6)$$

- the *torsion*

$$T^a = \mathrm{d}\theta^a + \overset{\bullet}{\omega}{}^a{}_b \wedge \theta^b\,, \quad (7)$$

- the *affine connections*

$$\overset{\bullet}{\Gamma}{}^\rho{}_{\mu\nu} = e_a{}^\rho(\partial_\nu \theta^a{}_\mu + \overset{\bullet}{\omega}{}^a{}_{b\nu}\theta^b{}_\mu)\,, \quad (8)$$

$$\overset{\circ}{\Gamma}{}^\rho{}_{\mu\nu} = e_a{}^\rho(\partial_\nu \theta^a{}_\mu + \overset{\circ}{\omega}{}^a{}_{b\nu}\theta^b{}_\mu) = \frac{1}{2}g^{\rho\sigma}(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu})\,. \quad (9)$$

We call a *teleparallel geometry* the triple

$$(M, \boldsymbol{\theta}, \overset{\bullet}{\boldsymbol{\omega}})\,. \quad (10)$$

References

[1] M. Hohmann, L. Järv, U. Ualikhanova, "Covariant formulation of scalar-torsion gravity," Phys. Rev. D **97** (2018) 104011 [arXiv:1801.05786 [gr-qc]].

[2] M. Hohmann, L. Järv, M. Krššák and C. Pfeifer, "Modified teleparallel theories of gravity in symmetric spacetimes," arXiv:1901.05472 [gr-qc].

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Symmetries in teleparallel gravity [2]

Symmetries under group actions

Definition. A *symmetry* of a teleparallel geometry $(M, \boldsymbol{\theta}, \overset{\bullet}{\boldsymbol{\omega}})$ is a group action $\varphi : G \times M \to M, x \mapsto x'$ of a Lie group G such that the induced metric (4) and affine connection (8) are invariant, i.e., $\varphi^*_u g = g$ and $\varphi^*_u \Gamma = \Gamma$ for all $u \in G$, where

$$(\varphi^*_u g)_{\mu\nu}(x) = g_{\rho\sigma}(x')\frac{\partial x'^\rho}{\partial x^\mu}\frac{\partial x'^\sigma}{\partial x^\nu}\,, \quad (\varphi^*_u \Gamma)^\mu{}_{\nu\rho}(x) = \Gamma^\tau{}_{\omega\sigma}(x')\frac{\partial x^\mu}{\partial x'^\tau}\frac{\partial x'^\omega}{\partial x^\nu}\frac{\partial x'^\sigma}{\partial x^\rho} + \frac{\partial x^\mu}{\partial x'^\sigma}\frac{\partial^2 x'^\sigma}{\partial x^\nu \partial x^\rho}\,. \quad (11)$$

The teleparallel geometry is then called *symmetric* under the group action φ.

Proposition. *A teleparallel geometry $(M, \boldsymbol{\theta}, \overset{\bullet}{\boldsymbol{\omega}})$ is symmetric under a group action $\varphi : G \times M \to M$ if and only if there exists a* local Lie group homomorphism $\boldsymbol{\Lambda} : G \times M \to \mathrm{SO}(1,3)$ *such that*

$$(\varphi^*_u \theta)^a{}_\mu = \left(\boldsymbol{\Lambda}_u^{-1}\right)^a{}_b \theta^b{}_\mu\,, \quad (\varphi^*_u \overset{\bullet}{\omega})^a{}_{b\mu} = \left(\boldsymbol{\Lambda}_u^{-1}\right)^a{}_c \boldsymbol{\Lambda}^d_{ub}\overset{\bullet}{\omega}{}^c{}_{d\mu} + \left(\boldsymbol{\Lambda}_u^{-1}\right)^a{}_c \partial_\mu \boldsymbol{\Lambda}^c_{ub} \quad (12)$$

for all $u \in G$, where

$$(\varphi^*_u \theta)^a{}_\mu(x) = \theta^a{}_\nu(x')\frac{\partial x'^\nu}{\partial x^\mu}\,, \quad (\varphi^*_u \overset{\bullet}{\omega})^a{}_{b\mu}(x) = \overset{\bullet}{\omega}{}^a{}_{b\nu}(x')\frac{\partial x'^\nu}{\partial x^\mu}\,. \quad (13)$$

Infinitesimal symmetries

A Lie group action $\varphi : G \times M \to M$ induces a *Lie algebra homomorphism* $X : \mathfrak{g} \to \mathrm{Vect}\,M$ (the *fundamental vector fields*). For $\xi \in \mathfrak{g}$ we then have

$$(\mathcal{L}_{X_\xi} g)_{\mu\nu} = X^\rho_\xi \partial_\rho g_{\mu\nu} + \partial_\mu X^\rho_\xi g_{\rho\nu} + \partial_\nu X^\rho_\xi g_{\mu\rho} \quad (14)$$

and

$$(\mathcal{L}_{X_\xi}\Gamma)^\mu{}_{\nu\rho} = X^\sigma_\xi \partial_\sigma \Gamma^\mu{}_{\nu\rho} - \partial_\sigma X^\mu_\xi \Gamma^\sigma{}_{\nu\rho} + \partial_\nu X^\sigma_\xi \Gamma^\mu{}_{\sigma\rho} + \partial_\rho X^\sigma_\xi \Gamma^\mu{}_{\nu\sigma} + \partial_\nu \partial_\rho X^\mu_\xi\,. \quad (15)$$

Tetrad and spin connection transform as

$$(\mathcal{L}_{X_\xi}\theta)^a{}_\mu = X^\nu_\xi \partial_\nu \theta^a{}_\mu + \partial_\mu X^\nu_\xi \theta^a{}_\nu\,, \quad (\mathcal{L}_{X_\xi}\overset{\bullet}{\omega})^a{}_{b\mu} = X^\nu_\xi \partial_\nu \overset{\bullet}{\omega}{}^a{}_{b\mu} + \partial_\mu X^\nu_\xi \overset{\bullet}{\omega}{}^a{}_{b\nu}\,. \quad (16)$$

Proposition. *For a teleparallel geometry $(M, \boldsymbol{\theta}, \overset{\bullet}{\boldsymbol{\omega}})$ which is symmetric under a group action, there exists a* local Lie algebra homomorphism $\boldsymbol{\lambda} : \mathfrak{g} \times M \to \mathfrak{so}(1,3)$ *defined by*

$$\boldsymbol{\lambda}_\xi(x) = \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\Lambda}_{\exp(\xi t)}(x)\bigg|_{t=0}\,, \quad (17)$$

such that

$$(\mathcal{L}_{X_\xi}\theta)^a{}_\mu = -\boldsymbol{\lambda}^a_{\xi b}\theta^b{}_\mu\,, \quad (\mathcal{L}_{X_\xi}\overset{\bullet}{\omega})^a{}_{b\mu} = \mathrm{D}_\mu \boldsymbol{\lambda}^a_{\xi b}\,, \quad (18)$$

where we used the total covariant derivative

$$\mathrm{D}_\mu \boldsymbol{\lambda}^a_{\xi b} = \partial_\mu \boldsymbol{\lambda}^a_{\xi b} + \overset{\bullet}{\omega}{}^a{}_{c\mu}\boldsymbol{\lambda}^c_{\xi b} - \overset{\bullet}{\omega}{}^c{}_{b\mu}\boldsymbol{\lambda}^a_{\xi c}\,. \quad (19)$$

Scalar-torsion gravity [1]

Action and field equations

A generic, but simple class of scalar-torsion theories of gravity is given by the *action*

$$S = \frac{1}{2\kappa^2}\int_M \left[f(T, \boldsymbol{\phi}) + Z_{AB}(\boldsymbol{\phi})g^{\mu\nu}\phi^A_{,\mu}\phi^B_{,\nu}\right]\theta d^4x + S_m[\theta^a, \chi^I]\,. \quad (20)$$

We call it *minimally coupled* if $f_{T\phi^A} \equiv 0$. The *field equations* are the symmetric part

$$\frac{1}{2}f g_{\mu\nu} + \overset{\circ}{\nabla}_\rho\left(f_T S_{(\mu\nu)}{}^\rho\right) - \frac{1}{2}f_T S_{(\mu}{}^{\rho\sigma}T_{\nu)\rho\sigma} - Z_{AB}\phi^A_{,\mu}\phi^B_{,\nu} + \frac{1}{2}Z_{AB}\phi^A_{,\rho}\phi^B_{,\sigma}g^{\rho\sigma}g_{\mu\nu} = \kappa^2\Theta_{\mu\nu}\,, \quad (21)$$

the antisymmetric part

$$\partial_{[\rho}f_T T^\rho{}_{\mu\nu]} = 0 \quad \Leftrightarrow \quad \partial_\mu f_T\left[\partial_\nu\left(\theta\, e_{[a}{}^\mu e_{b]}{}^\nu\right) + 2\theta\, e_c{}^{[\mu}e_{[a}{}^{\nu]}\overset{\bullet}{\omega}{}^c{}_{b]\nu}\right] = 0 \quad (22)$$

and the scalar field equation

$$f_{\phi^A} - \left(2Z_{AB,\phi^C} - Z_{BC,\phi^A}\right)g^{\mu\nu}\phi^B_{,\mu}\phi^C_{,\nu} - 2Z_{AB}\overset{\circ}{\Box}\phi^B = 0\,. \quad (23)$$

Solving the antisymmetric field equation

There are different ways to *solve the antisymmetric equations* (22):

1. For theories with $f_{TT} \equiv 0$ and $f_{T\phi^A} \equiv 0$, so that $f(T, \boldsymbol{\phi}) = kT - V(\boldsymbol{\phi})$, the equations (22) are solved identically for any field configuration.
2. Field configurations with $\partial_\mu T = 0$ and $\partial_\mu \phi^A = 0$, i.e., constant torsion scalar and constant scalar fields, always solve the equations (22), independently of the function f. The remaining field equations (21) reduce to general relativity with cosmological constant.
3. Field configurations where T and ϕ^A depend only on a single coordinate y satisfy $\partial_\mu f_T \propto \partial_\mu y$. They solve the equations (22) if the six vector fields, which are defined by the terms in square brackets in (22) for the six values of $[ab]$, are tangent to the hypersurfaces of constant y, independently of the function f.
4. In the general case, the solutions depend on f. A general field configuration solves the equations (22) if the six vector fields mentioned above are tangent to the hypersurfaces of constant f_T.

Cosmological dynamics

Scalar field equation

Scalar field equation for the shown cosmological tetrads (single scalar field case):

$$f_\phi - 2Z\ddot{\phi} - 6ZH\dot{\phi} - Z_\phi \dot{\phi}^2 = 0\,. \quad (24)$$

Tetrad field equations: $k = -1$, real tetrad

With the Weitzenböck tetrad (34) or diagonal tetrad (29) and spin connection (35) we have

$$\frac{1}{2}f + 6f_T H\left(H - \frac{1}{a}\right) - \frac{1}{2}Z\dot{\phi}^2 = \kappa^2\rho\,, \quad (25a)$$

$$\frac{1}{2}f + 2f_{T\phi}\left(H - \frac{1}{a}\right)\dot{\phi} - 24f_{TT}\left(\dot{H} + \frac{H}{a}\right)\left(H - \frac{1}{a}\right)^2 + 6f_T H\left(H - \frac{1}{a}\right) + 2f_T\left(\dot{H} + \frac{1}{a^2}\right) + \frac{1}{2}Z\dot{\phi}^2 = -\kappa^2 p\,. \quad (25b)$$

Tetrad field equations: $k = -1$, complex tetrad

With the Weitzenböck tetrad (36) or diagonal tetrad (29) and spin connection (37) we have

$$\frac{1}{2}f + 6f_T H^2 - \frac{1}{2}Z\dot{\phi}^2 = \kappa^2\rho\,, \quad (26a)$$

$$\frac{1}{2}f + 2f_{T\phi}H\dot{\phi} - 24f_{TT}\left(\dot{H} + \frac{1}{a^2}\right)H^2 + 6f_T H^2 + 2f_T\left(\dot{H} - \frac{1}{a^2}\right) + \frac{1}{2}Z\dot{\phi}^2 = -\kappa^2 p\,. \quad (26b)$$

Tetrads and spin connections with cosmological symmetry

Generating vector fields and metric

The *cosmological symmetry* is generated by the vector fields (with $\chi = \sqrt{1 - kr^2}$)

$$X_1 = \chi\sin\vartheta\cos\varphi\partial_r + \frac{\chi}{r}\cos\vartheta\cos\varphi\partial_\vartheta - \frac{\chi\sin\varphi}{r\sin\vartheta}\partial_\varphi\,, \quad (27a)$$

$$X_2 = \chi\sin\vartheta\sin\varphi\partial_r + \frac{\chi}{r}\cos\vartheta\sin\varphi\partial_\vartheta + \frac{\chi\cos\varphi}{r\sin\vartheta}\partial_\varphi\,, \quad (27b)$$

$$X_3 = \chi\cos\vartheta\partial_r - \frac{\chi}{r}\sin\vartheta\partial_\vartheta\,, \quad (27c)$$

$$X_x = \sin\varphi\partial_\vartheta + \frac{\cos\varphi}{\tan\vartheta}\partial_\varphi\,, \quad (27d)$$

$$X_y = -\cos\varphi\partial_\vartheta + \frac{\sin\varphi}{\tan\vartheta}\partial_\varphi\,, \quad (27e)$$

$$X_z = -\partial_\varphi\,. \quad (27f)$$

The most general compatible metric is the *Friedmann-Lemaître-Robertson-Walker* metric

$$g_{\mu\nu}dx^\mu dx^\nu = -n^2(t)dt^2 + a^2(t)\left[\frac{dr^2}{1 - kr^2} + r^2(d\vartheta^2 + \sin^2\vartheta d\varphi^2)\right]\,. \quad (28)$$

The symmetry conditions (18) can be solved either in the *Weitzenböck gauge* with non-diagonal tetrad θ^a and vanishing spin connection $\overset{\bullet}{\omega}{}^a{}_b \equiv 0$, or in the *diagonal gauge* by the tetrad

$$\theta'^0 = n(t)\mathrm{d}t\,, \quad \theta'^1 = \frac{a(t)}{\sqrt{1 - kr^2}}\mathrm{d}r\,, \quad \theta'^2 = a(t)r\mathrm{d}\vartheta\,, \quad \theta'^3 = a(t)r\sin\vartheta\mathrm{d}\varphi\,. \quad (29)$$

and non-vanishing spin connection $\overset{\bullet}{\omega}{}'^a{}_b$.

Proposition. *Any tetrad / spin connection combination which possesses cosmological symmetry under the vector fields (27), i.e., satisfies the invariance conditions (18), identically solves the antisymmetric part of the field equations of* any generic teleparallel gravity theory.

$\mathrm{ISO}(3)$: flat space $k = 0$

Weitzenböck gauge

$$\theta^0 = n(t)\mathrm{d}t\,, \quad (30a)$$

$$\theta^1 = a(t)\left[\sin\vartheta\cos\varphi\mathrm{d}r + r\cos\theta\cos\varphi\mathrm{d}\vartheta - r\sin\vartheta\sin\varphi\mathrm{d}\varphi\right]\,, \quad (30b)$$

$$\theta^2 = a(t)\left[\sin\vartheta\sin\varphi\mathrm{d}r + r\cos\theta\sin\varphi\mathrm{d}\vartheta + r\sin\vartheta\cos\varphi\mathrm{d}\varphi\right]\,, \quad (30c)$$

$$\theta^3 = a(t)\left[\cos\vartheta\mathrm{d}r - r\sin\vartheta\mathrm{d}\vartheta\right]\,. \quad (30d)$$

Diagonal gauge

$$\overset{\bullet}{\omega}{}'^1{}_{2\vartheta} = -\overset{\bullet}{\omega}{}'^2{}_{1\vartheta} = -1\,, \quad \overset{\bullet}{\omega}{}'^1{}_{3\varphi} = -\overset{\bullet}{\omega}{}'^3{}_{1\varphi} = -\sin\vartheta\,, \quad \overset{\bullet}{\omega}{}'^2{}_{3\varphi} = -\overset{\bullet}{\omega}{}'^3{}_{2\varphi} = -\cos\vartheta\,, \quad (31)$$

$\mathrm{SO}(4)$: positively curved space $k = 1$

Weitzenböck gauge

$$\theta^0_\pm = n(t)\mathrm{d}t\,, \quad (32a)$$

$$\theta^1_\pm = a(t)\left[\frac{\sin\vartheta\cos\varphi}{\chi}\mathrm{d}r + r(\chi\cos\theta\cos\varphi \pm r\sin\varphi)\mathrm{d}\vartheta - r\sin\vartheta(\chi\sin\varphi \mp r\cos\vartheta\cos\varphi)\mathrm{d}\varphi\right]\,, \quad (32b)$$

$$\theta^2_\pm = a(t)\left[\frac{\sin\vartheta\sin\varphi}{\chi}\mathrm{d}r + r(\chi\cos\theta\sin\varphi \mp r\cos\varphi)\mathrm{d}\vartheta + r\sin\vartheta(\chi\cos\varphi \pm r\cos\vartheta\sin\varphi)\mathrm{d}\varphi\right]\,, \quad (32c)$$

$$\theta^3_\pm = a(t)\left[\frac{\cos\vartheta}{\chi}\mathrm{d}r - r\chi\sin\vartheta\mathrm{d}\vartheta \mp r^2\sin^2\vartheta\mathrm{d}\varphi\right]\,, \quad (32d)$$

Diagonal gauge

$$\overset{\bullet}{\omega}{}'^1_{\pm\,2\vartheta} = -\overset{\bullet}{\omega}{}'^2_{\pm\,1\vartheta} = -\chi\,, \quad \overset{\bullet}{\omega}{}'^1_{\pm\,2\varphi} = -\overset{\bullet}{\omega}{}'^2_{\pm\,1\varphi} = \pm r\sin\vartheta\,, \quad \overset{\bullet}{\omega}{}'^1_{\pm\,3\vartheta} = -\overset{\bullet}{\omega}{}'^3_{\pm\,1\vartheta} = \mp r\,,$$

$$\overset{\bullet}{\omega}{}'^1_{\pm\,3\varphi} = -\overset{\bullet}{\omega}{}'^3_{\pm\,1\varphi} = -\chi\sin\vartheta\,, \quad \overset{\bullet}{\omega}{}'^2_{\pm\,3r} = -\overset{\bullet}{\omega}{}'^3_{\pm\,2r} = \pm\frac{1}{\chi}\,, \quad \overset{\bullet}{\omega}{}'^2_{\pm\,3\varphi} = -\overset{\bullet}{\omega}{}'^3_{\pm\,2\varphi} = -\cos\vartheta\,. \quad (33)$$

$\mathrm{SO}(1,3)$: negatively curved space $k = -1$, real solution

Weitzenböck gauge

$$\theta^0_\pm = \pm n(t)\chi\mathrm{d}t + \pm a(t)\frac{r}{\chi}\mathrm{d}r\,, \quad (34a)$$

$$\theta^1_\pm = a(t)\left[\sin\vartheta\cos\varphi\left(\mathrm{d}r + \frac{n(t)}{a(t)}r\mathrm{d}t\right) + r\cos\vartheta\cos\varphi\mathrm{d}\vartheta - r\sin\vartheta\sin\varphi\mathrm{d}\varphi\right]\,, \quad (34b)$$

$$\theta^2_\pm = a(t)\left[\sin\vartheta\sin\varphi\left(\mathrm{d}r + \frac{n(t)}{a(t)}r\mathrm{d}t\right) + r\cos\vartheta\sin\varphi\mathrm{d}\vartheta + r\sin\vartheta\cos\varphi\mathrm{d}\varphi\right]\,, \quad (34c)$$

$$\theta^3_\pm = a(t)\left[\cos\vartheta\left(\mathrm{d}r + \frac{n(t)}{a(t)}r\mathrm{d}t\right) - r\sin\vartheta\mathrm{d}\vartheta\right]\,, \quad (34d)$$

Diagonal gauge

$$\overset{\bullet}{\omega}{}'^0_{\pm\,1r} = \overset{\bullet}{\omega}{}'^1_{\pm\,0r} = \frac{1}{\chi}\,, \quad \overset{\bullet}{\omega}{}'^0_{\pm\,2\vartheta} = \overset{\bullet}{\omega}{}'^2_{\pm\,0\vartheta} = r\,, \quad \overset{\bullet}{\omega}{}'^0_{\pm\,3\varphi} = \overset{\bullet}{\omega}{}'^3_{\pm\,0\varphi} = r\sin\vartheta\,,$$

$$\overset{\bullet}{\omega}{}'^1_{\pm\,2\vartheta} = -\overset{\bullet}{\omega}{}'^2_{\pm\,1\vartheta} = -\chi\,, \quad \overset{\bullet}{\omega}{}'^1_{\pm\,3\varphi} = -\overset{\bullet}{\omega}{}'^3_{\pm\,1\varphi} = -\chi\sin\vartheta\,, \quad \overset{\bullet}{\omega}{}'^2_{\pm\,3\varphi} = -\overset{\bullet}{\omega}{}'^3_{\pm\,2\varphi} = -\cos\vartheta\,. \quad (35)$$

$\mathrm{SO}(1,3)$: negatively curved space $k = -1$, complex solution

Weitzenböck gauge

$$\theta^0_\pm = n(t)\mathrm{d}t\,, \quad (36a)$$

$$\theta^1_\pm = a(t)\left[\frac{\sin\vartheta\cos\varphi}{\chi}\mathrm{d}r + r(\chi\cos\theta\cos\varphi \pm ir\sin\varphi)\mathrm{d}\vartheta - r\sin\vartheta(\chi\sin\varphi \mp ir\cos\vartheta\cos\varphi)\mathrm{d}\varphi\right]\,, \quad (36b)$$

$$\theta^2_\pm = a(t)\left[\frac{\sin\vartheta\sin\varphi}{\chi}\mathrm{d}r + r(\chi\cos\theta\sin\varphi \mp ir\cos\varphi)\mathrm{d}\vartheta + r\sin\vartheta(\chi\cos\varphi \pm ir\cos\vartheta\sin\varphi)\mathrm{d}\varphi\right]\,, \quad (36c)$$

$$\theta^3_\pm = a(t)\left[\frac{\cos\vartheta}{\chi}\mathrm{d}r - r\chi\sin\vartheta\mathrm{d}\vartheta \mp ir^2\sin^2\vartheta\mathrm{d}\varphi\right]\,. \quad (36d)$$

Diagonal gauge

$$\overset{\bullet}{\omega}{}'^1_{\pm\,2\vartheta} = -\overset{\bullet}{\omega}{}'^2_{\pm\,1\vartheta} = -\chi\,, \quad \overset{\bullet}{\omega}{}'^1_{\pm\,2\varphi} = -\overset{\bullet}{\omega}{}'^2_{\pm\,1\varphi} = \pm ir\sin\vartheta\,, \quad \overset{\bullet}{\omega}{}'^1_{\pm\,3\vartheta} = -\overset{\bullet}{\omega}{}'^3_{\pm\,1\vartheta} = \mp ir\,,$$

$$\overset{\bullet}{\omega}{}'^1_{\pm\,3\varphi} = -\overset{\bullet}{\omega}{}'^3_{\pm\,1\varphi} = -\chi\sin\vartheta\,, \quad \overset{\bullet}{\omega}{}'^2_{\pm\,3r} = -\overset{\bullet}{\omega}{}'^3_{\pm\,2r} = \pm\frac{i}{\chi}\,, \quad \overset{\bullet}{\omega}{}'^2_{\pm\,3\varphi} = -\overset{\bullet}{\omega}{}'^3_{\pm\,2\varphi} = -\cos\vartheta\,, \quad (37)$$

Conclusion

Extended teleparallel gravity theories, such as scalar-torsion gravity, allow for *different tetrad / spin connection* combinations with cosmological symmetry, which yield the *same metric*, but *different cosmological dynamics*. Hence, the cosmological dynamics depend on degrees of freedom "hidden" from observations of the metric.