Hamiltonian formulation of teleparallel gravity
using the language of differential forms

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Outline

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- Consider globally hyperbolic manifold $M \cong \mathbb{R} \times \Sigma$.

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- Introduce time translation vector field:
  - Curve through every point $\mathbf{x} \in M$:

$$\begin{aligned} \gamma_{\mathbf{x}} : \mathbb{R} &\to M \\ t &\mapsto i\left(\mathfrak{t}(\mathbf{x}) + t, \mathfrak{s}(\mathbf{x})\right) \end{aligned}.$$

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$$(\partial_{\mathfrak{t}})_{\mathbf{x}} = \left.\frac{\mathrm{d}}{\mathrm{d}t}\gamma_{\mathbf{x}}(t)\right|_{t=0}\,.$$

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- NB! $\partial_{\mathfrak{t}}$ is *not* a unit normal vector field - no metric used!

Split of vector fields and differential forms

- Split of a vector field $\mathbf{X} \in \Gamma(TM)$:
  - Temporal part:
$$\check{\mathbf{X}} = \hat{\mathbf{X}}\partial_{\mathfrak{t}} = (\mathbf{X} \lrcorner \mathbf{d}\mathfrak{t})\partial_{\mathfrak{t}} \,.$$
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- Split of a differential k-form τ:
  - Temporal part:
$$\check{\tau} = \mathbf{d}\mathfrak{t} \wedge \hat{\tau} = \mathbf{d}\mathfrak{t} \wedge (\partial_{\mathfrak{t}} \lrcorner \tau) \,.$$
  - Spatial part:
$$\vec{\tau} = \tau - \check{\tau} = \partial_{\mathfrak{t}} \lrcorner (\mathbf{d}\mathfrak{t} \wedge \tau) \,.$$

Pullback to spatial geometry on Σ

- Definition of time slice Σ_t for $t \in \mathbb{R}$:

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- Time dependent spatial geometry on Σ:
  - Pullback of (spatial) vector fields $\mathbf{X} \in \Gamma(T^s M)$:

$$X(t) = \mathfrak{s}_* \circ \mathbf{X} \circ i_t \in \Gamma(T\Sigma)\,.$$

  - Pullback of (spatial) differential forms $\boldsymbol{\tau} \in \Omega^{ks}(M)$:

$$\tau(t) = i_t^* \boldsymbol{\tau} \in \Omega^k(\Sigma)\,.$$

Tetrads and teleparallel geometry

- Define the tetrad (coframe) and its dual (frame):

$$\theta^a \in \Omega^1(M)\,, \quad \mathbf{e}_a \in \Gamma(TM)\,, \quad \mathbf{e}_a \lrcorner \theta^b = \delta_a^b\,.$$

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$$\eta_{ab} = \mathrm{diag}(-1, 1, 1, 1)\,.$$

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- Here: use Weitzenböck gauge $\omega^a{}_b \equiv 0$.

Space-time split of tetrad geometry

- Decomposition and pullback of the tetrad:

$$\boldsymbol{\theta}^a = \hat{\boldsymbol{\theta}}^a \mathbf{d}\mathfrak{t} + \vec{\boldsymbol{\theta}}^a, \quad \hat{\theta}(t) = i_t^* \hat{\boldsymbol{\theta}}^a, \quad \vec{\theta}(t) = i_t^* \vec{\boldsymbol{\theta}}^a.$$

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- Properties of the basis complement:
  - Orthonormality:

$$\eta_{ab} \xi^a \vec{\theta}^b = 0 \,, \quad \eta_{ab} \xi^a \xi^b = -1 \,.$$

  - Completeness:

$$\eta^{ab} = -\xi^a \xi^b + q^{-1}(\vec{\theta}^a, \vec{\theta}^b) \,.$$

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$$\boldsymbol{\partial}_{\mathbf{t}} = \boldsymbol{\alpha}\mathbf{N} + \boldsymbol{\beta}\,.$$

- Relation between unit normal vector field and basis complement:

$$\xi^a(t) = i_t^*(\mathbf{N} \lrcorner \boldsymbol{\theta}^a)\,.$$

Irreducible decomposition

- For a time-dependent one-form $\tau^a : \mathbb{R} \to \Omega^1(\Sigma)$ define:

$$\overset{\odot}{\tau}{}^a = -\xi^a \xi_b \tau^b \,,$$

$$\overset{\ominus}{\tau}{}^a = \frac{1}{2}\left[q^{-1}(\vec{\theta}^a, \vec{\theta}_b)\tau^b - q^{-1}(\vec{\theta}^a, \tau^b)\vec{\theta}_b\right] ,$$

$$\overset{\oplus}{\tau}{}^a = \frac{1}{2}\left[q^{-1}(\vec{\theta}^a, \vec{\theta}_b)\tau^b + q^{-1}(\vec{\theta}^a, \tau^b)\vec{\theta}_b\right] - \frac{1}{3}q^{-1}(\vec{\theta}_b, \tau^b)\vec{\theta}^a \,,$$

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- Components are orthogonal and complete:

$$\tau^a \wedge *\sigma_a = \overset{\odot}{\tau}{}^a \wedge *\overset{\odot}{\sigma}_a + \overset{\ominus}{\tau}{}^a \wedge *\overset{\ominus}{\sigma}_a + \overset{\oplus}{\tau}{}^a \wedge *\overset{\oplus}{\sigma}_a + \overset{\otimes}{\tau}{}^a \wedge *\overset{\otimes}{\sigma}_a\,.$$

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Teleparallel action

- Irreducible decomposition of the torsion:

$$\mathcal{V}^a = \frac{1}{3}\theta^a \wedge (\mathbf{e}_b \lrcorner \mathbf{T}^b)\,,$$
$$\mathcal{A}^a = \frac{1}{3}\mathbf{e}^a \lrcorner (\theta_b \wedge \mathbf{T}^b)\,,$$
$$\mathcal{T}^a = \mathbf{T}^a - \mathcal{V}^a - \mathcal{A}^a\,.$$

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- Action with three free constants C_T, C_V, C_T:

$$S[\theta^a] = \int_M \mathcal{L} = \int_M \left(C_T \mathcal{T}^a \wedge \star\mathcal{T}_a + C_V \mathcal{V}^a \wedge \star\mathcal{V}_a + C_A \mathcal{A}^a \wedge \star\mathcal{A}_a\right)\,.$$

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- Equivalent expression for the Lagrangian:

$$\mathcal{L} = \frac{2C_T + C_V}{3}\mathbf{T}^a \wedge \star\mathbf{T}_a$$
$$+ \frac{C_T - C_V}{3}(\mathbf{T}^a \wedge \theta_b) \wedge \star(\mathbf{T}^b \wedge \theta_a) + \frac{C_A - C_T}{3}(\mathbf{T}^a \wedge \theta_a) \wedge \star(\mathbf{T}^b \wedge \theta_b)\,.$$

Space-time split of the Lagrangian

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$$\mathcal{L} = \check{\mathcal{L}} = \mathbf{d}t \wedge \hat{\mathcal{L}} = \mathbf{d}t \wedge (\partial_t \lrcorner \mathcal{L}) .$$

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- Pullback to Σ:

$$\hat{\mathcal{L}} = \frac{2C_T + C_V}{3} \left\{ -\frac{1}{\alpha} \left[\dot{\vec{\theta}}^a - \mathrm{d}(\alpha\xi^a) - \pounds_\beta \vec{\theta}^a \right] \wedge * \left[\dot{\vec{\theta}}_a - \mathrm{d}(\alpha\xi_a) - \pounds_\beta \vec{\theta}_a \right] + \alpha \mathrm{d}\vec{\theta}^a \wedge * \mathrm{d}\vec{\theta}_a \right.$$
$$+ \frac{C_T - C_V}{3} \left[-\frac{1}{\alpha} \left(\dot{\vec{\theta}}^a \wedge \vec{\theta}_b + E^a{}_b \right) \wedge * \left(\dot{\vec{\theta}}^b \wedge \vec{\theta}_a + E^b{}_a \right) + \alpha (\mathrm{d}\vec{\theta}^a \wedge \vec{\theta}_b) \wedge * (\mathrm{d}\vec{\theta}^b \wedge \vec{\theta}_a) \right.$$
$$+ \frac{C_A - C_T}{3} \left[-\frac{1}{\alpha} \left(\dot{\vec{\theta}}^a \wedge \vec{\theta}_a + E^a{}_a \right) \wedge * \left(\dot{\vec{\theta}}^b \wedge \vec{\theta}_b + E^b{}_b \right) + \alpha (\mathrm{d}\vec{\theta}^a \wedge \vec{\theta}_a) \wedge * (\mathrm{d}\vec{\theta}^b \wedge \vec{\theta}_b) \right.$$

where

$$E^b{}_a = -\mathrm{d}(\alpha\xi^b) \wedge \vec{\theta}_a + \alpha\xi_a \mathrm{d}\vec{\theta}^b - (\pounds_\beta \vec{\theta}^b) \wedge \vec{\theta}_a \, .$$

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- Simple relation between velocities and momenta:

$$\overset{\odot}{\pi}_a = -\frac{2}{3}(2C_T + C_V)\overset{\odot}{v}_a \,,$$

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- Solve for velocities and use momenta instead.

General structure of the Hamiltonian

- Bare Hamiltonian $\hat{\mathcal{H}}_0 = v^a \wedge p_a - \hat{\mathcal{L}}$ takes the form

$$
\begin{aligned}
\hat{\mathcal{H}}_0 = {} & \hat{\mathcal{H}}_0[\overset{\odot}{p}] + \hat{\mathcal{H}}_0[\overset{\ominus}{p}] + \hat{\mathcal{H}}_0[\overset{\oplus}{p}] + \hat{\mathcal{H}}_0[\overset{\otimes}{p}] \\
& + \frac{C_A - C_T}{3}\alpha\left[\xi_a\xi_b \mathrm{d}\vec{\theta}^a \wedge *\mathrm{d}\vec{\theta}^b - \mathrm{d}\vec{\theta}^a \wedge \theta_a \wedge *(\mathrm{d}\vec{\theta}^b \wedge \theta_b)\right] \\
& + \frac{C_T - C_V}{3}\alpha(\vec{\theta}^\sharp_a \lrcorner \mathrm{d}\vec{\theta}^a) \wedge *(\vec{\theta}^\sharp_b \lrcorner \mathrm{d}\vec{\theta}^b) - C_T\alpha \mathrm{d}\vec{\theta}^a \wedge *\mathrm{d}\vec{\theta}_a \\
& - (\alpha\xi^a + \beta \lrcorner \vec{\theta}^a)\mathrm{d}p_a - \mathrm{d}\vec{\theta}^a \wedge (\beta \lrcorner p_a) + \mathrm{d}\left[(\alpha\xi^a + \beta \lrcorner \vec{\theta}^a)p_a\right] .
\end{aligned}
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General structure of the Hamiltonian

- Bare Hamiltonian $\hat{\mathcal{H}}_0 = v^a \wedge p_a - \hat{\mathcal{L}}$ takes the form

$$\hat{\mathcal{H}}_0 = \hat{\mathcal{H}}_0[\overset{\odot}{p}] + \hat{\mathcal{H}}_0[\overset{\ominus}{p}] + \hat{\mathcal{H}}_0[\overset{\oplus}{p}] + \hat{\mathcal{H}}_0[\overset{\otimes}{p}]$$
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Terms quadratic in the momenta

- Vector part:

$$\hat{\mathcal{H}}_0[\overset{\odot}{p}] = \begin{cases} 0 & \text{for } 2C_T + C_V = 0\,, \\ -\frac{3\alpha}{4(2C_T+C_V)}\overset{\odot}{c}_a \wedge *\overset{\odot}{c}^a & \text{otherwise.} \end{cases}$$

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Possible additional primary constraints

- Vector constraint: if $2C_T + C_V = 0$, then

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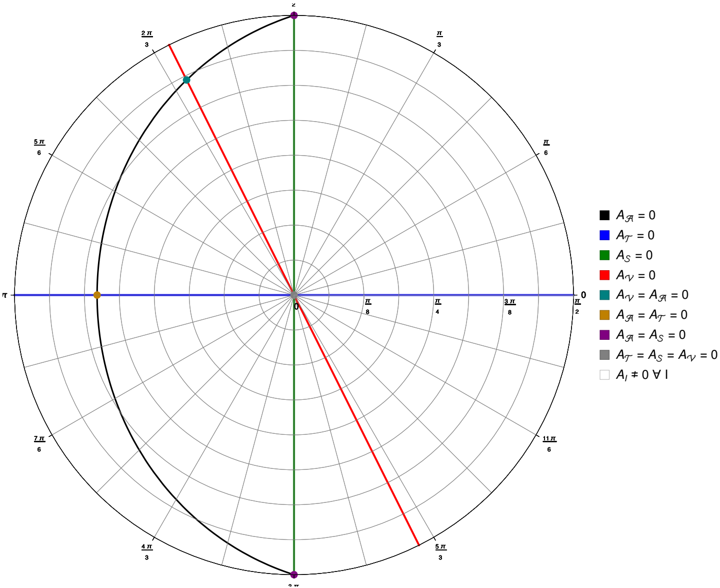
2π/3
π/3
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$A_{\mathcal{A}} = 0$
$A_{\mathcal{T}} = 0$
$A_{\mathcal{S}} = 0$
$A_{\mathcal{V}} = 0$
$A_{\mathcal{V}} = A_{\mathcal{A}} = 0$
$A_{\mathcal{A}} = A_{\mathcal{T}} = 0$
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$A_I \neq 0 \ \forall\ I$

Outline

Summary

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  - Tetrad yields $\xi^a \Rightarrow$ lapse α and shift β.
- Application to New General Relativity
  - Use eector-axial-tensor decomposition in the action.
  - Easy to derive, decompose and invert canonical momenta.
  - Kinematic Hamiltonian obtained from Similar decomposition.
  - Appearance of constraints depend on parameters.

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