Generalized scalar-torsion theories of gravity

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Motivation

- Open questions in cosmology and gravity:
  - Accelerating phases in the history of the Universe?
  - Relation between gravity and gauge theories?
  - How to quantize gravity?

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  - Based on tetrad and flat spin connection.
  - Describes gravity as gauge theory of the translation group.
  - Gravitational field strength is torsion.
  - First order action, second order field equations.
  - Spin connection as Lorentz gauge degree of freedom.

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  - Differs from non-minimal coupling to curvature.
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  - Possibly arises from more fundamental theory.
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- Arising questions:
  - Most general class of scalar-torsion gravity theories?
  - Behavior under conformal transformations?

Ingredients of scalar-torsion gravity

- Fundamental fields:
  - Coframe field $\theta^a = \theta^a{}_\mu \mathrm{d}x^\mu$.
  - Flat spin connection $\overset{\bullet}{\omega}{}^a{}_b = \overset{\bullet}{\omega}{}^a{}_{b\mu} \mathrm{d}x^\mu$.
  - N scalar fields $\phi = (\phi^A; A = 1, \ldots, N)$.
  - Arbitrary matter fields χ^I.

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- Derived quantities:
  - Frame field $e_a = e_a{}^\mu \partial_\mu$ with $\iota_{e_a}\theta^b = \delta^b_a$.
  - Metric $g_{\mu\nu} = \eta_{ab}\theta^a{}_\mu \theta^b{}_\nu$.
  - Volume form $\theta \mathrm{d}^4x = \theta^0 \wedge \theta^1 \wedge \theta^2 \wedge \theta^3$.
  - Levi-Civita connection

$$\overset{\circ}{\omega}_{ab} = -\frac{1}{2}(\iota_{e_b}\iota_{e_c}\mathrm{d}\theta_a + \iota_{e_c}\iota_{e_a}\mathrm{d}\theta_b - \iota_{e_a}\iota_{e_b}\mathrm{d}\theta_c)\theta^c\,.$$

  - Torsion $T^a = \mathrm{d}\theta^a + \overset{\bullet}{\omega}{}^a{}_b \wedge \theta^b$.

Overview

$$S_g\left[\theta^a, \overset{\bullet}{\omega}{}^a{}_b, \phi^A\right] + S_m\left[\theta^a, \phi^A, \chi^I\right]$$

$$L(T, X, Y, \phi)$$

$$\mathcal{A}(\phi)T + \mathcal{B}(\phi)\partial_\mu\phi\partial^\mu\phi + \mathcal{C}(\phi)T^\mu\partial_\mu\phi + \mathcal{V}(\phi)$$

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$$f(T, \phi) + Z(\phi)\partial_\mu\phi\partial^\mu\phi$$

General scalar-torsion gravity - action

- Structure of the action [MH '18]:

$$S\left[\theta^a, \overset{\bullet}{\omega}{}^a{}_b, \phi^A, \chi^I\right] = S_g\left[\theta^a, \overset{\bullet}{\omega}{}^a{}_b, \phi^A\right] + S_m\left[\theta^a, \phi^A, \chi^I\right] \,.$$

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- Variation of the action:
  - Gravitational part:

$$\begin{aligned} \delta S_g &= \int_M \left(\Delta_a \wedge \delta\theta^a + \frac{1}{2}\Xi_a{}^b \wedge \delta\overset{\bullet}{\omega}^a{}_b + \Phi_A \wedge \delta\phi^A\right) \\ &= \int_M \left(\Upsilon_a \wedge \delta\theta^a + \Pi_a \wedge \delta T^a + \Phi_A \wedge \delta\phi^A\right) . \end{aligned}$$

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  - Matter part:

$$\delta S_m = \int_M \left(\Sigma_a \wedge \delta\theta^a + \Psi_A \wedge \delta\phi^A + \Omega_I \wedge \delta\chi^I\right) .$$

General scalar-torsion gravity - field equations

- Local Lorentz invariance:
  - Gravitational part:

$$\Upsilon^{[a} \wedge \theta^{b]} + \Pi^{[a} \wedge T^{b]} = 0 \quad \Leftrightarrow \quad \Delta^{[a} \wedge \theta^{b]} - \frac{1}{2}\overset{\bullet}{\mathrm{D}}\Xi^{ab} = 0\,.$$

  - Matter part: $\Sigma^{[a} \wedge \theta^{b]} = 0$.

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- Field equations:
  - Tetrad field equations:

$$\Delta_a + \Sigma_a = 0 \quad \Leftrightarrow \quad \Upsilon_a - \overset{\bullet}{\mathrm{D}}\Pi_a + \Sigma_a = 0\,.$$

  - Antisymmetric part $\equiv$ connection field equations:

$$\overset{\bullet}{\mathrm{D}}\Xi^{ab} = 0 \quad \Leftrightarrow \quad \overset{\bullet}{\mathrm{D}}\Pi^{[a} \wedge \theta^{b]} + \Pi^{[a} \wedge T^{b]} = 0\,.$$

  - Scalar field equations: $\Phi_A + \Psi_A = 0$.
  - Matter field equations: $\Omega_I = 0$.

$L(T, X, Y, \phi)$ theory - action

- Gravitational part of the action [MH, C. Pfeifer '18]:

$$S_g\left[\theta^a, \overset{\bullet}{\omega}{}^a{}_b, \phi^A\right] = \int_M L\left(T, X^{AB}, Y^A, \phi^A\right)\theta \mathrm{d}^4x\,.$$

  - Torsion scalar: $T = \frac{1}{2}T^\rho{}_{\mu\nu}S_\rho{}^{\mu\nu}$.
  - Superpotential:

$$S_{\rho\mu\nu} = \frac{1}{2}\left(T_{\nu\mu\rho} + T_{\rho\mu\nu} - T_{\mu\nu\rho}\right) - g_{\rho\mu}T^\sigma{}_{\sigma\nu} + g_{\rho\nu}T^\sigma{}_{\sigma\mu}\,.$$

  - Scalar field kinetic term: $X^{AB} = -\frac{1}{2}g^{\mu\nu}\phi^A_{,\mu}\phi^B_{,\nu}$.
  - Kinetic coupling term: $Y^A = T_\mu{}^{\mu\nu}\phi^A_{,\nu}$.

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- Matter action variation expressed in components:

$$\delta S_m[\theta^a, \phi^A, \chi^I] = \int_M \left(\Theta_a{}^\mu\delta\theta^a{}_\mu + \vartheta_A\delta\phi^A + \varpi_I\delta\chi^I\right)\theta \mathrm{d}^4x\,.$$

$L(T, X, Y, \phi)$ theory - field equations

- Symmetric part of tetrad equations:

$$\overset{\circ}{\nabla}_{(\mu}\left(L_{Y^A}\phi^A_{,\nu)}\right) - \overset{\circ}{\nabla}_{\sigma}\left(L_{Y^A}\phi^A_{,\rho}\right)g^{\rho\sigma}g_{\mu\nu} + L_{Y^A}\left(T_{(\mu\nu)}{}^{\rho}\phi^A_{,\rho} + T^{\rho}{}_{\rho(\mu}\phi^A_{,\nu)}\right)$$
$$- Lg_{\mu\nu} - 2\overset{\circ}{\nabla}_{\rho}\left(L_T S_{(\mu\nu)}{}^{\rho}\right) + L_T S_{(\mu}{}^{\rho\sigma}T_{\nu)\rho\sigma} - L_{X^{AB}}\phi^A_{,\mu}\phi^B_{,\nu} = \Theta_{\mu\nu}\,.$$

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- Antisymmetric part of tetrad equations $\equiv$ connection equations:

$$3\partial_{[\rho}L_T T^{\rho}{}_{\mu\nu]} + \partial_{[\mu}L_{Y^A}\phi^A_{,\nu]} - \frac{3}{2}L_{Y^A}T^{\rho}{}_{[\mu\nu}\phi^A_{,\rho]} = 0\,.$$

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- Scalar field equations:

$$g^{\mu\nu}\overset{\circ}{\nabla}_{\mu}\left(L_{Y^A}T^{\rho}{}_{\rho\nu} - L_{X^{AB}}\phi^B_{,\nu}\right) - L_{\phi^A} = \vartheta_A\,.$$

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- Matter field equations: $\varpi_I = 0$.

"Scalar-curvature"-like class - action

- Action [MH '18]:
  - Gravitational part:

$$S_g\left[\theta^a, \overset{\bullet}{\omega}{}^a{}_b, \phi^A\right] = \frac{1}{2\kappa^2}\int_M \left[-\mathcal{A}(\phi)T + 2\mathcal{B}_{AB}(\phi)X^{AB} + 2\mathcal{C}_A(\phi)Y^A - 2\kappa^2\mathcal{V}(\phi)\right]\theta \mathrm{d}^4x\,.$$

  - Matter part:

$$S_m[\theta^a, \phi, \chi^I] = S_m^{\mathfrak{J}}\left[e^{\alpha(\phi)}\theta^a, \chi^I\right]\,.$$

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- Free functions $\mathcal{A}, \mathcal{B}_{AB}, \mathcal{C}_A, \mathcal{V}, \alpha$ of scalar fields.

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- Free functions $\mathcal{A}, \mathcal{B}_{AB}, \mathcal{C}_A, \mathcal{V}, \alpha$ of scalar fields.
- $\mathcal{C}_A \equiv -\mathcal{A}_{,A} \Leftrightarrow$ theory reduces to scalar-curvature gravity.

“Scalar-curvature”-like class - field equation

- Symmetric part of the tetrad equations:

$$\left(\mathcal{A}_{,A} + \mathcal{C}_A\right) S_{(\mu\nu)}{}^{\rho}\phi^A_{,\rho} + \mathcal{A}\left(\overset{\circ}{R}_{\mu\nu} - \frac{1}{2}\overset{\circ}{R}g_{\mu\nu}\right) + \left(\frac{1}{2}\mathcal{B}_{AB} - \mathcal{C}_{(A,B)}\right)\phi^A_{,\rho}\phi^B_{,\sigma}g^{\rho\sigma}g_{\mu\nu}$$
$$- \left(\mathcal{B}_{AB} - \mathcal{C}_{(A,B)}\right)\phi^A_{,\mu}\phi^B_{,\nu} + \mathcal{C}_A\left(\overset{\circ}{\nabla}_{\mu}\overset{\circ}{\nabla}_{\nu}\phi^A - \overset{\circ}{\square}\phi^A g_{\mu\nu}\right) + \kappa^2\mathcal{V}g_{\mu\nu} = \kappa^2\Theta_{\mu\nu}\,,$$

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- Antisymmetric part of the tetrad field equations:

$$3(\mathcal{A}_{,A} + \mathcal{C}_A)\,T^{\rho}{}_{[\mu\nu}\phi^A_{,\rho]} + 2\mathcal{C}_{[A,B]}\phi^A_{,\mu}\phi^B_{,\nu} = 0\,.$$

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- Scalar field equation:

$$\frac{1}{2}\mathcal{A}_{,A}T-\mathcal{B}_{AB}\overset{\circ}{\square}\phi^B-\left(\mathcal{B}_{AB,C}-\frac{1}{2}\mathcal{B}_{BC,A}\right)g^{\mu\nu}\phi^B_{,\mu}\phi^C_{,\nu}$$
$$+\mathcal{C}_A\overset{\circ}{\nabla}_\mu T_\nu{}^{\nu\mu}+2\mathcal{C}_{[A,B]}T_\mu{}^{\mu\nu}\phi^B_{,\nu}+\kappa^2\mathcal{V}_{,A}=\kappa^2\alpha_{,A}\Theta\,.$$

“Scalar-curvature”-like class - conformal transf.

- Conformal transformation and scalar field redefinition:

$$\bar{\theta}^a = e^{\gamma(\phi)}\theta^a\,, \quad \bar{e}_a = e^{-\gamma(\phi)}e_a\,, \quad \bar{\phi}^A = f^A(\phi)\,.$$

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- Transformation of geometry:

$$\bar{T} = e^{-2\gamma}\left(T + 4\gamma_{,A}Y^A + 12\gamma_{,A}\gamma_{,B}X^{AB}\right)\,, \quad \bar{\phi}^A = f^A\,,$$

$$\bar{X}^{AB} = e^{-2\gamma}\frac{\partial\bar{\phi}^A}{\partial\phi^C}\frac{\partial\bar{\phi}^B}{\partial\phi^D}X^{CD}\,, \quad \bar{Y}^A = e^{-2\gamma}\frac{\partial\bar{\phi}^A}{\partial\phi^B}\left(Y^B + 6\gamma_{,C}X^{BC}\right)\,,$$

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- Transformation of parameter functions to preserve action:

$$\mathcal{A} = e^{2\gamma}\bar{\mathcal{A}}\,,$$
$$\mathcal{B} = e^{2\gamma}\left(\bar{\mathcal{B}}f'^2 - 6\bar{\mathcal{A}}\gamma'^2 + 6\bar{\mathcal{C}}f'\gamma'\right)\,,$$
$$\mathcal{C} = e^{2\gamma}\left(\bar{\mathcal{C}}f' - 2\bar{\mathcal{A}}\gamma'\right)\,,$$
$$\mathcal{V} = e^{4\gamma}\bar{\mathcal{V}}\,,$$
$$\alpha = \bar{\alpha} + \gamma\,.$$

“Scalar-curvature”-like class - invariants

- Quantities invariant under conformal transformations γ:
  - “Scalar” quantities:

$$\mathcal{I}_1 = \frac{e^{2\alpha}}{\mathcal{A}}\,, \quad \mathcal{I}_2 = \frac{\mathcal{V}}{\mathcal{A}^2}\,.$$

  - “Covector” quantities:

$$\mathcal{H}_A = \frac{\mathcal{C}_A + \mathcal{A}_{,A}}{2\mathcal{A}}\,, \quad \mathcal{K}_A = \frac{\mathcal{C}_A + 2\alpha_{,A}\mathcal{A}}{2e^{2\alpha}}\,.$$

  - “Metric” quantities:

$$\mathcal{F}_{AB} = \frac{2\mathcal{A}\mathcal{B}_{AB} - 6\mathcal{A}_{,(A}\mathcal{C}_{B)} - 3\mathcal{A}_{,A}\mathcal{A}_{,B}}{4\mathcal{A}^2}\,,$$

$$\mathcal{G}_{AB} = \frac{\mathcal{B}_{AB} - 6\alpha_{,(A}\mathcal{C}_{B)} - 6\alpha_{,A}\alpha_{,B}\mathcal{A}}{2e^{2\alpha}}\,.$$

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- Covariance under scalar field redefinitions:

$$\bar{\mathcal{I}}_{1,2} = \mathcal{I}_{1,2}\,, \quad (\bar{\mathcal{H}}, \bar{\mathcal{K}})_A = \frac{\partial\phi^B}{\partial\bar{\phi}^A}(\mathcal{H}, \mathcal{K})_B\,, \quad (\bar{\mathcal{F}}, \bar{\mathcal{G}})_{AB} = \frac{\partial\phi^C}{\partial\bar{\phi}^A}\frac{\partial\phi^D}{\partial\bar{\phi}^B}(\mathcal{F}, \mathcal{G})_{CD}\,.$$

“Scalar-curvature”-like class - special frames

- Jordan frame: minimal coupling to matter.

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- “Debraiding frame” (for $\mathcal{H}_{[A,B]} \equiv 0$): minimal coupling to torsion.

$$\left(\ln\mathcal{A}^{\mathfrak{D}}\right)_{,A} = 2\mathcal{H}_A\,, \quad \left(\ln\mathcal{B}^{\mathfrak{D}}\right)^{A}{}_{B,C} = [\ln(\mathcal{F} + 3\mathcal{H}\otimes\mathcal{H})]^{A}{}_{B,C} + 2\delta^A_B\mathcal{H}_C\,,$$

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Scalar-torsion gravity without derivative coupling

- Gravitational action [MH, L. Järv, U. Ualikhanova '18]:

$$S = \frac{1}{2\kappa^2}\int_M \left[f(T,\phi) + Z_{AB}(\phi)g^{\mu\nu}\phi^A_{,\mu}\phi^B_{,\nu}\right]\theta d^4x + S_m[\theta^a, \chi^I]\,.$$

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  - Symmetric part of the tetrad field equations:

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$$f_{\phi^A} - \left(2Z_{AB,\phi^C} - Z_{BC,\phi^A}\right)g^{\mu\nu}\phi^B_{,\mu}\phi^C_{,\nu} - 2Z_{AB}\overset{\circ}{\Box}\phi^B = 0\,.$$

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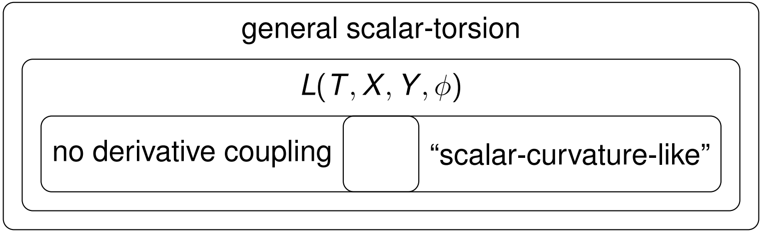
- Equation possesses generic solutions with spacetime symmetries.

Conclusion

- Summary:
  - Four new classes of scalar-torsion theories.



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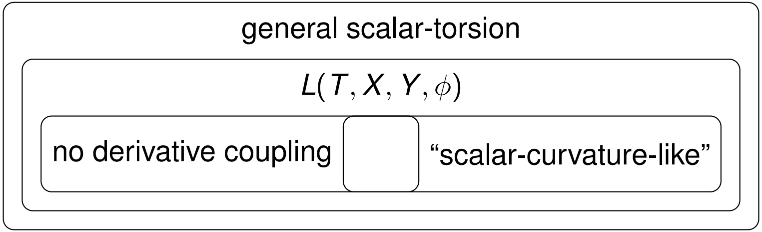

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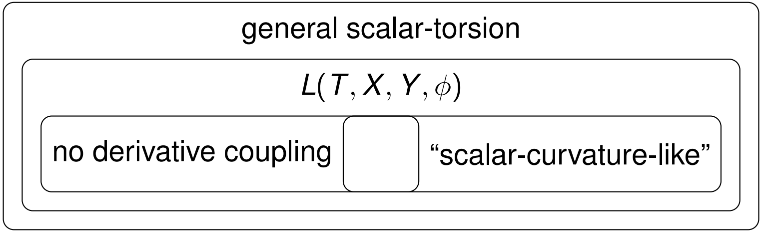

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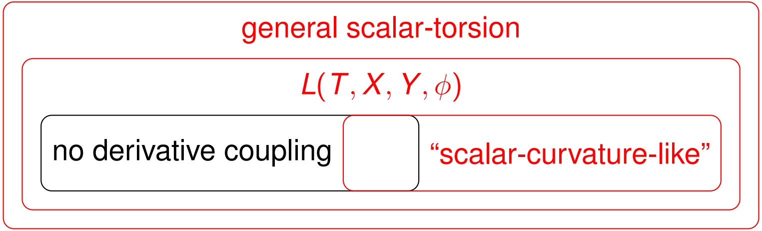

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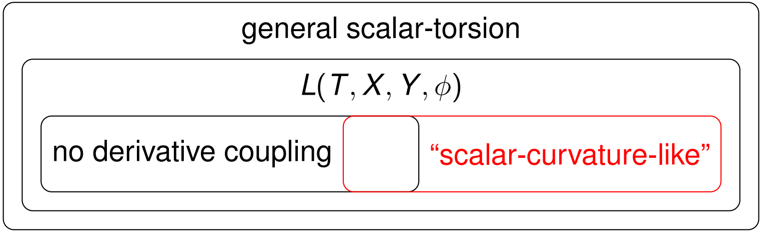

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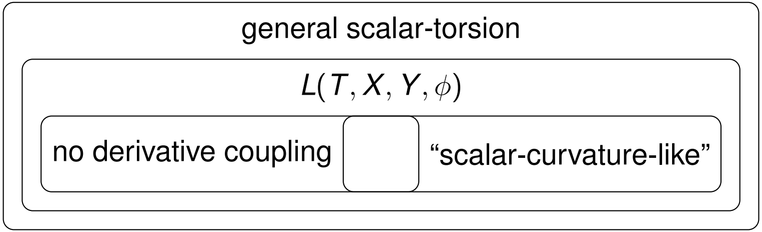

- Outlook - analyze various aspects of these theories:
  - Cosmological dynamics / dynamical systems analysis.
  - Post-Newtonian limit.
  - Gravitational waves - speed and polarisations.
  - Hamiltonian formulation - degrees of freedom.

Literature

- MH, L. Järv, U. Ualikhanova; Covariant formulation of scalar-torsion gravity; Phys. Rev. D **97** (2018) 104011 [arXiv:1801.05786].
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