Field transformations and invariant quantities in scalar-teleparallel theories of gravity

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Outline

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- Constraints:
  1. General teleparallel gravity: impose only vanishing curvature

$$R^{\mu}{}_{\nu\rho\sigma} = \partial_\rho \Gamma^{\mu}{}_{\nu\sigma} - \partial_\sigma \Gamma^{\mu}{}_{\nu\rho} + \Gamma^{\mu}{}_{\tau\rho}\Gamma^{\tau}{}_{\nu\sigma} - \Gamma^{\mu}{}_{\tau\sigma}\Gamma^{\tau}{}_{\nu\rho} \equiv 0\,. \tag{2}$$

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  3. Metric teleparallel gravity: vanishing curvature and nonmetricity:

$$Q_{\mu\nu\rho} = \nabla_\mu g_{\nu\rho} = \partial_\mu g_{\nu\rho} - \Gamma^{\sigma}{}_{\nu\mu} g_{\sigma\rho} - \Gamma^{\sigma}{}_{\rho\mu} g_{\nu\sigma} \equiv 0\,. \tag{4}$$

Matter action

- Assume structure of matter action with scalar field coupling:

$$S_{\mathrm{m}}\left[g_{\mu\nu}, \Gamma^{\mu}{}_{\nu\rho}, \phi, \chi^I\right] = \hat{S}_{\mathrm{m}}\left[g_{\mu\nu}e^{2\alpha(\phi)}, \Gamma^{\mu}{}_{\nu\rho} + \beta(\phi)\delta^{\mu}_{\nu}\phi_{,\rho}, \chi^I\right]. \tag{5}$$

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- General form of matter action variation:

$$\delta S_{\mathrm{m}} = \int_M \left(\frac{1}{2}\Theta^{\mu\nu}\delta g_{\mu\nu} + H_{\mu}{}^{\nu\rho}\delta\Gamma^{\mu}{}_{\nu\rho} + \Phi\delta\phi + \varpi_I\delta\chi^I\right)\sqrt{-g}\mathrm{d}^4x\,. \tag{6}$$

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$\Rightarrow$ Scalar field variation is determined by coupling functions:

$$\Phi = \alpha'\Theta - \beta\mathring{\nabla}_{\nu}H_{\mu}{}^{\mu\nu}. \tag{7}$$

General teleparallel gravity

- Decomposition of general scalar-teleparallel gravity action:

$$S_{\text{gen}} = S_G + S_{\mathfrak{r}} + S_{\text{m}} \,. \tag{8}$$

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$$S_G[g_{\mu\nu}, \Gamma^{\mu}{}_{\nu\rho}, \phi^a] = \frac{1}{2\kappa^2} \int_M \mathrm{d}^4x \sqrt{-g} \left[-\mathcal{A}(\phi)G + 2\mathcal{B}(\phi)X + 2\mathcal{C}(\phi)U + 2\mathcal{D}(\phi)V + 2\mathcal{E}(\phi)W - 2\kappa^2\mathcal{V}(\phi)\right] \,. \tag{9}$$

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  - Action terms:

$$G = 2M^{\mu}{}_{\rho[\mu}M^{\rho\nu}{}_{\nu]} \,, \quad X = -\frac{1}{2}g^{\mu\nu}\phi_{,\mu}\phi_{,\nu} \,, \quad U = T_{\mu}{}^{\mu\nu}\phi_{,\nu} \,, \quad V = Q^{\nu\mu}{}_{\mu}\phi_{,\nu} \,, \quad W = Q_{\mu}{}^{\mu\nu}\phi_{,\nu} \,. \tag{10}$$

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  - Scalar field coupling functions determine particular theory.

- Lagrange multiplier with tensor density $\mathfrak{r}_{\mu}{}^{\nu\rho\sigma}$ to impose vanishing curvature:

$$S_{\mathfrak{r}}[\mathfrak{r}_{\mu}{}^{\nu\rho\sigma}, \Gamma^{\mu}{}_{\nu\rho}] = \int_M \mathfrak{r}_{\mu}{}^{\nu\rho\sigma} R^{\mu}{}_{\nu\rho\sigma}\mathrm{d}^4x \,. \tag{11}$$

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$$S_{\mathfrak{t}}[\mathfrak{t}_{\mu}{}^{\nu\rho}, \Gamma^{\mu}{}_{\nu\rho}] = \int_M \mathfrak{t}_{\mu}{}^{\nu\rho} T^{\mu}{}_{\nu\rho} \mathrm{d}^4x \,. \tag{13}$$

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- Nonmetricity scalar:

$$Q = \frac{1}{4} Q^{\mu\nu\rho} Q_{\mu\nu\rho} - \frac{1}{2} Q^{\mu\nu\rho} Q_{\rho\mu\nu} - \frac{1}{4} Q^{\rho\mu}{}_{\mu} Q_{\rho\nu}{}^{\nu} + \frac{1}{2} Q^{\mu}{}_{\mu\rho} Q^{\rho\nu}{}_{\nu} \,. \tag{15}$$

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$$S_{\mathfrak{q}}[\mathfrak{q}^{\mu\nu\rho}, g_{\mu\nu}, \Gamma^{\mu}{}_{\nu\rho}] = \int_M \mathfrak{q}^{\mu\nu\rho} Q_{\mu\nu\rho} \mathrm{d}^4x \,. \tag{17}$$

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- Torsion scalar:

$$T = \frac{1}{4}T^{\mu\nu\rho}T_{\mu\nu\rho} + \frac{1}{2}T^{\mu\nu\rho}T_{\rho\nu\mu} - T^{\mu}{}_{\mu\rho}T_{\nu}{}^{\nu\rho}\,. \tag{19}$$

Outline

Transformation of scalar-teleparallel geometry

- Transformation of dynamical fields:
  - Scalar field:
$$\bar{\phi} = f(\phi)\,. \tag{20}$$
  - Metric:
$$\bar{g}_{\mu\nu} = g_{\mu\nu} e^{2\gamma(\phi)}\,. \tag{21}$$
  - Affine connection:
$$\bar{\Gamma}^{\mu}{}_{\nu\rho} = \Gamma^{\mu}{}_{\nu\rho} + \zeta(\phi)\delta^{\mu}_{\nu}\phi_{,\rho}\,. \tag{22}$$

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- Transformation of characteristic tensor fields:
  - Curvature remains vanishing:
$$\bar{R}^{\mu}{}_{\nu\rho\sigma} = \partial_{\rho}\bar{\Gamma}^{\mu}{}_{\nu\sigma} - \partial_{\sigma}\bar{\Gamma}^{\mu}{}_{\nu\rho} + \bar{\Gamma}^{\mu}{}_{\tau\rho}\bar{\Gamma}^{\tau}{}_{\nu\sigma} - \bar{\Gamma}^{\mu}{}_{\tau\sigma}\bar{\Gamma}^{\tau}{}_{\nu\rho} = R^{\mu}{}_{\nu\rho\sigma} \equiv 0\,. \tag{23}$$
  - Torsion:
$$\bar{T}^{\mu}{}_{\nu\rho} = \bar{\Gamma}^{\mu}{}_{\rho\nu} - \bar{\Gamma}^{\mu}{}_{\nu\rho} = T^{\mu}{}_{\nu\rho} - 2\zeta(\phi)\delta^{\mu}_{[\nu}\phi_{,\rho]}\,. \tag{24}$$
  - Nonmetricity:
$$\bar{Q}_{\mu\nu\rho} = \bar{\nabla}_{\mu}\bar{g}_{\nu\rho} = e^{2\gamma(\phi)}\left[Q_{\mu\nu\rho} - 2(\zeta(\phi) - \gamma'(\phi))g_{\nu\rho}\phi_{,\mu}\right]. \tag{25}$$

Transformation of matter action

- Equivalence between transformed and original matter actions:

$$\bar{S}_{\text{m}}\left[\bar{g}_{\mu\nu}, \bar{\Gamma}^{\mu}{}_{\nu\rho}, \bar{\phi}, \chi^{I}\right] = S_{\text{m}}\left[g_{\mu\nu}, \Gamma^{\mu}{}_{\nu\rho}, \phi, \chi^{I}\right] . \tag{26}$$

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- Transformed action must have same structure with new functions $\bar{\alpha}, \bar{\beta}$:

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⇒ Action is form-invariant with parameter functions transforming as

$$\alpha = \bar{\alpha} + \gamma\,, \quad \beta = f'\bar{\beta} + \zeta\,. \tag{28}$$

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⇒ Transformation of matter action variation:

$$\Theta^{\mu\nu} = e^{6\gamma}\bar{\Theta}^{\mu\nu}\,, \quad H_{\mu}{}^{\nu\rho} = e^{4\gamma}\bar{H}_{\mu}{}^{\nu\rho}\,, \quad \Phi = e^{4\gamma}\left(\gamma'\bar{\Theta}-\zeta\mathring{\bar{\nabla}}_{\nu}\bar{H}_{\mu}{}^{\mu\nu}+f'\bar{\Phi}\right), \quad \varpi_{I} = e^{4\gamma}\bar{\varpi}_{I}\,. \tag{29}$$

Transformation of general teleparallel gravity

- Transformation of Lagrange multiplier term:

$$\bar{S}_{\mathfrak{r}}[\bar{\mathfrak{r}}_{\mu}{}^{\nu\rho\sigma}, \bar{\Gamma}^{\mu}{}_{\nu\rho}] = \int_M \bar{\mathfrak{r}}_{\mu}{}^{\nu\rho\sigma} \bar{R}^{\mu}{}_{\nu\rho\sigma} \mathrm{d}^4x = \int_M \bar{\mathfrak{r}}_{\mu}{}^{\nu\rho\sigma} R^{\mu}{}_{\nu\rho\sigma} \mathrm{d}^4x \,. \tag{30}$$

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- Transformation of general teleparallel gravity action:

$$\bar{S}_G[\bar{g}_{\mu\nu}, \bar{\Gamma}^{\mu}{}_{\nu\rho}, \bar{\phi}] = S_G[g_{\mu\nu}, \Gamma^{\mu}{}_{\nu\rho}, \phi]\,. \tag{31}$$

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⇒ Action is form-invariant with parameter functions transforming as

$$\mathcal{A} = e^{2\gamma}\bar{\mathcal{A}} \,, \tag{32a}$$

$$\mathcal{B} = e^{2\gamma}[f'^2\bar{\mathcal{B}} - 6\gamma'^2\bar{\mathcal{A}} + 6\zeta f'\bar{\mathcal{C}} + 4(\zeta - \gamma')f'(4\bar{\mathcal{D}} + \bar{\mathcal{E}})] \,, \tag{32b}$$

$$\mathcal{C} = e^{2\gamma}(f'\bar{\mathcal{C}} - 2\gamma'\bar{\mathcal{A}}) \,, \tag{32c}$$

$$\mathcal{D} = e^{2\gamma}(f'\bar{\mathcal{D}} + \gamma'\bar{\mathcal{A}}) \,, \tag{32d}$$

$$\mathcal{E} = e^{2\gamma}(f'\bar{\mathcal{E}} - \gamma'\bar{\mathcal{A}}) \,, \tag{32e}$$

$$\mathcal{V} = e^{4\gamma}\bar{\mathcal{V}} \,. \tag{32f}$$

Symmetric teleparallel gravity?

- Lagrange multiplier term is not invariant:

$$\bar{S}_{\mathfrak{t}}[\bar{\mathfrak{t}}_{\mu}{}^{\nu\rho}, \bar{\Gamma}^{\mu}{}_{\nu\rho}] = \int_M \bar{\mathfrak{t}}_{\mu}{}^{\nu\rho} \bar{T}^{\mu}{}_{\nu\rho} \mathrm{d}^4x = \int_M \bar{\mathfrak{t}}_{\mu}{}^{\nu\rho} \left(T^{\mu}{}_{\nu\rho} - 2\zeta\delta^{\mu}_{[\nu}\phi_{,\rho]} \right) \mathrm{d}^4x \,. \tag{33}$$

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$$\bar{S}_{\mathfrak{t}}[\bar{\mathfrak{t}}_{\mu}{}^{\nu\rho}, \bar{\Gamma}^{\mu}{}_{\nu\rho}] = \int_M \bar{\mathfrak{t}}_{\mu}{}^{\nu\rho} \bar{T}^{\mu}{}_{\nu\rho} \mathrm{d}^4x = \int_M \bar{\mathfrak{t}}_{\mu}{}^{\nu\rho} \left(T^{\mu}{}_{\nu\rho} - 2\zeta\delta^{\mu}_{[\nu}\phi_{,\rho]} \right) \mathrm{d}^4x \,. \quad (33)$$

- ⇝ Generalized Lagrange multiplier with non-vanishing torsion:

$$S'_{\mathfrak{t}}[\mathfrak{t}_{\mu}{}^{\nu\rho}, \Gamma^{\mu}{}_{\nu\rho}, \phi] = \int_M \mathfrak{t}_{\mu}{}^{\nu\rho} \left[T^{\mu}{}_{\nu\rho} - 2\mathcal{T}(\phi)\delta^{\mu}_{[\nu}\phi_{,\rho]} \right] \mathrm{d}^4x \,. \quad (34)$$

Symmetric teleparallel gravity?

- ↯ Lagrange multiplier term is not invariant:

$$\bar{S}_{\mathfrak{t}}[\bar{\mathfrak{t}}_{\mu}{}^{\nu\rho}, \bar{\Gamma}^{\mu}{}_{\nu\rho}] = \int_M \bar{\mathfrak{t}}_{\mu}{}^{\nu\rho} \bar{T}^{\mu}{}_{\nu\rho} \mathrm{d}^4x = \int_M \bar{\mathfrak{t}}_{\mu}{}^{\nu\rho} \left(T^{\mu}{}_{\nu\rho} - 2\zeta\delta^{\mu}_{[\nu}\phi_{,\rho]} \right) \mathrm{d}^4x \,. \tag{33}$$

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- ⇒ Lagrange multiplier is form-invariant with transformation

$$\mathfrak{t}_{\mu}{}^{\nu\rho} = \bar{\mathfrak{t}}_{\mu}{}^{\nu\rho} \,, \quad \mathcal{T} = f'\bar{\mathcal{T}} + \zeta \,. \tag{35}$$

Generalized symmetric teleparallel gravity

- Relations imposed by generalized Lagrange multiplier:

$$G = Q + 2\mathcal{T}(V - W) + 12\mathcal{T}^2 X\,, \quad U = -6\mathcal{T}X\,. \tag{36}$$

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⇒ Generalized scalar-nonmetricity gravity action:

$$S'_Q[g_{\mu\nu}, \Gamma^\mu{}_{\nu\rho}, \phi] = \frac{1}{2\kappa^2}\mathrm{d}^4x \int_M \sqrt{-g} \left[-\underset{Q}{\mathcal{A}}(\phi)Q + 2\underset{Q}{\mathcal{B}}(\phi)X + 2\underset{Q}{\mathcal{D}}(\phi)V + 2\underset{Q}{\mathcal{E}}(\phi)W - 2\kappa^2\underset{Q}{\mathcal{V}}(\phi)\right]\,. \tag{37}$$

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⇒ New parameter functions:

$$\underset{Q}{\mathcal{A}} = \mathcal{A}\,, \quad \underset{Q}{\mathcal{B}} = \mathcal{B} - 6\mathcal{C}\mathcal{T} - 6\mathcal{A}\mathcal{T}^2\,, \quad \underset{Q}{\mathcal{D}} = \mathcal{D} - \mathcal{A}\mathcal{T}\,, \quad \underset{Q}{\mathcal{E}} = \mathcal{E} + \mathcal{A}\mathcal{T}\,, \quad \underset{Q}{\mathcal{V}} = \mathcal{V}\,. \tag{38}$$

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$\Rightarrow$ Complete symmetric teleparallel gravity action:

$$S'_{\mathrm{sym}} = S'_Q + S_{\mathfrak{r}} + S'_{\mathfrak{t}} + S_{\mathrm{m}}\,. \tag{39}$$

Transformation of generalized symmetric teleparallel gravity

- Transformation of generalized symmetric teleparallel gravity action:

$$\bar{S}'_Q[\bar{g}_{\mu\nu}, \bar{\Gamma}^{\mu}{}_{\nu\rho}, \bar{\phi}] = S'_Q[g_{\mu\nu}, \Gamma^{\mu}{}_{\nu\rho}, \phi]\,. \tag{40}$$

Transformation of generalized symmetric teleparallel gravity

- Transformation of generalized symmetric teleparallel gravity action:

$$\bar{S}'_Q[\bar{g}_{\mu\nu}, \bar{\Gamma}^{\mu}{}_{\nu\rho}, \bar{\phi}] = S'_Q[g_{\mu\nu}, \Gamma^{\mu}{}_{\nu\rho}, \phi]\,. \tag{40}$$

⇒ Action is form-invariant with parameter functions transforming as

$$\underset{Q}{\mathcal{A}} = e^{2\gamma}\underset{Q}{\bar{\mathcal{A}}}\,, \tag{41a}$$

$$\underset{Q}{\mathcal{B}} = e^{2\gamma}\left[f'^2\underset{Q}{\bar{\mathcal{B}}} - 6(\zeta - \gamma')^2\underset{Q}{\bar{\mathcal{A}}} + 4(\zeta - \gamma')f'\left(4\underset{Q}{\bar{\mathcal{D}}} + \underset{Q}{\bar{\mathcal{E}}}\right)\right], \tag{41b}$$

$$\underset{Q}{\mathcal{D}} = e^{2\gamma}\left[f'\underset{Q}{\bar{\mathcal{D}}} - (\zeta - \gamma')\underset{Q}{\bar{\mathcal{A}}}\right], \tag{41c}$$

$$\underset{Q}{\mathcal{E}} = e^{2\gamma}\left[f'\underset{Q}{\bar{\mathcal{E}}} + (\zeta - \gamma')\underset{Q}{\bar{\mathcal{A}}}\right], \tag{41d}$$

$$\underset{Q}{\mathcal{V}} = e^{4\gamma}\underset{Q}{\bar{\mathcal{V}}}\,. \tag{41e}$$

Metric teleparallel gravity?

- ↯ Lagrange multiplier term is not invariant:

$$\bar{S}_{\mathfrak{q}}[\bar{\mathfrak{q}}^{\mu\nu\rho}, \bar{g}_{\mu\nu}, \bar{\Gamma}^{\mu}{}_{\nu\rho}] = \int_M \bar{\mathfrak{q}}^{\mu\nu\rho} \bar{Q}_{\mu\nu\rho} \mathrm{d}^4x = \int_M \bar{\mathfrak{q}}^{\mu\nu\rho} \left[Q_{\mu\nu\rho} - 2(\zeta - \gamma') g_{\nu\rho} \phi_{,\mu} \right] e^{2\gamma} \mathrm{d}^4x \,. \quad (42)$$

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$$S'_{\mathfrak{q}}[\mathfrak{q}^{\mu\nu\rho}, g_{\mu\nu}, \Gamma^{\mu}{}_{\nu\rho}, \phi] = \int_M \mathfrak{q}^{\mu\nu\rho}\left[Q_{\mu\nu\rho} - 2\mathcal{Q}(\phi)g_{\nu\rho}\phi_{,\mu}\right]\mathrm{d}^4x\,, \quad (43)$$

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⇒ Lagrange multiplier is form-invariant with transformation

$$\mathfrak{q}^{\mu\nu\rho} = e^{2\gamma}\bar{\mathfrak{q}}^{\mu\nu\rho}\,, \quad \mathcal{Q} = f'\bar{\mathcal{Q}} + \zeta - \gamma'\,. \quad (44)$$

Generalized metric teleparallel gravity

- Relations imposed by generalized Lagrange multiplier:

$$G = T + 4\mathcal{Q}U + 12\mathcal{Q}^2 X\,, \quad V = -16\mathcal{Q}X\,, \quad W = -4\mathcal{Q}X\,. \tag{45}$$

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$$S'_T[g_{\mu\nu}, \Gamma^{\mu}{}_{\nu\rho}, \phi] = \frac{1}{2\kappa^2}\mathrm{d}^4x \int_M \sqrt{-g} \left[-\underset{T}{\mathcal{A}}(\phi)T + 2\underset{T}{\mathcal{B}}(\phi)X + 2\underset{T}{\mathcal{C}}(\phi)U - 2\kappa^2\underset{T}{\mathcal{V}}(\phi)\right]. \tag{46}$$

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⇒ New parameter functions:

$$\underset{T}{\mathcal{A}} = \mathcal{A}\,, \quad \underset{T}{\mathcal{B}} = \mathcal{B} - 16\mathcal{D}\mathcal{Q} - 4\mathcal{E}\mathcal{Q} - 6\mathcal{A}\mathcal{Q}^2\,, \quad \underset{T}{\mathcal{C}} = \mathcal{C} - 2\mathcal{A}\mathcal{Q}\,, \quad \underset{T}{\mathcal{V}} = \mathcal{V}\,. \tag{47}$$

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⇒ Complete symmetric teleparallel gravity action:

$$S'_{\mathrm{met}} = S'_T + S_{\mathfrak{t}} + S'_{\mathfrak{q}} + S_{\mathrm{m}}\,. \tag{48}$$

Transformation of generalized metric teleparallel gravity

- Transformation of generalized metric teleparallel gravity action:

$$\bar{S}'_T[\bar{g}_{\mu\nu}, \bar{\Gamma}^{\mu}{}_{\nu\rho}, \bar{\phi}] = S'_T[g_{\mu\nu}, \Gamma^{\mu}{}_{\nu\rho}, \phi]\,. \tag{49}$$

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⇒ Action is form-invariant with parameter functions transforming as

$$\underset{T}{\mathcal{A}} = e^{2\gamma} \underset{T}{\bar{\mathcal{A}}} \,, \tag{50a}$$

$$\underset{T}{\mathcal{B}} = e^{2\gamma} \left(f'^2 \underset{T}{\bar{\mathcal{B}}} - 6\zeta^2 \underset{T}{\bar{\mathcal{A}}} + 6\zeta f' \underset{T}{\bar{\mathcal{C}}} \right) \,, \tag{50b}$$

$$\underset{T}{\mathcal{C}} = e^{2\gamma} \left(f' \underset{T}{\bar{\mathcal{C}}} - 2\zeta \underset{T}{\bar{\mathcal{A}}} \right) \,, \tag{50c}$$

$$\underset{T}{\mathcal{V}} = e^{4\gamma} \underset{T}{\bar{\mathcal{V}}} \,. \tag{50d}$$

Outline

Scalar type invariants

- Invariant combination of parameter functions:

$$\mathcal{I} = \frac{e^{2\alpha}}{\mathcal{A}}\,, \quad \mathcal{U} = \frac{\mathcal{V}}{\mathcal{A}^2}\,. \tag{51}$$

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- Notion of invariance:
  - Functional form of $\phi \mapsto \mathcal{I}(\phi)$ and $\bar{\phi} \mapsto \bar{\mathcal{I}}(\bar{\phi})$ differs in general.
  - Values of functions at each spacetime point $x \in M$ agree:

$$\bar{\mathcal{I}}(\bar{\phi}(x)) = \bar{\mathcal{I}}(f(\phi(x))) = \mathcal{I}(\phi(x)). \tag{52}$$

$\Rightarrow$ Change of functional form compensates scalar field transformation.

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$\Rightarrow$ Change of functional form compensates scalar field transformation.

- Transformation of derivatives:

$$\mathcal{I}' = \frac{d\mathcal{I}}{d\phi} = \frac{df}{d\phi}\frac{d\bar{\mathcal{I}}}{d\bar{\phi}} = f'\bar{\mathcal{I}}'\,. \tag{53}$$

Vector type invariants

- Invariants from general scalar-teleparallel gravity action:

$$\mathcal{K} = \frac{\mathcal{C} + 2\alpha'\mathcal{A}}{2e^{2\alpha}}, \qquad \mathcal{M} = \frac{\mathcal{D} - \alpha'\mathcal{A}}{2e^{2\alpha}}, \qquad \mathcal{N} = \frac{\mathcal{E} + \alpha'\mathcal{A}}{2e^{2\alpha}}, \tag{54a}$$

$$\mathcal{H} = \frac{\mathcal{C} + \mathcal{A}'}{2\mathcal{A}}, \qquad \mathcal{J} = \frac{2\mathcal{D} - \mathcal{A}'}{4\mathcal{A}}, \qquad \mathcal{L} = \frac{2\mathcal{E} + \mathcal{A}'}{4\mathcal{A}}. \tag{54b}$$

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- Invariants from Lagrange multiplier terms:

$$\mathcal{S} = \mathcal{T} - \beta\,, \quad \mathcal{P} = \mathcal{Q} + \alpha' - \beta\,. \tag{55}$$

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$$\underset{T}{\mathcal{K}} = \frac{\underset{T}{\mathcal{C}} + 2\beta\underset{T}{\mathcal{A}}}{2e^{2\alpha}}\,, \qquad \underset{Q}{\mathcal{M}} = \frac{\underset{Q}{\mathcal{D}} - (\alpha' - \beta)\underset{Q}{\mathcal{A}}}{2e^{2\alpha}}\,, \qquad \underset{Q}{\mathcal{N}} = \frac{\underset{Q}{\mathcal{E}} + (\alpha' - \beta)\underset{Q}{\mathcal{A}}}{2e^{2\alpha}}\,, \tag{56a}$$

$$\underset{T}{\mathcal{H}} = \frac{\underset{T}{\mathcal{C}} + \underset{T}{\mathcal{A}}' + 2\underset{T}{\mathcal{A}}\mathcal{Q}}{2\underset{T}{\mathcal{A}}}\,, \qquad \underset{Q}{\mathcal{J}} = \frac{2\underset{Q}{\mathcal{D}} - \underset{Q}{\mathcal{A}}' + 2\underset{Q}{\mathcal{A}}\mathcal{T}}{4\underset{Q}{\mathcal{A}}}\,, \qquad \underset{Q}{\mathcal{L}} = \frac{2\underset{Q}{\mathcal{E}} + \underset{Q}{\mathcal{A}}' - 2\underset{Q}{\mathcal{A}}\mathcal{T}}{4\underset{Q}{\mathcal{A}}}\,. \tag{56b}$$

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- Transformation of vector type invariants:

$$\mathcal{K} = f'\bar{\mathcal{K}}. \tag{57}$$

Tensor type invariants

- Invariant from general scalar-teleparallel gravity action:

$$\mathcal{G} = \frac{\mathcal{B} - 6\alpha'^2\mathcal{A} - 6\beta\mathcal{C} + 4(\alpha' - \beta)(4\mathcal{D} + \mathcal{E})}{2e^{2\alpha}}\,. \tag{58}$$

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- Invariants from metric and symmetric teleparallel gravity actions:

$$\underset{Q}{\mathcal{G}} = \frac{\underset{Q}{\mathcal{B}} - 6(\alpha' - \beta)^2\underset{Q}{\mathcal{A}} + 4(\alpha' - \beta)(4\underset{Q}{\mathcal{D}} + \underset{Q}{\mathcal{E}})}{2e^{2\alpha}}\,, \quad \underset{T}{\mathcal{G}} = \frac{\underset{T}{\mathcal{B}} - 6\beta^2\underset{T}{\mathcal{A}} - 6\beta\underset{T}{\mathcal{C}}}{2e^{2\alpha}}\,. \tag{59}$$

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- Alternative construction of invariants:

$$\underset{Q}{\mathcal{F}} = \frac{2\underset{Q}{\mathcal{A}}\underset{Q}{\mathcal{B}} - 3(\underset{Q}{\mathcal{A}}' - 2\underset{Q}{\mathcal{A}}\mathcal{T})^2 + 4(\underset{Q}{\mathcal{A}}' - 2\underset{Q}{\mathcal{A}}\mathcal{T})(4\underset{Q}{\mathcal{D}} + \underset{Q}{\mathcal{E}})}{4\underset{Q}{\mathcal{A}}^2} , \tag{60a}$$

$$\underset{T}{\mathcal{F}} = \frac{2\underset{T}{\mathcal{A}}\underset{T}{\mathcal{B}} - 3(\underset{T}{\mathcal{A}}' + 2\underset{T}{\mathcal{A}}\mathcal{Q})^2 - 6(\underset{T}{\mathcal{A}}' + 2\underset{T}{\mathcal{A}}\mathcal{Q})\underset{T}{\mathcal{C}}}{4\underset{T}{\mathcal{A}}^2} . \tag{60b}$$

Tensor type invariants

- Invariant from general scalar-teleparallel gravity action:

$$\mathcal{G} = \frac{\mathcal{B} - 6\alpha'^2\mathcal{A} - 6\beta\mathcal{C} + 4(\alpha' - \beta)(4\mathcal{D} + \mathcal{E})}{2e^{2\alpha}}. \tag{58}$$

- Invariants from metric and symmetric teleparallel gravity actions:

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$$\underset{T}{\mathcal{F}} = \frac{2\underset{T}{\mathcal{A}}\underset{T}{\mathcal{B}} - 3(\underset{T}{\mathcal{A}}' + 2\underset{T}{\mathcal{A}}\mathcal{Q})^2 - 6(\underset{T}{\mathcal{A}}' + 2\underset{T}{\mathcal{A}}\mathcal{Q})\underset{T}{\mathcal{C}}}{4\underset{T}{\mathcal{A}}^2}. \tag{60b}$$

- Transformation of tensor type invariants:

$$\mathcal{G} = f'^2\bar{\mathcal{G}}. \tag{61}$$

Invariant characterization of theories

- Analogues of scalar-curvature gravity:
  - General teleparallel case:
$$\mathcal{H} \equiv \mathcal{J} \equiv \mathcal{L} \equiv 0\,. \tag{62}$$
  - Symmetric teleparallel case:
$$\underset{Q}{\mathcal{J}} \equiv \underset{Q}{\mathcal{L}} \equiv 0\,. \tag{63}$$
  - Metric teleparallel case:
$$\underset{T}{\mathcal{H}} \equiv 0\,. \tag{64}$$

Invariant characterization of theories

- Analogues of scalar-curvature gravity:
  - General teleparallel case:

$$\mathcal{H} \equiv \mathcal{J} \equiv \mathcal{L} \equiv 0\,. \tag{62}$$

  - Symmetric teleparallel case:

$$\underset{Q}{\mathcal{J}} \equiv \underset{Q}{\mathcal{L}} \equiv 0\,. \tag{63}$$

  - Metric teleparallel case:

$$\underset{T}{\mathcal{H}} \equiv 0\,. \tag{64}$$

- Minimally coupled theories:
  - General teleparallel case:

$$\mathcal{K} \equiv \mathcal{M} \equiv \mathcal{N} \equiv 0\,. \tag{65}$$

  - Symmetric teleparallel case:

$$\underset{Q}{\mathcal{M}} \equiv \underset{Q}{\mathcal{N}} \equiv 0\,. \tag{66}$$

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$$\underset{T}{\mathcal{K}} \equiv 0\,. \tag{67}$$

Outline

Definition of frames

- Frame $\mathfrak{F}$ defined by conditions c on parameter functions $\boldsymbol{\mathcal{X}} = (\alpha, \beta, \mathcal{A}, \ldots)$:

$$c(\overset{\mathfrak{F}}{\boldsymbol{\mathcal{X}}}) \equiv 0\,. \tag{68}$$

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⇒ Invariant metric and connection in "transformed" frame:

$$\overset{\mathfrak{F}}{g}_{\mu\nu} = e^{2\overset{\mathfrak{F}}{\gamma}} g_{\mu\nu}\,, \quad \overset{\mathfrak{F}}{\Gamma}{}^{\mu}{}_{\nu\rho} = \Gamma^{\mu}{}_{\nu\rho} + \overset{\mathfrak{F}}{\zeta}\delta^{\mu}_{\nu}\phi_{,\rho}\,. \tag{70}$$

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⇒ Frame-fixed parameter functions $\overset{\mathfrak{F}}{\boldsymbol{\mathcal{X}}}$ are invariants.

Jordan-like frame

- Vanishing coupling between scalar field and matter fields:

$$\overset{\mathfrak{J}}{\alpha} \equiv \overset{\mathfrak{J}}{\beta} \equiv 0\,. \tag{71}$$

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⇒ Parameter functions:

$$\overset{\mathfrak{J}}{\mathcal{A}} = \frac{1}{\mathcal{I}}\,, \quad \overset{\mathfrak{J}}{\mathcal{B}} = 2\mathcal{G}\,, \quad \overset{\mathfrak{J}}{\mathcal{C}} = 2\mathcal{K}\,, \quad \overset{\mathfrak{J}}{\mathcal{D}} = 2\mathcal{M}\,, \quad \overset{\mathfrak{J}}{\mathcal{E}} = 2\mathcal{N}\,, \quad \overset{\mathfrak{J}}{\mathcal{V}} = \frac{\mathcal{U}}{\mathcal{I}^2}\,. \tag{74}$$

Einstein-like frame (symmetric teleparallel gravity)

- Vanishing coupling between scalar field and geometry:

$$\underset{Q}{\overset{\mathfrak{E}}{\mathcal{A}}} \equiv 1\,, \quad \overset{\mathfrak{E}}{\mathcal{T}} \equiv 0\,. \tag{75}$$

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Einstein-like frame (metric teleparallel gravity)

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