A geometric view on local Lorentz transformations in teleparallel gravity

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Motivation: problems to solve

- So far unexplained cosmological observations:
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- Idea here: modification of the geometric structure of spacetime!
  - Study classical gravity theories based on modified geometry.
  - Consider geometries as effective models of quantum gravity.
  - Derive observable effects to test modified geometry.

Field variables in teleparallel gravity

- Metric teleparallelism conventionally formulated using:
  - Tetrad / coframe: $\theta^a = \theta^a{}_\mu \mathrm{d}x^\mu$ with inverse $e_a = e_a{}^\mu \partial_\mu$.
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- Induced metric-affine geometry:
  - Metric:

$$g_{\mu\nu} = \eta_{ab} \theta^a{}_\mu \theta^b{}_\nu \,. \tag{1}$$

  - Affine connection:

$$\Gamma^\mu{}_{\nu\rho} = e_a{}^\mu \left(\partial_\rho \theta^a{}_\nu + \omega^a{}_{b\rho} \theta^b{}_\nu \right) \,. \tag{2}$$

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- Conditions on the spin connection:
  - Flatness $R = 0$:

$$\partial_\mu \omega^a{}_{b\nu} - \partial_\nu \omega^a{}_{b\mu} + \omega^a{}_{c\mu}\omega^c{}_{b\nu} - \omega^a{}_{c\nu}\omega^c{}_{b\mu} = 0 \,. \tag{3}$$

  - Metric compatibility $Q = 0$:

$$\eta_{ac}\omega^c{}_{b\mu} + \eta_{bc}\omega^c{}_{a\mu} = 0 \,. \tag{4}$$

Local Lorentz transformations

- Local Lorentz transformation of the tetrad only:

$$\theta^a{}_\mu \mapsto \theta'^a{}_\mu = \Lambda^a{}_b \theta^b{}_\mu \,. \tag{5}$$

✓ Metric is invariant: $g'_{\mu\nu} = g_{\mu\nu}$.
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- Is LLI broken if teleparallel gravity action depends on $\Gamma^\mu{}_{\nu\rho}$?

The Weitzenböck gauge

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$$\partial_\mu \omega^a{}_{b\nu} - \partial_\nu \omega^a{}_{b\mu} + \omega^a{}_{c\mu}\omega^c{}_{b\nu} - \omega^a{}_{c\nu}\omega^c{}_{b\mu} \equiv 0\,. \tag{7}$$

⇒ *The spin connection can always be written in the form*

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- Remark: this holds also in symmetric and general teleparallelism.

How to obtain the Weitzenböck gauge?

- Recall that we have gauge invariant quantities:
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  - ✓ Global Lorentz invariance encoded in freedom of choice for $\overset{w}{\theta}{}^a{}_\mu(x)$.

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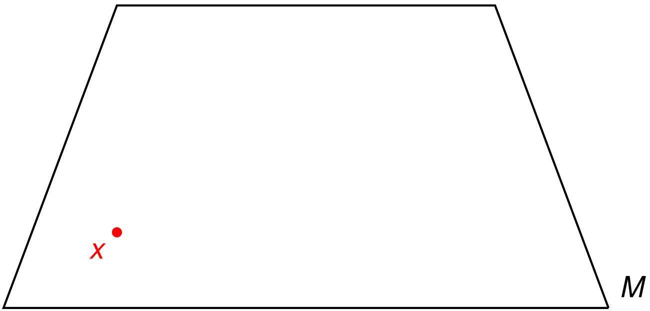

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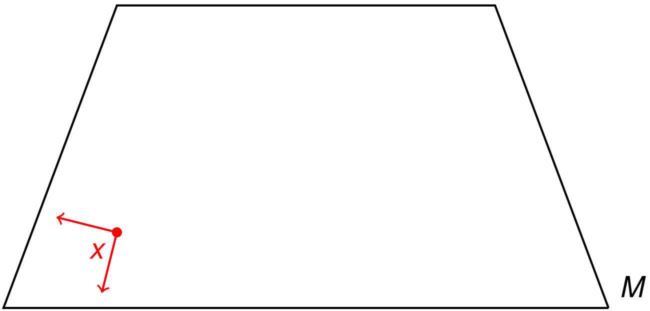

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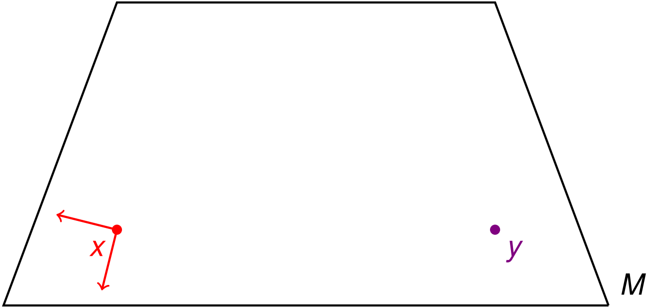

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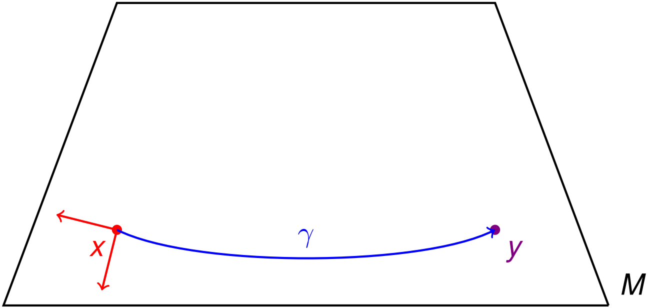

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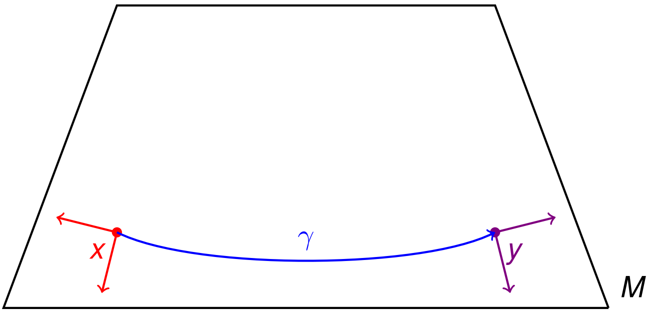

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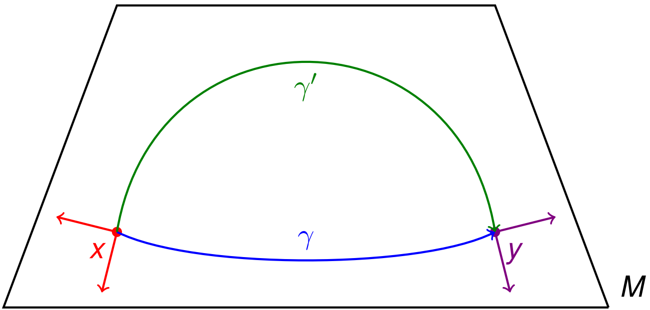

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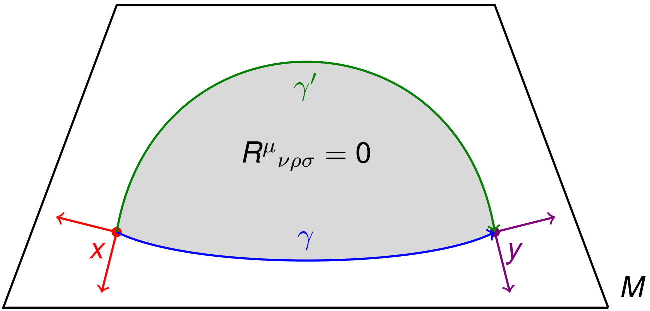

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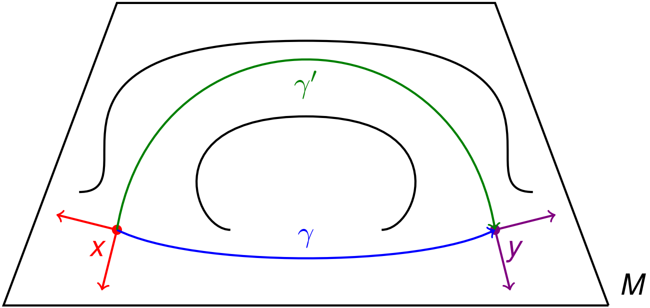
  - ↯ But only if γ and γ' are homotopic paths!



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- Consider local Lorentz transformations $\Lambda : M \to \mathrm{O}(1,3)$:
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- Physical geometry: $\mathrm{SO}_0(1,3)$ reduction of the frame bundle & Γ.

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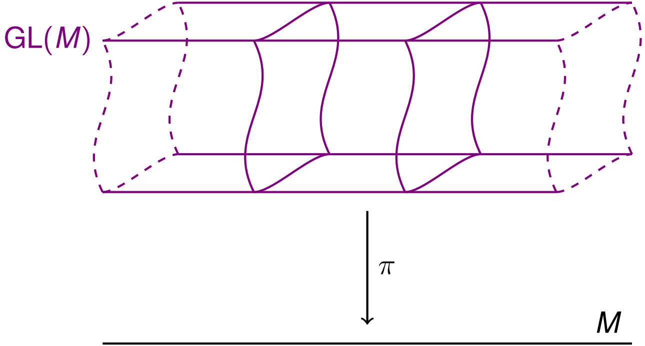
$\Rightarrow$ Most fundamental variables found in geometric picture.

The geometric picture

1. Start with the general linear frame bundle $\pi : \mathrm{GL}(M) \to M$.



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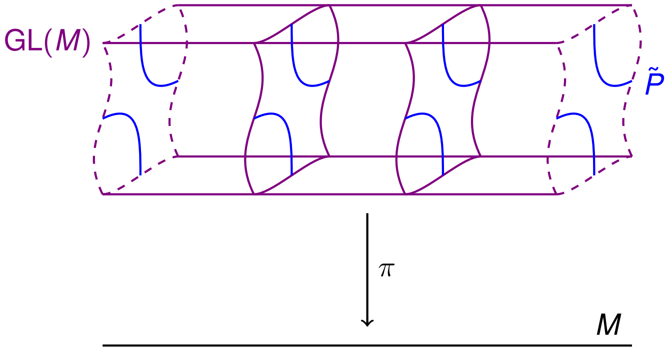

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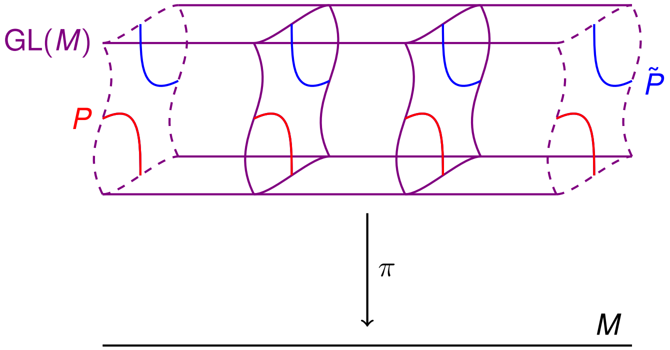

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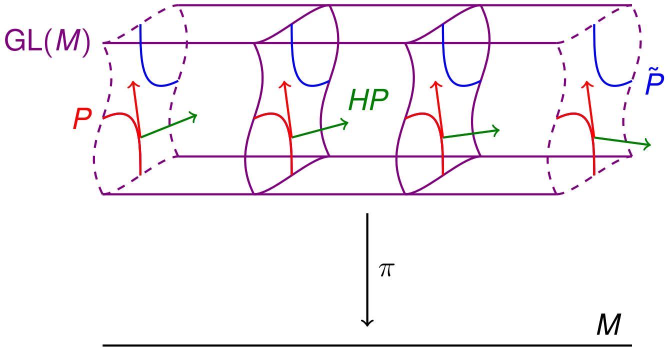

Tetrads and spin structure

- How to obtain a spin structure from a tetrad $e : M \to P$?
  1. Spin structure obtained from trivial bundle $Q = M \times \mathrm{SL}(2, \mathbb{C})$.
  2. Use covering map $\sigma : \mathrm{SL}(2, \mathbb{C}) \to \mathrm{SO}_0(1, 3)$.
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- Do different tetrads e, e' define the same spin structure?
  - Consider non-simply connected manifold M.
  - Let $\gamma : [0, 1] \to M$ with $\gamma(0) = \gamma(1)$ non-contractible.
  - Let $\Lambda : M \to \mathrm{SO}_0(1, 3)$ such that $\Lambda \circ \gamma$ has odd winding.
  - Tetrads $e = e' \cdot \Lambda$ define spin structures φ, φ'.
  - Assume existence of bundle isomorphism $\mu : Q \to Q$, $\varphi = \varphi' \circ \mu$.

  $\Rightarrow$ Curve connects antipodes: $\mu(\gamma(1), \mathbb{1}) = -\mu(\gamma(0), \mathbb{1})$.

  ↯ Contradicts $\gamma(0) = \gamma(1)$.

  $\Rightarrow$ Spin structures φ, φ' are inequivalent.

Tetrads vs observers

- Clash of two notions of orthonormal frames:
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$\Rightarrow$ Observer geometry defined by metric: LLI holds.

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   - Orthonormal - by using clocks, measuring rods, simultaneity.
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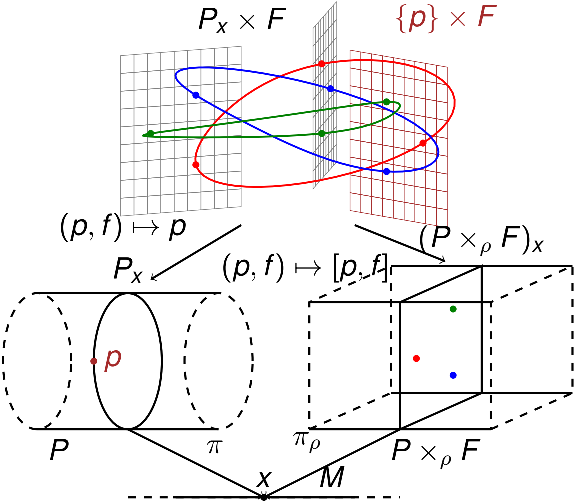
Extra: the associated bundle



Extra: the many faces of connections