General cosmologies and their perturbations in teleparallel gravity

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Metric-Affine Gravity - Tartu, 30. June 2022

Outline

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Metric-affine geometry and spacetime symmetries

- Fundamental fields in metric-affine geometry:
  - Metric tensor $g_{\mu\nu}$:
    - Defines length of and angle between tangent vectors.
    - Defines length of curves and proper time.
    - Defines causality (spacelike and timelike directions).

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  - Connection with coefficients $\Gamma^{\mu}{}_{\nu\rho}$:
    - Defines covariant derivative ∇_{μ} of tensor fields.
    - Defines parallel transport along arbitrary curves.
    - Defines autoparallel curves via parallel transport of tangent vector.

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    - Defines covariant derivative ∇_{μ} of tensor fields.
    - Defines parallel transport along arbitrary curves.
    - Defines autoparallel curves via parallel transport of tangent vector.
- Symmetry under action of a vector field X^{μ}:
  - Metric:

$$0 = (\mathcal{L}_X g)_{\mu\nu} = X^{\rho}\partial_{\rho}g_{\mu\nu} + \partial_{\mu}X^{\rho}g_{\rho\nu} + \partial_{\nu}X^{\rho}g_{\mu\rho}\,. \tag{1}$$

  - Connection coefficients:

$$\begin{aligned} 0 = (\mathcal{L}_X\Gamma)^{\mu}{}_{\nu\rho} &= X^{\sigma}\partial_{\sigma}\Gamma^{\mu}{}_{\nu\rho} - \partial_{\sigma}X^{\mu}\Gamma^{\sigma}{}_{\nu\rho} + \partial_{\nu}X^{\sigma}\Gamma^{\mu}{}_{\sigma\rho} + \partial_{\rho}X^{\sigma}\Gamma^{\mu}{}_{\nu\sigma} + \partial_{\nu}\partial_{\rho}X^{\mu} \\ &= \nabla_{\rho}\nabla_{\nu}X^{\mu} - X^{\sigma}R^{\mu}{}_{\nu\rho\sigma} - \nabla_{\rho}(X^{\sigma}T^{\mu}{}_{\nu\sigma})\,. \end{aligned} \tag{2}$$

Properties of metric-affine geometry

- Three characteristic quantities:
  - Curvature:
$$R^{\mu}{}_{\nu\rho\sigma} = \partial_\rho \Gamma^{\mu}{}_{\nu\sigma} - \partial_\sigma \Gamma^{\mu}{}_{\nu\rho} + \Gamma^{\mu}{}_{\tau\rho}\Gamma^{\tau}{}_{\nu\sigma} - \Gamma^{\mu}{}_{\tau\sigma}\Gamma^{\tau}{}_{\nu\rho}\,. \tag{3}$$
  - Torsion:
$$T^{\mu}{}_{\nu\rho} = \Gamma^{\mu}{}_{\rho\nu} - \Gamma^{\mu}{}_{\nu\rho}\,. \tag{4}$$
  - Nonmetricity:
$$Q_{\mu\nu\rho} = \nabla_\mu g_{\nu\rho} = \partial_\mu g_{\nu\rho} - \Gamma^{\sigma}{}_{\nu\mu} g_{\sigma\rho} - \Gamma^{\sigma}{}_{\rho\mu} g_{\nu\sigma}\,. \tag{5}$$

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- Some special classes of connections used in gravity theory:
  - Levi-Civita connection: $T = Q = 0$.
  - Metric teleparallelism: $R = Q = 0$.
  - Symmetric teleparallelism: $R = T = 0$.

Decomposition of the connection

- Affine connection can be decomposed:

$$\Gamma^{\mu}{}_{\nu\rho} = \mathring{\Gamma}^{\mu}{}_{\nu\rho} + K^{\mu}{}_{\nu\rho} + L^{\mu}{}_{\nu\rho} \,. \tag{6}$$

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- Parts of the decomposition:
  - Levi-Civita connection of the metric:

$$\overset{\circ}{\Gamma}{}^{\mu}{}_{\nu\rho} = \frac{1}{2}g^{\mu\sigma}\left(\partial_{\nu}g_{\sigma\rho} + \partial_{\rho}g_{\nu\sigma} - \partial_{\sigma}g_{\nu\rho}\right)\,. \tag{7}$$

  - Contortion:

$$K^{\mu}{}_{\nu\rho} = \frac{1}{2}\left(T_{\nu}{}^{\mu}{}_{\rho} + T_{\rho}{}^{\mu}{}_{\nu} - T^{\mu}{}_{\nu\rho}\right)\,. \tag{8}$$

  - Disformation:

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- All three components depend on the metric.

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$$R_1 = \sin\varphi\partial_\vartheta + \frac{\cos\varphi}{\tan\vartheta}\partial_\varphi\,, \tag{10a}$$

$$R_2 = -\cos\varphi\partial_\vartheta + \frac{\sin\varphi}{\tan\vartheta}\partial_\varphi\,, \tag{10b}$$

$$R_3 = -\partial_\varphi\,, \tag{10c}$$

  - Translations:

$$T_1 = \chi\sin\vartheta\cos\varphi\partial_r + \frac{\chi}{r}\cos\vartheta\cos\varphi\partial_\vartheta - \frac{\chi\sin\varphi}{r\sin\vartheta}\partial_\varphi\,, \tag{11a}$$

$$T_2 = \chi\sin\vartheta\sin\varphi\partial_r + \frac{\chi}{r}\cos\vartheta\sin\varphi\partial_\vartheta + \frac{\chi\cos\varphi}{r\sin\vartheta}\partial_\varphi\,, \tag{11b}$$

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- Here $\chi = \sqrt{1-(ur)^2}$, and u can be real or imaginary.

Cosmologically symmetric metric-affine geometry

1. Most general metric with cosmological symmetry:
   - Metric in space-time split:
$$g_{\mu\nu} = -n_\mu n_\nu + h_{\mu\nu}\,. \tag{12}$$
   - Unit normal covector field:
$$n_\mu \mathrm{d}x^\mu = -N\,\mathrm{d}t\,. \tag{13}$$
   - Spatial metric (gives projection onto spatial slices):
$$h_{\mu\nu}\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu = A^2\left[\frac{\mathrm{d}r\otimes\mathrm{d}r}{\chi^2} + r^2(\mathrm{d}\vartheta\otimes\mathrm{d}\vartheta + \sin^2\vartheta\mathrm{d}\varphi\otimes\mathrm{d}\varphi)\right]\,. \tag{14}$$

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   ⇒ Metric depends on lapse $N(t)$ and scale factor $A(t)$.
2. Most general affine connection with cosmological symmetry:
   - Connection characterized by cosmologically symmetric torsion and nonmetricity:
   $$T^\mu{}_{\nu\rho} = \frac{2}{A}(\mathcal{T}_1 h^\mu_{[\nu} n_{\rho]} + \mathcal{T}_2 n_\sigma \varepsilon^{\sigma\mu}{}_{\nu\rho})\,, \quad Q_{\rho\mu\nu} = \frac{2}{A}(\mathcal{Q}_1 n_\rho n_\mu n_\nu + 2\mathcal{Q}_2 n_\rho h_{\mu\nu} + 2\mathcal{Q}_3 h_{\rho(\mu} n_{\nu)})\,. \tag{15}$$

   ⇒ Connection depends on five free functions $\mathcal{T}_1(t), \mathcal{T}_2(t), \mathcal{Q}_1(t), \mathcal{Q}_2(t), \mathcal{Q}_3(t)$.

Cosmologically symmetric teleparallel geometry

- Define conformal time derivative and conformal Hubble parameter:

$$f' = \frac{A}{N}\frac{\mathrm{d}f}{\mathrm{d}t}\,, \quad \mathcal{H} = \frac{A'}{A}\,. \tag{16}$$

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- $R^{\mu}{}_{\nu\rho\sigma} = 0$ if and only if:

$$(\mathcal{Q}_1 + \mathcal{Q}_2)(\mathcal{H} - \mathcal{T}_1 + \mathcal{Q}_2) + (\mathcal{H} - \mathcal{T}_1 + \mathcal{Q}_2)' = 0\,, \tag{17a}$$

$$\mathcal{T}_2(\mathcal{H} - \mathcal{T}_1 + \mathcal{Q}_2) = 0\,, \tag{17b}$$

$$(\mathcal{Q}_1 + \mathcal{Q}_2)(\mathcal{H} - \mathcal{T}_1 + \mathcal{Q}_2 - \mathcal{Q}_3) - (\mathcal{H} - \mathcal{T}_1 + \mathcal{Q}_2 - \mathcal{Q}_3)' = 0\,, \tag{17c}$$

$$\mathcal{T}_2(\mathcal{H} - \mathcal{T}_1 + \mathcal{Q}_2 - \mathcal{Q}_3) = 0\,, \tag{17d}$$

$$(\mathcal{H} - \mathcal{T}_1 + \mathcal{Q}_2)(\mathcal{H} - \mathcal{T}_1 + \mathcal{Q}_2 - \mathcal{Q}_3) - \mathcal{T}_2^2 + u^2 = 0\,, \tag{17e}$$

$$\mathcal{T}_2' = 0\,. \tag{17f}$$

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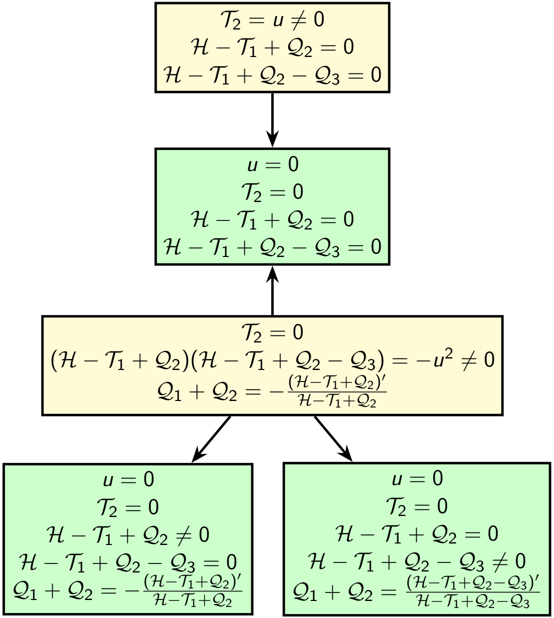
$\Rightarrow$ Different branches of solutions for $u = 0$ and $u \neq 0$.

Solution branches



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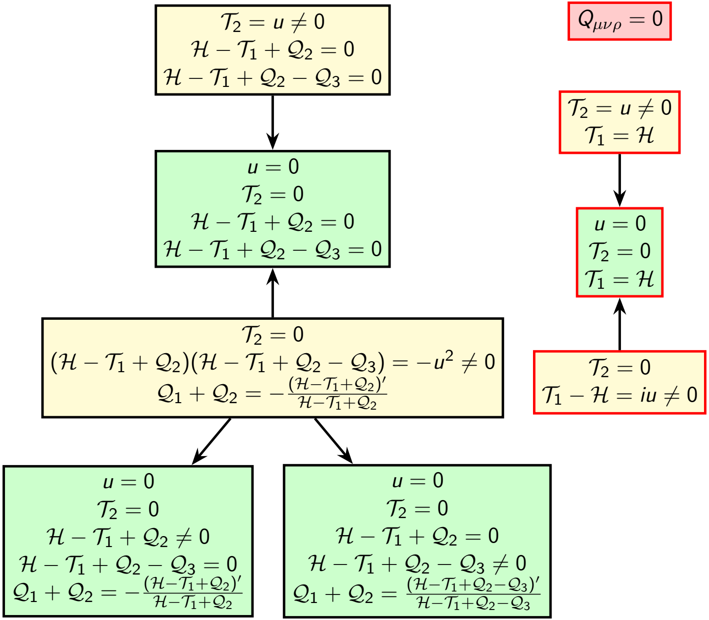

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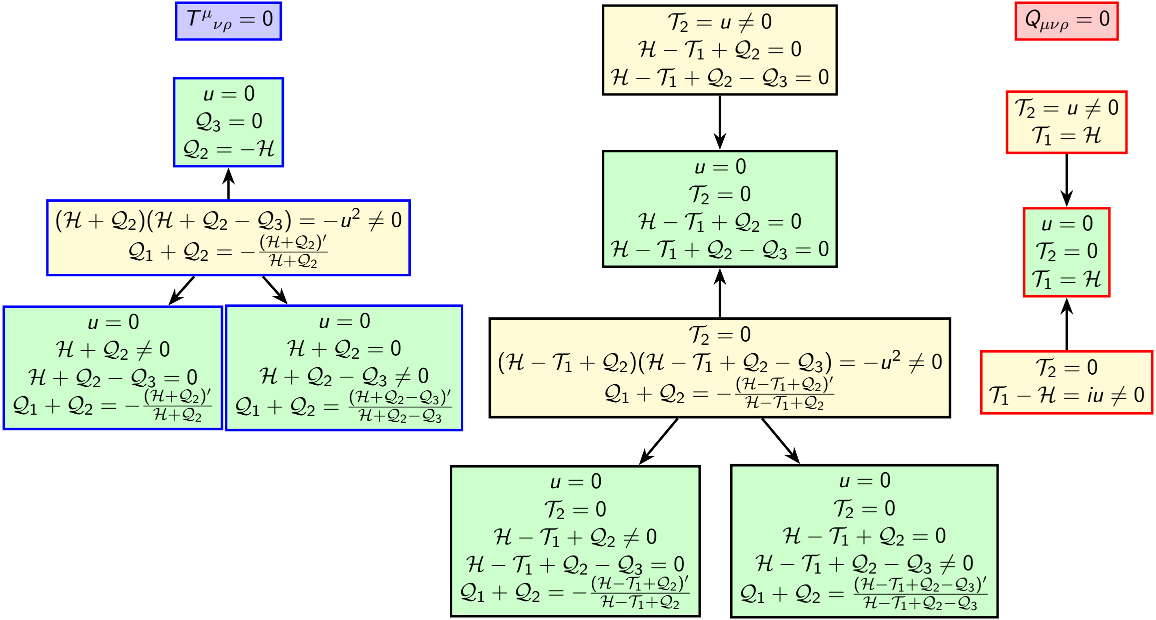

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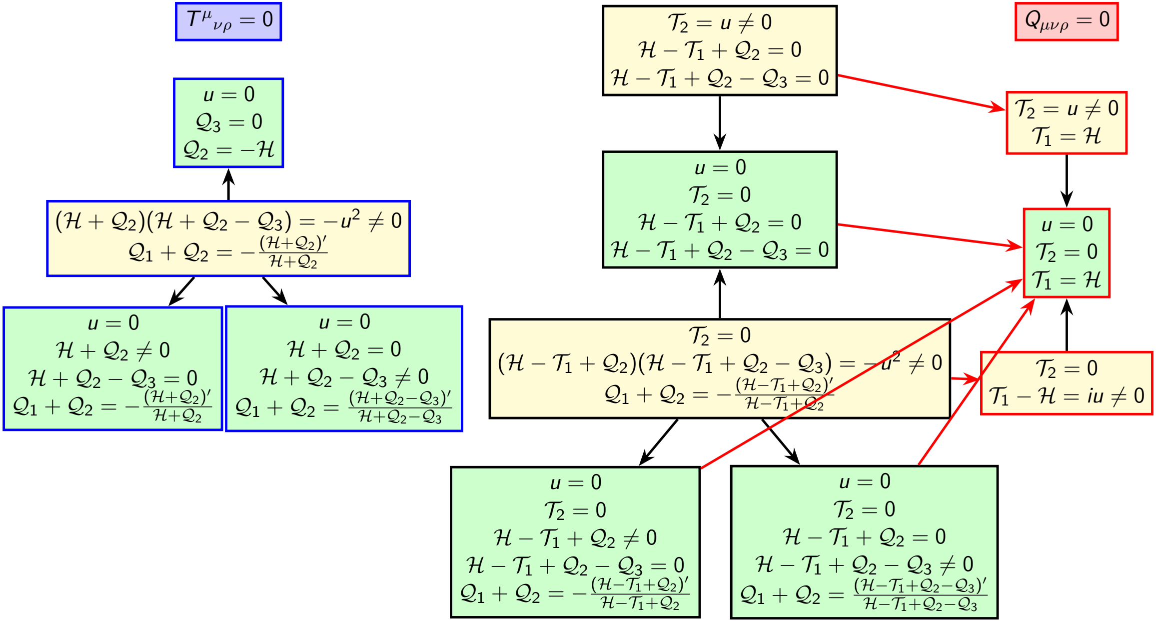

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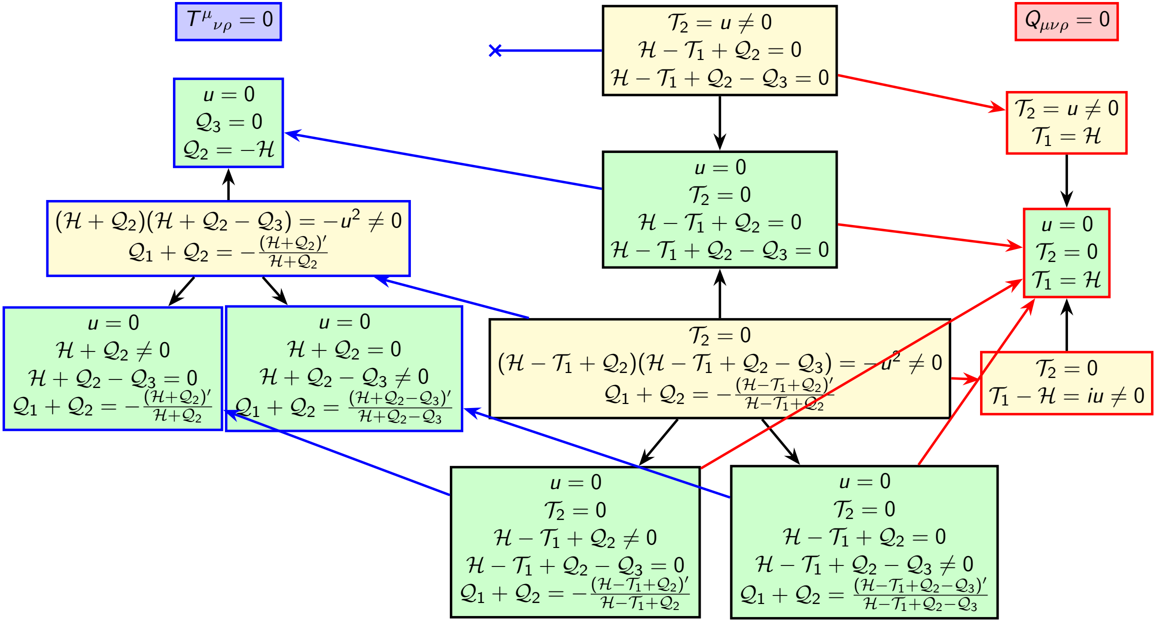

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Linear perturbations in teleparallel gravity

- Consider linear perturbation $\delta g_{\mu\nu}, \delta\Gamma^{\mu}{}_{\nu\rho}$ around background $\bar{g}_{\mu\nu}, \bar{\Gamma}^{\mu}{}_{\nu\rho}$:

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu}\,, \quad \Gamma^{\mu}{}_{\nu\rho} = \bar{\Gamma}^{\mu}{}_{\nu\rho} + \delta\Gamma^{\mu}{}_{\nu\rho}\,. \tag{18}$$

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- Further conditions in metric and symmetric teleparallel gravity:
  - Metric case $\delta Q_{\mu\nu\rho} = 0$:

$$\lambda_{\mu\nu} + \lambda_{\nu\mu} = \delta g_{\mu\nu} \quad \Rightarrow \quad \lambda_{\mu\nu} = \frac{1}{2}(\delta g_{\mu\nu} + a_{\mu\nu})\,. \tag{20}$$

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  - Symmetric case $\delta T^{\mu}{}_{\nu\rho} = 0$:

$$\nabla_{[\rho}\lambda^{\mu}{}_{\nu]} = 0 \quad \Rightarrow \quad \lambda^{\mu}{}_{\nu} = \nabla_{\nu}\zeta^{\mu}\,. \tag{21}$$

$3+1$ split of background geometry

- Maximally symmetric metric γ_{ab} used to move spatial indices:

$$h_{\mu\nu}\mathrm{d}x^{\mu} \otimes \mathrm{d}x^{\nu} = A^2\gamma_{ab}\mathrm{d}x^{a} \otimes \mathrm{d}x^{b}\,. \tag{22}$$

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$\Rightarrow$ Spatial Levi-Civita tensor υ_{abc} associated to γ_{ab}:

$$n^{\sigma}\epsilon_{\sigma\mu\nu\rho}\mathrm{d}x^{\mu} \otimes \mathrm{d}x^{\nu} \otimes \mathrm{d}x^{\rho} = A^3\upsilon_{abc}\mathrm{d}x^{a} \otimes \mathrm{d}x^{b} \otimes \mathrm{d}x^{c} \,. \tag{23}$$

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$\Rightarrow$ Levi-Civita covariant derivative d_a of γ_{ab} acting on spatial tensors:

$$\digamma^a{}_{bc} = \frac{1}{2}\gamma^{ad}(\partial_b\gamma_{dc} + \partial_c\gamma_{bd} - \partial_d\gamma_{bc}) \,. \tag{24}$$

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! Affine connection ∇_{μ} need not preserve $3+1$ split - no “induced spatial connection”.

$3+1$ split of perturbations

- Introduce projector fields:

$$\Pi^a_\mu \partial_a \otimes \mathrm{d}x^\mu = A\delta^a_b \, \partial_a \otimes \mathrm{d}x^b \,, \quad \Pi^\mu_a \partial_\mu \otimes \mathrm{d}x^a = A^{-1}\delta^b_a \, \partial_b \otimes \mathrm{d}x^a \,. \tag{25}$$

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- Decomposition of tensor fields on metric background:
  - Vector field X^μ:

$$X = N^{-1}\hat{X}^0\partial_t + A^{-1}\hat{X}^a\partial_a \quad \Leftrightarrow \quad \hat{X}^0 = -n_\mu X^\mu = NX^0 \,, \quad \hat{X}^a = \Pi^a_\mu X^\mu = AX^a \tag{26}$$

  - Covector field α:

$$\alpha = N\hat{\alpha}_0 \, \mathrm{d}t + A\hat{\alpha}_a \, \mathrm{d}x^a \quad \Leftrightarrow \quad \hat{\alpha}_0 = n^\mu \alpha_\mu = N^{-1}\alpha_0 \,, \quad \hat{\alpha}_a = \Pi^\mu_a \alpha_\mu = A^{-1}\alpha_a \,. \tag{27}$$

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- Raising and lowering indices of decomposed components:

$$\hat{X}^0 = -\hat{X}_0 \,, \quad \hat{X}^a = \gamma^{ab}\hat{X}_b \,. \tag{28}$$

3 + 1 split of covariant derivatives

- Space-time split of Levi-Civita covariant derivative:

$$\overset{\circ}{\nabla}_{\alpha} X^{\beta} = \\ = \tag{29}$$

$3+1$ split of covariant derivatives

- Space-time split of Levi-Civita covariant derivative:

$$\overset{\circ}{\nabla}_\alpha X^\beta = (h_\alpha^\gamma - n_\alpha n^\gamma)(h_\delta^\beta - n^\beta n_\delta)\overset{\circ}{\nabla}_\gamma(n^\delta \hat{X}^0 + \Pi_a^\delta \hat{X}^a) \\ = \tag{29}$$

- Introduce projectors for space-time split.

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- Introduce projectors for space-time split.
- Identify components originating from index choice:
  1. Derivative in time direction yields time derivatives.

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- Introduce projectors for space-time split.
- Identify components originating from index choice:
  1. Derivative in time direction yields time derivatives.
  2. Derivative in spatial direction yields spatial derivatives.

$3+1$ split of covariant derivatives

- Space-time split of Levi-Civita covariant derivative:

$$\begin{aligned}\overset{\circ}{\nabla}_{\alpha}X^{\beta} &= (h^{\gamma}_{\alpha} - n_{\alpha}n^{\gamma})(h^{\beta}_{\delta} - n^{\beta}n_{\delta})\overset{\circ}{\nabla}_{\gamma}(n^{\delta}\hat{X}^{0} + \Pi^{\delta}_{a}\hat{X}^{a}) \\ &= -\frac{n_{\alpha}}{N}(n^{\beta}\partial_{t}\hat{X}^{0} + \Pi^{\beta}_{a}\partial_{t}\hat{X}^{a}) + \frac{\Pi^{a}_{\alpha}}{A}(n^{\beta}\mathrm{d}_{a}\hat{X}^{0} + \Pi^{\beta}_{b}\mathrm{d}_{a}\hat{X}^{b}) + H(h^{\beta}_{\alpha}\hat{X}^{0} + \gamma_{ab}\Pi^{a}_{\alpha}n^{\beta}\hat{X}^{b})\end{aligned} \tag{29}$$

- Introduce projectors for space-time split.
- Identify components originating from index choice:
  1. Derivative in time direction yields time derivatives.
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  1. Derivative in time direction yields time derivatives.
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  3. Mixed Christoffel symbols contain Hubble parameter.
- Hubble parameter enters through derivative of projectors:
  - Eulerian observers move on geodesics $\Rightarrow$ acceleration vanishes:

$$a_\mu = n^\nu \overset{\circ}{\nabla}_\nu n_\mu = 0\,. \tag{30}$$

  - Spatial geometry is maximally symmetric $\Rightarrow$ extrinsic curvature:

$$K_{\mu\nu} = \overset{\circ}{\nabla}_\mu n_\nu + n_\mu a_\nu = H h_{\mu\nu}\,. \tag{31}$$

Irreducible decomposition of perturbations

- Algebraic $3+1$ split of perturbation tensor fields:
  - Metric: $\widehat{\delta g}_{00}, \widehat{\delta g}_{0a}, \widehat{\delta g}_{ab}$.
  - General teleparallel: $\hat{\lambda}_{00}, \hat{\lambda}_{0a}, \hat{\lambda}_{a0}, \hat{\lambda}_{ab}$.
  - Metric teleparallel: $\hat{a}_{0a}, \hat{a}_{ab}$.
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- Differential decomposition of spatial algebraic components:
  - Vector $U_a = \mathrm{d}_a\tilde{U} + \hat{U}_a$, $\mathrm{d}_a\hat{U}^a = 0 \rightsquigarrow$ scalar + divergence-free vector.
  - Symmetric tensor $U_{ab} = \bar{U}\gamma_{ab} + (\mathrm{d}_a\mathrm{d}_b - \gamma_{ab}\triangle/3)\tilde{U} + \mathrm{d}_{(a}\hat{U}_{b)} + \check{U}_{ab}$.
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$\Rightarrow$ Number of irreducible components:

	scalar	pseudoscalar	vector	pseudovector	tensor
$\delta g_{\mu\nu}$	4	0	2	0	1
$\lambda_{\mu\nu}$	5	1	3	1	1
$a_{\mu\nu}$	1	1	1	1	0
ζ_μ	2	0	1	0	0

Infinitesimal coordinate transformations

- Transformation of perturbations under coordinate changes:
  - Fields transform under infinitesimal coordinate change $x'^{\mu} = x^{\mu} + X^{\mu}(x)$:

$$g_{\mu\nu} - g'_{\mu\nu} = (\mathcal{L}_X g)_{\mu\nu}\,, \quad \Gamma^{\mu}{}_{\nu\rho} - \Gamma'^{\mu}{}_{\nu\rho} = (\mathcal{L}_X \Gamma)^{\mu}{}_{\nu\rho}\,. \tag{32}$$

  - Linear perturbation expansion of fields around common background:

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu}\,, \qquad \Gamma^{\mu}{}_{\nu\rho} = \bar{\Gamma}^{\mu}{}_{\nu\rho} + \delta\Gamma^{\mu}{}_{\nu\rho}\,, \tag{33a}$$

$$g'_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g'_{\mu\nu}\,, \qquad \Gamma'^{\mu}{}_{\nu\rho} = \bar{\Gamma}^{\mu}{}_{\nu\rho} + \delta\Gamma'^{\mu}{}_{\nu\rho}\,. \tag{33b}$$

  - Consider X^{μ} to be of same order as linear perturbations:

$$\delta_X \delta g_{\mu\nu} = \delta g_{\mu\nu} - \delta g'_{\mu\nu} = (\mathcal{L}_X \bar{g})_{\mu\nu}\,, \quad \delta_X \delta\Gamma^{\mu}{}_{\nu\rho} = \delta\Gamma^{\mu}{}_{\nu\rho} - \delta\Gamma'^{\mu}{}_{\nu\rho} = (\mathcal{L}_X \bar{\Gamma})^{\mu}{}_{\nu\rho}\,. \tag{34}$$

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- Transformation of connection perturbations:
  - Use Lie derivative of flat connection:

$$(\mathcal{L}_X \bar{\Gamma})^{\mu}{}_{\nu\rho} = \bar{\nabla}_{\rho}\bar{\nabla}_{\nu} X^{\mu} - \bar{\nabla}_{\rho}(X^{\sigma}\,\bar{T}^{\mu}{}_{\nu\sigma})\,. \tag{35}$$

  ⇒ Transformation of perturbation tensor fields:

$$\delta_X \lambda^{\mu}{}_{\nu} = \lambda^{\mu}{}_{\nu} - \lambda'^{\mu}{}_{\nu} = \bar{\nabla}_{\nu} X^{\mu} - X^{\sigma}\,\bar{T}^{\mu}{}_{\nu\sigma}\,. \tag{36}$$

$3+1$ split and gauge transformations

- Perform $3+1$ decomposition of coordinate transformation:
  - Metric transformation:

$$A\delta_X\widehat{\delta g}_{00} = 2\hat{X}'_\perp\,, \tag{37a}$$

$$A\delta_X\widehat{\delta g}_{a0} = \mathrm{d}_a\hat{X}_\perp + \mathrm{d}_a\hat{X}'_\parallel + \hat{Z}'_a - \mathcal{H}(\mathrm{d}_a\hat{X}_\parallel + \hat{Z}_a)\,, \tag{37b}$$

$$A\delta_X\widehat{\delta g}_{ab} = 2\mathrm{d}_a\mathrm{d}_b\hat{X}_\parallel + 2\mathrm{d}_{(a}\hat{Z}_{b)} - 2\mathcal{H}\hat{X}_\perp\gamma_{ab}\,. \tag{37c}$$

  - Connection transformation:

$$A\delta_X\hat{\lambda}_{00} = \hat{X}'_\perp - \mathcal{Q}_1\hat{X}_\perp\,, \tag{38a}$$

$$A\delta_X\hat{\lambda}_{0b} = \mathrm{d}_b\hat{X}_\perp - (\mathcal{H} + \mathcal{Q}_2 - \mathcal{Q}_3 - \mathcal{T}_1)(\mathrm{d}_b\hat{X}_\parallel + \hat{Z}_b)\,, \tag{38b}$$

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⇝ Further decompose into transformation of irreducible components.

Gauge fixing and gauge-invariant variables

- Construction of gauge-invariant quantities for gauge G:
  - Decompose irreducible components into gauge-invariant and gauge-dependent part:

$$\hat{Y} = \underset{\mathrm{G}}{\hat{\boldsymbol{Y}}} + \delta_{\underset{\mathrm{G}}{\hat{X}}} \hat{Y} \,. \tag{39}$$

  - Gauge condition fixing $\underset{\mathrm{G}}{\hat{\boldsymbol{Y}}} \Leftrightarrow$ gauge transformation $\underset{\mathrm{G}}{\hat{X}}$ from arbitrary gauge:

$$0 = \underset{\mathrm{G}}{\hat{C}}(\underset{\mathrm{G}}{\hat{\boldsymbol{Y}}}) = \underset{\mathrm{G}}{\hat{C}}(\hat{Y} - \delta_{\underset{\mathrm{G}}{\hat{X}}} \hat{Y}) \quad \Leftrightarrow \quad \underset{\mathrm{G}}{\hat{X}} = \underset{\mathrm{G}}{\hat{f}}(\hat{Y}) \,. \tag{40}$$

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- Number of independent components:
  - n perturbation components $\hat{Y}$ before gauge fixing.
  - 4 components of gauge-defining vector field $\underset{\mathrm{G}}{\hat{X}}$.
  - 4 gauge conditions $\underset{\mathrm{G}}{\hat{C}}$.

  $\Rightarrow$ $n - 4$ independent gauge-invariant components $\underset{\mathrm{G}}{\hat{\boldsymbol{Y}}}$.

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- Example: coincident (perturbation) gauge:

$$(\underset{\mathrm{C}}{\hat{\boldsymbol{\zeta}}}_0, \underset{\mathrm{C}}{\hat{\boldsymbol{\zeta}}}_a) \equiv 0 \quad \Leftrightarrow \quad (\underset{\mathrm{C}}{\hat{X}}_0, \underset{\mathrm{C}}{\hat{X}}_a) = (\hat{\zeta}_0, \hat{\zeta}_a) \,. \tag{41}$$

Outline

Example: equivalent branches of $f(X)$ theories

- Consider similarly constructed gravity theories:

$$\int_M \frac{f(Q)}{2\kappa^2}\sqrt{-g}\mathrm{d}^4x$$

$$\int_M \frac{f(G)}{2\kappa^2}\sqrt{-g}\mathrm{d}^4x$$

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- Consider flat branch of cosmological teleparallel geometries:

$$\begin{matrix} u = 0 \\ \mathcal{Q}_3 = 0 \\ \mathcal{Q}_2 = -\mathcal{H} \end{matrix} \longleftarrow \begin{matrix} u = 0 \\ \mathcal{T}_2 = 0 \\ \mathcal{H} - \mathcal{T}_1 + \mathcal{Q}_2 = 0 \\ \mathcal{H} - \mathcal{T}_1 + \mathcal{Q}_2 - \mathcal{Q}_3 = 0 \end{matrix} \longrightarrow \begin{matrix} u = 0 \\ \mathcal{T}_2 = 0 \\ \mathcal{T}_1 = \mathcal{H} \end{matrix}$$

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⇒ Identical dynamics for cosmological background evolution:

$$\kappa^2\rho = -\frac{1}{2}f + 6f'H^2\,, \tag{42a}$$

$$\kappa^2 p = \frac{1}{2}f - 2f'(\dot{H} + 3H^2) - 24f''H^2\dot{H}\,. \tag{42b}$$

Example: inequivalent branches of $f(X)$ theories

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$$G = \frac{3}{A^2}[2\mathcal{T}_2^2 - 2(\mathcal{Q}_2 - \mathcal{T}_1)^2 - \mathcal{Q}_3(\mathcal{Q}_1 - \mathcal{Q}_2 + 2\mathcal{T}_1)]\,, \tag{43a}$$

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↯ Some background scalars decouple, but enter in perturbations.

Outline

Summary

- Cosmologically symmetric teleparallel background geometry:
  - Metric takes familiar Robertson-Walker form.
  - Different branches for flat connection:

	general	symmetric	metric
spatially flat	3	3	1
spatially curved	2	1	2
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  - $3 + 1$-split and irreducible decomposition.
  - Gauge transformations: universal prescription for gauge-invariant variables.
- Application to teleparallel gravity (example):
  - $f(G)$, $f(T)$, $f(Q)$ yield same cosmological dynamics on one branch.
  - $f(G)$, $f(T)$, $f(Q)$ cosmological dynamics differ for other branches.
  - Strong coupling problem in $f(T)$ gravity.
  - Even stronger coupling problem in $f(Q)$ and $f(G)$?

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