Cosmological perturbations in teleparallel gravity

Manuel Hohmann

Laboratory of Theoretical Physics - Institute of Physics - University of Tartu
Center of Excellence "The Dark Side of the Universe"

European Union
European Regional
Development Fund

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Space Science @ Drop Tower Seminar - 15. November 2021

Outline

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Motivation

? Open questions in cosmology and gravity:
  - Accelerating phases in the history of the Universe (inflation, dark energy)?
  - Relation between gravity and gauge theories / particle physics?
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  - Based on tetrad and flat spin connection.
  - Alternative formulation using metric and flat, metric-compatible connection.
  - First order action, second order field equations.
  - Spin connection as Lorentz gauge degree of freedom.

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  - Consider approximation of gravitational field around FLRW background solution.
  - Characterizes gravity theories by dynamics of the perturbations.
  - Dynamics of perturbations related to observations (CMB, gravitational waves. . .).

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  - Characterizes gravity theories by dynamics of the perturbations.
  - Dynamics of perturbations related to observations (CMB, gravitational waves. . .).
  - ⇝ General framework invites for generic computer algebra implementation.
- Example discussed here: $f(T)$ class of gravity theories:
  - Simple yet general class of teleparallel gravity theories.
  - Well-studied cosmological background dynamics and viable models.
  - Consistent with post-Newtonian and gravitational wave experiments.
  - ↯ Strong coupling around flat FLRW background: missing perturbative modes.
  - ⇝ Need to study perturbations around spatially non-flat FLRW background.

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Definition of metric-affine geometry

- Metric tensor $g_{\mu\nu}$:
  - Defines length of and angle between tangent vectors.
  - Defines length of curves and proper time.
  - Defines causality (spacelike and timelike directions).

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- Three characteristic quantities:
  - Curvature:

$$R^{\mu}{}_{\nu\rho\sigma} = \partial_{\rho}\Gamma^{\mu}{}_{\nu\sigma} - \partial_{\sigma}\Gamma^{\mu}{}_{\nu\rho} + \Gamma^{\mu}{}_{\tau\rho}\Gamma^{\tau}{}_{\nu\sigma} - \Gamma^{\mu}{}_{\tau\sigma}\Gamma^{\tau}{}_{\nu\rho} \,. \tag{1}$$

  - Torsion:

$$T^{\mu}{}_{\nu\rho} = \Gamma^{\mu}{}_{\rho\nu} - \Gamma^{\mu}{}_{\nu\rho} \,. \tag{2}$$

  - Nonmetricity:

$$Q_{\mu\nu\rho} = \nabla_{\mu}g_{\nu\rho} = \partial_{\mu}g_{\nu\rho} - \Gamma^{\sigma}{}_{\nu\mu}g_{\sigma\rho} - \Gamma^{\sigma}{}_{\rho\mu}g_{\nu\sigma} \,. \tag{3}$$

Teleparallel geometries

- Fundamental fields in the Palatini / metric-affine formulation:
  - Metric tensor $g_{\mu\nu}$.
  - Flat affine connection $\Gamma^{\mu}{}_{\nu\rho} = 0$: vanishing curvature

$$R^{\rho}{}_{\sigma\mu\nu} = \partial_{\mu}\Gamma^{\rho}{}_{\sigma\nu} - \partial_{\nu}\Gamma^{\rho}{}_{\sigma\mu} + \Gamma^{\rho}{}_{\lambda\mu}\Gamma^{\lambda}{}_{\sigma\nu} - \Gamma^{\rho}{}_{\lambda\nu}\Gamma^{\lambda}{}_{\sigma\mu} = 0\,. \tag{4}$$

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- The flavors of teleparallel geometries: vanishing curvature
  - Metric teleparallel geometry: vanishing nonmetricity

$$Q_{\rho\mu\nu} = \nabla_{\rho}g_{\mu\nu} = 0\,. \tag{5}$$

  - Symmetric teleparallel geometry: vanishing torsion

$$T^{\rho}{}_{\mu\nu} = \Gamma^{\rho}{}_{\nu\mu} - \Gamma^{\rho}{}_{\mu\nu} = 0\,. \tag{6}$$

  - General teleparallel geometry: allow both torsion $T^{\rho}{}_{\mu\nu}$ and nonmetricity $Q_{\rho\mu\nu}$.

Metric teleparallel geometry: tetrad and spin connection

- Metric teleparallelism conventionally formulated using:
  - Tetrad / coframe: $\theta^A = \theta^A{}_\mu \mathrm{d}x^\mu$ with inverse $e_A = e_A{}^\mu \partial_\mu$.
  - Spin connection: $\omega^A{}_B = \omega^A{}_{B\mu} \mathrm{d}x^\mu$.

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- Induced metric-affine geometry:
  - Metric:

$$g_{\mu\nu} = \eta_{AB} \theta^A{}_\mu \theta^B{}_\nu \,. \tag{7}$$

  - Affine connection:

$$\Gamma^\mu{}_{\nu\rho} = e_A{}^\mu \left(\partial_\rho \theta^A{}_\nu + \omega^A{}_{B\rho} \theta^B{}_\nu \right) \,. \tag{8}$$

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- Conditions on the spin connection:
  - Flatness $R = 0$:
$$\partial_\mu \omega^A{}_{B\nu} - \partial_\nu \omega^A{}_{B\mu} + \omega^A{}_{C\mu} \omega^C{}_{B\nu} - \omega^A{}_{C\nu} \omega^C{}_{B\mu} = 0 \,. \tag{9}$$
  - Metric compatibility $Q = 0$:
$$\eta_{AC} \omega^C{}_{B\mu} + \eta_{BC} \omega^C{}_{A\mu} = 0 \,. \tag{10}$$

Local Lorentz invariance

- Local Lorentz transformation of the tetrad only:

$$\theta^A{}_\mu \mapsto \theta'^A{}_\mu = \Lambda^A{}_B \theta^B{}_\mu \,. \tag{11}$$

✓ Metric is invariant: $g'_{\mu\nu} = g_{\mu\nu}$.
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- Perform also transformation of the spin connection:

$$\omega^A{}_{B\mu} \mapsto \omega'^A{}_{B\mu} = \Lambda^A{}_C (\Lambda^{-1})^D{}_B \omega^C{}_{D\mu} + \Lambda^A{}_C \partial_\mu (\Lambda^{-1})^C{}_B \,. \tag{12}$$

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⇒ Metric-affine geometry equivalently described by:

  - Metric $g_{\mu\nu}$ and affine connection $\Gamma^\mu{}_{\nu\rho}$.
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  - Equivalence defined with respect to local Lorentz transformations.

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  - Equivalence class of tetrad $\theta^A{}_\mu$ and spin connection $\omega^A{}_{B\mu}$.
  - Equivalence defined with respect to local Lorentz transformations.
- Teleparallel geometry admits Weitzenböck gauge: $\omega^A{}_{B\mu} \equiv 0$.

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Linear perturbations of affine connections

- General affine connection perturbation: $\Gamma^{\mu}{}_{\nu\rho} = \bar{\Gamma}^{\mu}{}_{\nu\rho} + \delta\Gamma^{\mu}{}_{\nu\rho}$.
  - ⇒ Curvature perturbation:
$$\delta R^{\rho}{}_{\sigma\mu\nu} = \bar{\nabla}_{\mu}\delta\Gamma^{\rho}{}_{\sigma\nu} - \bar{\nabla}_{\nu}\delta\Gamma^{\rho}{}_{\sigma\mu} + \bar{T}^{\omega}{}_{\mu\nu}\delta\Gamma^{\rho}{}_{\sigma\omega}\,. \tag{13}$$
  - ⇒ Torsion perturbation:
$$\delta T^{\mu}{}_{\nu\rho} = \delta\Gamma^{\mu}{}_{\rho\nu} - \delta\Gamma^{\mu}{}_{\nu\rho}\,. \tag{14}$$

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- Restriction to particular geometries:
  - Vanishing torsion $T^{\mu}{}_{\nu\rho} \equiv 0$:

$$0 = \delta T^{\mu}{}_{\nu\rho} \quad \Leftrightarrow \quad \delta\Gamma^{\mu}{}_{\nu\rho} = \delta\Gamma^{\mu}{}_{\rho\nu}\,. \tag{15}$$

  - Vanishing curvature $R^{\rho}{}_{\sigma\mu\nu} \equiv 0$:

$$0 = \delta R^{\rho}{}_{\sigma\mu\nu} \quad \Leftrightarrow \quad \delta\Gamma^{\mu}{}_{\nu\rho} = \bar{\nabla}_{\rho}\tau^{\mu}{}_{\nu}\,. \tag{16}$$

  - Vanishing torsion $T^{\mu}{}_{\nu\rho} \equiv 0$ and curvature $R^{\rho}{}_{\sigma\mu\nu} \equiv 0$:

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- Restriction to particular geometries:
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$$0 = \delta T^{\mu}{}_{\nu\rho} \quad \Leftrightarrow \quad \delta\Gamma^{\mu}{}_{\nu\rho} = \delta\Gamma^{\mu}{}_{\rho\nu}\,. \tag{15}$$

  - Vanishing curvature $R^{\rho}{}_{\sigma\mu\nu} \equiv 0$: 16 components

$$0 = \delta R^{\rho}{}_{\sigma\mu\nu} \quad \Leftrightarrow \quad \delta\Gamma^{\mu}{}_{\nu\rho} = \bar{\nabla}_{\rho}\tau^{\mu}{}_{\nu}\,. \tag{16}$$

  - Vanishing torsion $T^{\mu}{}_{\nu\rho} \equiv 0$ and curvature $R^{\rho}{}_{\sigma\mu\nu} \equiv 0$: 4 components

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- In the following, focus on teleparallel case.

Linear perturbations of metric-affine geometry

- General metric perturbation: $g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu}$.

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- Restriction to particular geometries:
  - Riemann-Cartan geometry $Q_{\rho\mu\nu} \equiv 0$:

$$0 = \delta Q_{\rho\mu\nu} \quad \Leftrightarrow \quad \bar{g}_{\sigma\nu}\delta\Gamma^\sigma{}_{\mu\rho} + \bar{g}_{\mu\sigma}\delta\Gamma^\sigma{}_{\nu\rho} = \bar{\nabla}_\rho \delta g_{\mu\nu} \,. \tag{19}$$

  - Riemannian geometry $Q_{\rho\mu\nu} \equiv 0$ and $T^\mu{}_{\nu\rho} \equiv 0$:

$$0 = \delta T^\mu{}_{\nu\rho} \wedge 0 = \delta Q_{\rho\mu\nu} \quad \Leftrightarrow \quad \delta\Gamma^\rho{}_{\mu\nu} = \frac{1}{2}\bar{g}^{\rho\sigma}\left(\bar{\nabla}_\mu \delta g_{\sigma\nu} + \bar{\nabla}_\nu \delta g_{\mu\sigma} - \bar{\nabla}_\sigma \delta g_{\mu\nu}\right) \,. \tag{20}$$

  - Metric teleparallel geometry $Q_{\rho\mu\nu} \equiv 0$ and $R^\rho{}_{\sigma\mu\nu} \equiv 0$:

$$0 = \delta R^\rho{}_{\sigma\mu\nu} \wedge 0 = \delta Q_{\rho\mu\nu} \quad \Leftrightarrow \quad \delta g_{\mu\nu} = \tau_{\mu\nu} + \tau_{\nu\mu} \,. \tag{21}$$

Linear perturbations of metric-affine geometry

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Cosmological metric teleparallel background geometry

- Friedmann-Lemaître-Robertson-Walker metric:

$$\bar{g}_{\mu\nu}\mathrm{d}x^{\mu} \otimes \mathrm{d}x^{\nu} = -n_{\mu}n_{\nu} + h_{\mu\nu} = -N^2\mathrm{d}t \otimes \mathrm{d}t + A^2\gamma_{ab}\mathrm{d}x^a \otimes \mathrm{d}x^b \,. \tag{22}$$

⇒ Scale factor $A(t)$ and lapse function $N(t)$ depend on time t, metric γ_{ab} does not.

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- Cosmologically symmetric torsion and contortion tensors:

$$\bar{T}_{\mu\nu\rho} = \frac{2\mathscr{V} h_{\mu[\nu}n_{\rho]} + 2\mathscr{A}\varepsilon_{\mu\nu\rho}}{A} \,, \quad \bar{K}_{\mu\nu\rho} = \frac{2\mathscr{V} h_{\rho[\mu}n_{\nu]} - \mathscr{A}\varepsilon_{\mu\nu\rho}}{A} \,. \tag{23}$$

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- Two branches of cosmologically symmetric teleparallel geometries: [MH '20]
  1. "Vector" branch:

$$\mathscr{V} = \mathcal{H} \pm iu \,, \quad \mathscr{A} = 0 \,, \tag{24}$$

  2. "Axial" branch:

$$\mathscr{V} = \mathcal{H} \,, \quad \mathscr{A} = \pm u \,. \tag{25}$$

⇒ Torsion depends on constant $k = u^2$ and conformal Hubble parameter $\mathcal{H} = N^{-1}\partial_t A$.

Spatial geometry and $3 + 1$ decomposition

- Decomposition of Friedmann-Lemaître-Robertson-Walker metric:

$$\bar{g}_{\mu\nu} = -n_\mu n_\nu + h_{\mu\nu} \,. \tag{26}$$

Spatial geometry and $3 + 1$ decomposition

- Decomposition of Friedmann-Lemaître-Robertson-Walker metric:

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- Unit normal (co-)vector field:

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Projectors and tensor decomposition

- Introduce covariant and contravariant spatial projectors:

$$\Pi^a_\mu \partial_a \otimes \mathrm{d}x^\mu = A\delta^a_b\,\partial_a \otimes \mathrm{d}x^b\,, \quad \Pi^\mu_a \partial_\mu \otimes \mathrm{d}x^a = A^{-1}\delta^b_a\,\partial_b \otimes \mathrm{d}x^a\,. \tag{30}$$

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⇒ Relation of projectors with temporal and spatial metric components:

$$n_\mu \Pi^\mu_a = 0 \,, \quad n^\mu \Pi^a_\mu = 0 \,, \quad h_{\mu\nu}\Pi^\mu_a \Pi^\nu_b = \gamma_{ab} \,, \quad \gamma_{ab}\Pi^a_\mu \Pi^b_\nu = h_{\mu\nu} \,. \tag{31}$$

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$$\delta^\mu_\nu = -n^\mu n_\nu + h^\mu_\nu = -n^\mu n_\nu + \Pi^\mu_a \Pi^a_\nu\,, \quad \Pi^a_\mu \Pi^\mu_b = \delta^a_b\,. \tag{32}$$

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$$X = N^{-1}\hat{X}^0\partial_t + A^{-1}\hat{X}^a\partial_a \quad \Leftrightarrow \quad \hat{X}^0 = -n_\mu X^\mu = NX^0\,, \quad \hat{X}^a = \Pi^a_\mu X^\mu = AX^a\,, \tag{33a}$$

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⇒ Indices of decomposed components are raised and lowered with Minkowski metric:

$$X^\mu = g^{\mu\nu}X_\nu \quad \Leftrightarrow \quad \hat{X}^0 = -\hat{X}_0\,, \quad \hat{X}^a = \gamma^{ab}\hat{X}_b\,. \tag{34}$$

Derivative decomposition

- Space-time split of Levi-Civita covariant derivative:

$$\overset{\circ}{\nabla}_\alpha X^\beta =$$

$$=$$

(35)

Derivative decomposition

- Space-time split of Levi-Civita covariant derivative:

$$\overset{\circ}{\nabla}_\alpha X^\beta = (h^\gamma_\alpha - n_\alpha n^\gamma)(h^\beta_\delta - n^\beta n_\delta)\overset{\circ}{\nabla}_\gamma(n^\delta \hat{X}^0 + \Pi^\delta_a \hat{X}^a)$$

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- Introduce projectors for space-time split.

Derivative decomposition

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- Introduce projectors for space-time split.
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- Introduce projectors for space-time split.
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- Hubble parameter enters through derivative of projectors:
  - Eulerian observers move on geodesics ⇒ acceleration vanishes:

$$a_\mu = n^\nu \overset{\circ}{\nabla}_\nu n_\mu = 0\,. \tag{36}$$

  - Spatial geometry is maximally symmetric ⇒ extrinsic curvature:

$$K_{\mu\nu} = \overset{\circ}{\nabla}_\mu n_\nu + n_\mu a_\nu = H h_{\mu\nu}\,. \tag{37}$$

Time coordinate and derivatives

- Lapse function N can be fixed by choice of time coordinate:
  - Cosmological time $t \equiv \hat{t}$: lapse function $N \equiv 1$.
  - Conformal time $t \equiv \mathfrak{t}$: lapse function $N \equiv A$.

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- Common notation for derivatives of scalar function $f = f(t)$:
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$$\dot{f} = \frac{df}{d\hat{t}} = \frac{1}{N}\partial_t f = \mathcal{L}_n f\,. \tag{39}$$

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- Example: cosmological and conformal Hubble parameters $H, \mathcal{H}$:

$$\mathcal{H} = \frac{A'}{A} = \dot{A} = AH\,. \tag{41}$$

Outline

Perturbation decomposition

1. Consider linear perturbation $\tau_{\mu\nu} = \eta_{AB}\bar{\theta}^A{}_\mu \delta\theta^B{}_\nu$ of tetrad.

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3. Irreducible decomposition of tetrad components:

$$\hat{\tau}_{00} = \hat{\phi}\,, \tag{42a}$$

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5. Note that the term $\mathrm{d}_b\hat{c}_a$ is not symmetrized: [Golovnev, Koivisto '18]
   - Antisymmetric part $\mathrm{d}_{[a}\hat{c}_{b]} = \frac{1}{2}\upsilon_{abc}\upsilon^{dec}\mathrm{d}_d\hat{c}_e$ can be absorbed into $\hat{w}^a$.
   - Vanishing divergence follows from Bianchi identity

$$\mathrm{d}_c(\upsilon^{dec}\mathrm{d}_d\hat{c}_e) = \upsilon^{dec}\mathrm{d}_{[c}\mathrm{d}_{d]}\hat{c}_e = \frac{1}{2}\upsilon^{dec}R^f{}_{ecd}\hat{c}_f = 0\,. \tag{44}$$

Perturbed gravitational field equations

- Perturbative expansion of gravitational field equations:

$$\bar{E}_A{}^{\mu} + \mathfrak{E}_A{}^{\mu} = E_A{}^{\mu} = \Theta_A{}^{\mu} = \bar{\Theta}_A{}^{\mu} + \mathfrak{T}_A{}^{\mu}\,, \tag{45}$$

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- Structure of background equations determined by cosmological symmetry:

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- Quantities $\mathfrak{N}$, $\mathfrak{H}$ and $\mathfrak{E}_{\mu\nu}$ determined from gravity theory.

Irreducible decomposition of perturbed equations

- Decomposition of perturbed gravitational field tensor similar to tetrad:

$$\hat{\mathfrak{E}}_{00} = \hat{\Phi}\,, \tag{49a}$$

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- Decomposition of perturbed energy-momentum around perfect fluid:

$$\hat{\mathfrak{T}}_{00} = \hat{\mathcal{E}} + \bar{\rho}\hat{\phi}\,, \tag{50a}$$

$$\hat{\mathfrak{T}}_{0b} = -\left[(\bar{\rho}+\bar{p})(\mathrm{d}_b\hat{\mathcal{L}} + \hat{\mathcal{X}}_b) + \bar{p}(\hat{v}_b + \mathrm{d}_b\hat{y})\right]\,, \tag{50b}$$

$$\hat{\mathfrak{T}}_{a0} = -\left[(\bar{\rho}+\bar{p})(\mathrm{d}_a\hat{\mathcal{L}} + \hat{\mathcal{X}}_a + \hat{v}_a + \mathrm{d}_a\hat{y}) + \bar{p}(\hat{b}_a + \mathrm{d}_a\hat{j})\right]\,, \tag{50c}$$

$$\begin{aligned}\hat{\mathfrak{T}}_{ab} = {}& \hat{\mathcal{P}}\gamma_{ab} + \mathrm{d}_a\mathrm{d}_b\hat{\mathcal{S}} - \frac{1}{3}\triangle\hat{\mathcal{S}}\gamma_{ab} + \mathrm{d}_{(a}\hat{\mathcal{V}}_{b)} + \hat{\mathcal{T}}_{ab} \\ & - \bar{p}\left[\hat{\psi}\gamma_{ab} + \mathrm{d}_b\mathrm{d}_a\hat{\sigma} + \mathrm{d}_a\hat{c}_b - \upsilon_{abc}(\mathrm{d}^c\hat{\xi} + \hat{w}^c) + \frac{1}{2}\hat{q}_{ab}\right]\,.\end{aligned} \tag{50d}$$

Outline

Gauge transformations and gauge-invariant variables

- Consider infinitesimal coordinate transformation:

$$x'^{\mu} = x^{\mu} + X^{\mu}\,. \tag{51}$$

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$$\delta\theta^A{}_{\mu} - \delta\theta'^A{}_{\mu} = (£_X\bar{\theta})^A{}_{\mu}\,. \tag{52}$$

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$$\tau_{\mu\nu} - \tau'_{\mu\nu} = \bar{\nabla}_\nu X_\mu - \bar{T}_{\mu\nu}{}^\rho X_\rho = \bar{\mathring{\nabla}}_\nu X_\mu + \bar{K}_{\mu\nu}{}^\rho X_\rho\,. \tag{53}$$

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⇒ Similar relations for gravitational field equations:

$$\mathfrak{E}_{\mu\nu} - \mathfrak{E}'_{\mu\nu} = X^{\rho}\bar{\nabla}_{\rho}\bar{E}_{\mu\nu} + \bar{E}_{\alpha\gamma}\bar{\nabla}_{\beta}X^{\gamma} + T^{\gamma}{}_{\delta\beta}X^{\delta}\bar{E}_{\alpha\gamma} \,, \tag{54a}$$

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⇝ Gauge condition on perturbation variables ⇒ fixed choice of X.

Gauge choices

1. "Zero gauge":

$$\underset{0}{\hat{\boldsymbol{j}}} = \underset{0}{\hat{\boldsymbol{\sigma}}} = 0\,, \quad \underset{0}{\hat{\boldsymbol{c}}}_a = 0\,. \tag{56}$$

$$\Rightarrow A^{-1}\underset{0}{\hat{X}}_\perp = \hat{j} + (\mathcal{H} - \mathscr{V})\hat{\sigma}\,, \quad A^{-1}\underset{0}{\hat{X}}_\| = \hat{\sigma}\,, \quad A^{-1}\underset{0}{\hat{Z}}_a = \hat{c}_a\,. \tag{57}$$

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2. Newtonian gauge:

$$\underset{\mathrm{N}}{\hat{\boldsymbol{j}}} + \underset{\mathrm{N}}{\hat{\boldsymbol{y}}} = \underset{\mathrm{N}}{\hat{\boldsymbol{\sigma}}} = 0\,, \quad \underset{\mathrm{N}}{\hat{\boldsymbol{b}}}_a + \underset{\mathrm{N}}{\hat{\boldsymbol{v}}}_a = 0\,. \tag{58}$$

$$\Rightarrow A^{-1}\underset{\mathrm{N}}{\hat{X}}_{\perp} = \hat{j} + \hat{y} - \hat{\sigma}'\,, \quad A^{-1}\underset{\mathrm{N}}{\hat{X}}_{\parallel} = \hat{\sigma}\,, \quad (A^{-1}\underset{\mathrm{N}}{\hat{Z}}_a)' = \hat{b}_a + \hat{v}_a\,. \tag{59}$$

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3. Fluid comoving gauge:

$$\underset{\mathsf{C}}{\hat{\boldsymbol{j}}} + \underset{\mathsf{C}}{\hat{\boldsymbol{y}}} = \underset{\mathsf{C}}{\hat{\boldsymbol{\mathcal{L}}}} = 0\,, \quad \underset{\mathsf{C}}{\hat{\boldsymbol{\mathcal{X}}}}_a = 0\,. \tag{60}$$

$$\Rightarrow A^{-1}\underset{\mathsf{C}}{\hat{X}}_\perp = \hat{j} + \hat{y} + \hat{\mathcal{L}}\,, \quad (A^{-1}\underset{\mathsf{C}}{\hat{X}}_\parallel)' = -\hat{\mathcal{L}}\,, \quad (A^{-1}\underset{\mathsf{C}}{\hat{Z}}_a)' = -\hat{\mathcal{X}}_a\,. \tag{61}$$

Gauge-invariant perturbations

- Scalar and pseudo-scalar perturbations:

$$\underset{?}{\hat{\boldsymbol{\psi}}} = \hat{\psi} + A^{-1}\mathcal{H}\underset{?}{\hat{X}}_{\perp}\,, \qquad \underset{?}{\hat{\boldsymbol{\sigma}}} = \hat{\sigma} - A^{-1}\underset{?}{\hat{X}}_{\parallel}\,, \tag{62a}$$

$$\underset{?}{\hat{\boldsymbol{y}}} = \hat{y} - A^{-1}(\underset{?}{\hat{X}}'_{\parallel} - \mathscr{V}\underset{?}{\hat{X}}_{\parallel})\,, \qquad \underset{?}{\hat{\boldsymbol{j}}} = \hat{j} - A^{-1}[\underset{?}{\hat{X}}_{\perp} + (\mathscr{V} - \mathcal{H})\underset{?}{\hat{X}}_{\parallel}]\,, \tag{62b}$$

$$\underset{?}{\hat{\boldsymbol{\xi}}} = \hat{\xi} + A^{-1}\mathscr{A}\underset{?}{\hat{X}}_{\parallel}\,, \qquad \underset{?}{\hat{\boldsymbol{\phi}}} = \hat{\phi} - A^{-1}\underset{?}{\hat{X}}'_{\perp}\,, \tag{62c}$$

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- Vector and pseudo-vector perturbations:

$$\underset{?}{\hat{\boldsymbol{c}}}_a = \hat{c}_a - A^{-1}\underset{?}{\hat{Z}}_a\,, \qquad \underset{?}{\hat{\boldsymbol{v}}}_a = \hat{v}_a - A^{-1}(\underset{?}{\hat{Z}}'_a - \mathscr{V}\underset{?}{\hat{Z}}_a)\,, \tag{63a}$$

$$\underset{?}{\hat{\boldsymbol{b}}}_a = \hat{b}_a - A^{-1}(\mathscr{V} - \mathcal{H})\underset{?}{\hat{Z}}_a\,, \qquad \underset{?}{\hat{\boldsymbol{w}}}_a = \hat{w}_a + A^{-1}\mathscr{A}\underset{?}{\hat{Z}}_a\,, \tag{63b}$$

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- Tensor perturbation:

$$\underset{?}{\hat{\boldsymbol{q}}} = \hat{q}_{ab}\,. \tag{64}$$

Gauge-invariant components of gravitational side

- Scalar components:

$$\hat{\underset{?}{\mathbf{\Phi}}} = \hat{\Phi} - A^{-1}(\mathfrak{N}\underset{?}{\hat{X}}'_{\perp} - \underset{?}{\hat{X}}_{\perp}\mathfrak{N}')\,, \qquad \hat{\underset{?}{\mathbf{\Psi}}} = \hat{\Psi} - A^{-1}(\mathfrak{H}\mathcal{H} - \mathfrak{H}')\underset{?}{\hat{X}}_{\perp}\,, \tag{65a}$$

$$\hat{\underset{?}{\boldsymbol{Y}}} = \hat{Y} - A^{-1}[(\mathscr{V} - \mathcal{H})\mathfrak{N}\underset{?}{\hat{X}}_{\parallel} - \mathfrak{H}\underset{?}{\hat{X}}_{\perp}]\,, \qquad \hat{\underset{?}{\mathbf{\Xi}}} = \hat{\Xi} + A^{-1}\mathfrak{H}\mathscr{A}\underset{?}{\hat{X}}_{\parallel}\,, \tag{65b}$$

$$\hat{\underset{?}{\boldsymbol{J}}} = \hat{J} - A^{-1}\{[(\mathscr{V} - \mathcal{H})\mathfrak{H} - \mathcal{H}\mathfrak{N}]\underset{?}{\hat{X}}_{\parallel} + \mathfrak{N}\underset{?}{\hat{X}}'_{\parallel}\}\,, \qquad \hat{\underset{?}{\mathbf{\Sigma}}} = \hat{\Sigma} + A^{-1}\mathfrak{H}\underset{?}{\hat{X}}_{\parallel}\,, \tag{65c}$$

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- Tensor component:

$$\underset{?}{\hat{\boldsymbol{Q}}}_{ab} = \hat{Q}_{ab} \tag{67}$$

Gauge-invariant matter variables

- Gauge-invariant density perturbation:

$$\underset{?}{\hat{\boldsymbol{\mathcal{E}}}} = \hat{\mathcal{E}} + A^{-1}\underset{?}{\hat{X}}_{\perp}\bar{\rho}' \,. \tag{68}$$

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- Decompose velocity perturbation into transverse and longitudinal part:

$$\underset{?}{\hat{\boldsymbol{\mathcal{L}}}} = \hat{\mathcal{L}} + (A^{-1}\underset{?}{\hat{X}}_{\|})' \,, \quad \underset{?}{\hat{\boldsymbol{\mathcal{X}}}}_a = \hat{\mathcal{X}}_a + (A^{-1}\underset{?}{\hat{Z}}_a)' \,. \tag{70}$$

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- Anisotropic stress is gauge-invariant; decompose into scalar, vector, tensor:

$$\underset{?}{\hat{\boldsymbol{\mathcal{S}}}} = \hat{\mathcal{S}} \,, \quad \underset{?}{\hat{\boldsymbol{\mathcal{V}}}}_a = \hat{\mathcal{V}}_a \,, \quad \underset{?}{\hat{\boldsymbol{\mathcal{T}}}}_{ab} = \hat{\mathcal{T}}_{ab} \,. \tag{71}$$

Gauge-invariant field equations

- Decompose perturbed field equations into irreducible components:
  - Scalar components:

$$\underset{?}{\hat{\boldsymbol{J}}} = -(\bar{\rho} + \bar{p})\underset{?}{\hat{\boldsymbol{\mathcal{L}}}} - \bar{p}\underset{?}{\hat{\boldsymbol{y}}}\,, \qquad \underset{?}{\hat{\boldsymbol{Y}}} = -(\bar{\rho} + \bar{p})(\underset{?}{\hat{\boldsymbol{\mathcal{L}}}} + \underset{?}{\hat{\boldsymbol{y}}})\,, \tag{72a}$$

$$\underset{?}{\hat{\boldsymbol{\Sigma}}} = \underset{?}{\hat{\boldsymbol{\mathcal{S}}}}\,, \qquad \underset{?}{\hat{\boldsymbol{\Xi}}} = \bar{p}\underset{?}{\hat{\boldsymbol{\xi}}}\,, \tag{72b}$$

$$\underset{?}{\hat{\boldsymbol{\Psi}}} = \underset{?}{\hat{\boldsymbol{\mathcal{P}}}} - \frac{1}{3}\triangle\underset{?}{\hat{\boldsymbol{\mathcal{S}}}} - \bar{p}\underset{?}{\hat{\boldsymbol{\psi}}}\,, \qquad \underset{?}{\hat{\boldsymbol{\Phi}}} = \underset{?}{\hat{\boldsymbol{\mathcal{E}}}} + \bar{\rho}\underset{?}{\hat{\boldsymbol{\phi}}}\,. \tag{72c}$$

  - Vector components:

$$\underset{?}{\hat{\boldsymbol{V}}}_a = -(\bar{\rho} + \bar{p})(\underset{?}{\hat{\boldsymbol{\mathcal{X}}}}_a + \underset{?}{\hat{\boldsymbol{v}}}_a) - \bar{p}\underset{?}{\hat{\boldsymbol{b}}}_a\,, \qquad \underset{?}{\hat{\boldsymbol{W}}}_a = \bar{p}\underset{?}{\hat{\boldsymbol{w}}}_a - \frac{1}{2}\upsilon_{abc}\mathrm{d}^b\underset{?}{\hat{\boldsymbol{\mathcal{V}}}}^c\,, \tag{73a}$$

$$\underset{?}{\hat{\boldsymbol{B}}}_a = -(\bar{\rho} + \bar{p})\underset{?}{\hat{\boldsymbol{\mathcal{X}}}}_b - \bar{p}\underset{?}{\hat{\boldsymbol{v}}}_b\,, \qquad \underset{?}{\hat{\boldsymbol{C}}}_a = \underset{?}{\hat{\boldsymbol{\mathcal{V}}}}_a\,. \tag{73b}$$

  - Tensor component:

$$\underset{?}{\hat{\boldsymbol{Q}}}_{ab} = 2\underset{?}{\hat{\boldsymbol{\mathcal{T}}}}_{ab} - \bar{p}\underset{?}{\hat{\boldsymbol{q}}}_{ab}\,. \tag{74}$$

Gauge-invariant field equations

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✓ Equations are fully gauge-invariant.

Gauge-invariant field equations

- Decompose perturbed field equations into irreducible components:
  - Scalar components:

$$\underset{?}{\hat{\boldsymbol{J}}} = -(\bar{\rho} + \bar{p})\underset{?}{\hat{\boldsymbol{\mathcal{L}}}} - \bar{p}\underset{?}{\hat{\boldsymbol{y}}}\,, \qquad \underset{?}{\hat{\boldsymbol{Y}}} = -(\bar{\rho} + \bar{p})(\underset{?}{\hat{\boldsymbol{\mathcal{L}}}} + \underset{?}{\hat{\boldsymbol{y}}})\,, \tag{72a}$$

$$\underset{?}{\hat{\boldsymbol{\Sigma}}} = \underset{?}{\hat{\boldsymbol{\mathcal{S}}}}\,, \qquad \underset{?}{\hat{\boldsymbol{\Xi}}} = \bar{p}\underset{?}{\hat{\boldsymbol{\xi}}}\,, \tag{72b}$$

$$\underset{?}{\hat{\boldsymbol{\Psi}}} = \underset{?}{\hat{\boldsymbol{\mathcal{P}}}} - \frac{1}{3}\triangle\underset{?}{\hat{\boldsymbol{\mathcal{S}}}} - \bar{p}\underset{?}{\hat{\boldsymbol{\psi}}}\,, \qquad \underset{?}{\hat{\boldsymbol{\Phi}}} = \underset{?}{\hat{\boldsymbol{\mathcal{E}}}} + \bar{\rho}\underset{?}{\hat{\boldsymbol{\phi}}}\,. \tag{72c}$$

  - Vector components:

$$\underset{?}{\hat{\boldsymbol{V}}}_a = -(\bar{\rho} + \bar{p})(\underset{?}{\hat{\boldsymbol{\mathcal{X}}}}_a + \underset{?}{\hat{\boldsymbol{v}}}_a) - \bar{p}\underset{?}{\hat{\boldsymbol{b}}}_a\,, \qquad \underset{?}{\hat{\boldsymbol{W}}}_a = \bar{p}\underset{?}{\hat{\boldsymbol{w}}}_a - \frac{1}{2}\upsilon_{abc}\mathrm{d}^b\underset{?}{\hat{\boldsymbol{\mathcal{V}}}}^c\,, \tag{73a}$$

$$\underset{?}{\hat{\boldsymbol{B}}}_a = -(\bar{\rho} + \bar{p})\underset{?}{\hat{\boldsymbol{\mathcal{X}}}}_b - \bar{p}\underset{?}{\hat{\boldsymbol{v}}}_b\,, \qquad \underset{?}{\hat{\boldsymbol{C}}}_a = \underset{?}{\hat{\boldsymbol{\mathcal{V}}}}_a\,. \tag{73b}$$

  - Tensor component:

$$\underset{?}{\hat{\boldsymbol{Q}}}_{ab} = 2\underset{?}{\hat{\boldsymbol{\mathcal{T}}}}_{ab} - \bar{p}\underset{?}{\hat{\boldsymbol{q}}}_{ab}\,. \tag{74}$$

✓ Equations are fully gauge-invariant.

⇝ Remaining task: determine components of gravity side from gravity theory.

Outline

Key features needed from implementation (WIP)

1. Pre-defined geometric objects:
   - Tetrad with cosmological symmetry and its perturbation.
   - Different connections: Levi-Civita and metric teleparallel.
   - Tensors related to curvature and torsion.

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2. Variables specific to cosmological perturbations:
   - Energy-momentum variables: density, pressure, velocity, anisotropic stress.
   - Spatial geometry with metric γ_{ab} and Levi-Civita derivative d_a.
   - Projectors Π_a^μ and Π^a_μ to facilitate $3+1$ split.
   - Time-dependent scalar functions: $N, A, H, \mathcal{H}, \mathscr{V}, \mathscr{A}, \ldots$
   - Irreducible components of tetrad perturbation and perturbed field equations.

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   - Irreducible components of tetrad perturbation and perturbed field equations.
3. Algorithms typically used in cosmological perturbations:
   - Linear perturbation of all quantities with respect to tetrad perturbation.
   - $3+1$ decomposition of tensors and connection coefficients into time and space.
   - Substitution of background values for cosmologically symmetric tensors.
   - Irreducible decomposition of perturbations.
   - Transformation from and to gauge-invariant variables.
   - Transformation between different choice of time coordinate.

Work in progress: some known quantities

1. Scalar functions of time:

```
In[]:= {LapseF[], ScaleF[], Hubble[],
    CHubble[], VecTor[], AxiTor[]}
```

Out[]= $\{N, A, H, \mathcal{H}, \mathscr{V}, \mathscr{A}\}$

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2. Background metric and its decomposition:

```
In[]:= SMet[-T4α, -T4β] - Orth[-T4α] * Orth[-T4β]
```

Out[]= $-n_\alpha n_\beta + h_{\alpha\beta}$

```
In[]:= ProjectorToMetric[%]
```

Out[]= $g_{\alpha\beta}$

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```

Out[]= $g_{\alpha\beta}$

3. Projector fields:

```
In[]:= {ProjCon[-T4α, T3a], ProjCov[T4α, -T3a]}
```

Out[]= $\{\Pi^a{}_\alpha, \Pi_a{}^\alpha\}$

Work in progress: $3+1$ decomposition of tensors

1. Usual $3+1$ decomposition $g_{\mu\nu} \rightsquigarrow g_{00}, g_{a0}, g_{0b}, g_{ab}$ uses lapse and scale factor:

```
In[]:= SpaceTimeSplits[Met[-T4α, -T4β],
    {-T4α → -T3a, -T4β → -T3b}]
```

Out[]= $\{\{N^2\hat{g}_{00}, NA\hat{g}_{0b}\}, \{NA\hat{g}_{a0}, A^2\hat{g}_{ab}\}\}$

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2. Alternative approach using projectors and without explicit factors:

```
In[]:= SpaceTimeExpand[Met[-T4α, -T4β]]
```

Out[]= $n_\alpha n_\beta \hat{g}_{00} - n_\beta \Pi^a_\alpha \hat{g}_{0a} - n_\alpha \Pi^a_\beta \hat{g}_{0a} + \Pi^a_\alpha \Pi^b_\beta \hat{g}_{ab}$

```
In[]:= SpaceTimeSplits[%, {-T4α → -T3a, -T4β → -T3b}]
```

Out[]= $\{\{N^2\hat{g}_{00}, NA\hat{g}_{0b}\}, \{NA\hat{g}_{a0}, A^2\hat{g}_{ab}\}\}$

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```
In[]:= SpaceTimeSplits[%, {-T4α → -T3a, -T4β → -T3b}]
```

Out[]= $\{\{N^2\hat{g}_{00}, NA\hat{g}_{0b}\}, \{NA\hat{g}_{a0}, A^2\hat{g}_{ab}\}\}$

3. Use automatic background substitution $\hat{g}_{00} = -1, \hat{g}_{0a} = 0, \hat{g}_{ab} = \gamma_{ab}$:

```
In[]:= SpaceTimeSplits[Met[-T4α, -T4β],
    {-T4α → -T3a, -T4β → -T3b}, UseCosmoRules → True]
```

Out[]= $\{\{N^2, 0\}, \{0, A^2\gamma_{ab}\}\}$

```
In[]:= SpaceTimeExpand[Met[-T4α, -T4β], UseCosmoRules → True]
```

Out[]= $-n_\alpha n_\beta + \Pi^a_\alpha \Pi^b_\beta \gamma_{ab}$

Work in progress: $3+1$ decomposition of derivatives

1. Partial derivative of scalar:

```
In[]:= DefTensor[S[], {MfSpacetime}]
In[]:= SpaceTimeSplits[PD[-T4α][S[]], {-T4α → -T3a}]
Out[]= {∂₀Ŝ, ∂ₐŜ}
```

Work in progress: $3+1$ decomposition of derivatives

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In[]:= DefTensor[S[], {MfSpacetime}]
In[]:= SpaceTimeSplits[PD[-T4α][S[]], {-T4α → -T3a}]
```

Out[]= $\{\partial_0\hat{S}, \partial_a\hat{S}\}$

2. Levi-Civita covariant derivative of vector field:

```
In[]:= DefTensor[X[T4α], {MfSpacetime}]
In[]:= SpaceTimeSplits[CD[-T4α][X[T4β]],
    {-T4α → -T3a, T4β→ T3b}]
```

Out[]= $\left\{\left\{\frac{\partial_0\hat{X}^0}{N}, \frac{\partial_0\hat{X}^b}{A}\right\}, \left\{\frac{\mathrm{d}_a\hat{X}^0}{N} + \frac{\gamma_{ab}HA\hat{X}^b}{N}, \frac{\mathrm{d}_a\hat{X}^b}{A} + \delta_a^b H\hat{X}^0\right\}\right\}$

Work in progress: $3+1$ decomposition of derivatives

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Out[]= $\left\{\left\{\frac{\partial_0\hat{X}^0}{N}, \frac{\partial_0\hat{X}^b}{A}\right\}, \left\{\frac{\mathrm{d}_a\hat{X}^0}{N} + \frac{\gamma_{ab}HA\hat{X}^b}{N}, \frac{\mathrm{d}_a\hat{X}^b}{A} + \delta_a^b H\hat{X}^0\right\}\right\}$

3. Purely spatial part:

```
In[]:= SpaceTimeSplits[SD[-T4α][ProjectorSMet[X[T4β]]],
    {-T4α → -T3a, T4β→ T3b}]
```

Out[]= $\left\{\{0,0\}, \left\{0, \frac{\mathrm{d}_a\hat{X}^b}{A}\right\}\right\}$

Work in progress: calculating perturbations

1. Tetrad perturbation is expanded into $\tau_{\alpha\beta}$:

```
In[]:= Perturbation[Tet[L4Γ, -T4α]]
```

Out[]= $\tau^{\beta}{}_{\alpha}\theta^{\Gamma}{}_{\beta}$

```
In[]:= Perturbation[InvTet[-L4Γ, T4α]]
```

Out[]= $-e_{\Gamma}{}^{\beta}\tau^{\alpha}{}_{\beta}$

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In[]:= Perturbation[Tet[L4Γ, -T4α]]
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In[]:= Perturbation[InvTet[-L4Γ, T4α]]
```
Out[]= $-e_{\Gamma}{}^{\beta}\tau^{\alpha}{}_{\beta}$

2. Perturbations of common tensors:

```
In[]:= Perturbation[Met[-T4α, -T4β]]
```
Out[]= $\tau_{\alpha\beta} + \tau_{\beta\alpha}$
```
In[]:= Perturbation[TorsionFD[T4α, -T4β, -T4γ]]
```
Out[]= $\dot{\nabla}_{\beta}\tau^{\alpha}{}_{\gamma} - \dot{\nabla}_{\gamma}\tau^{\alpha}{}_{\beta}$

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In[]:= Perturbation[Met[-T4α, -T4β]]
```

Out[]= $\tau_{\alpha\beta} + \tau_{\beta\alpha}$

```
In[]:= Perturbation[TorsionFD[T4α, -T4β, -T4γ]]
```

Out[]= $\dot{\nabla}_{\beta}\tau^{\alpha}{}_{\gamma} - \dot{\nabla}_{\gamma}\tau^{\alpha}{}_{\beta}$

3. Perturbation of field equations defined from mixed form:

```
In[]:= Perturbation[GravField[-T4α, -T4β]]
```

Out[]= $\mathfrak{E}_{\alpha\beta} + E_{\alpha}{}^{\gamma}\tau_{\beta\gamma} + E^{\gamma}{}_{\beta}\tau_{\gamma\alpha} + E_{\alpha}{}^{\gamma}\tau_{\gamma\beta}$

Work in progress: irreducible decomposition

1. Spatial part of tetrad perturbation:

```
In[]:= ExpandTau[CT[Tau][-T3a, -T3b]]
```

Out[]= $\hat{\psi}\gamma_{ab} + \mathrm{d}_a\mathrm{d}_b\hat{\sigma} + \mathrm{d}_b\hat{c}_a + \upsilon_{abc}(\mathrm{d}^c\hat{\xi} + \hat{w}^c) + \frac{1}{2}\hat{q}_{ab}$

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2. Properties of irreducible components:

```
In[]:= {BD[T3a][CT[TauSSt][-T3a, -T3b]], CT[TauSSt][T3a, -T3a],
    CT[TauSSt][-T3a, -T3b] - CT[TauSSt][-T3b, -T3a]}
```

Out[]= $\{\mathrm{d}^a\hat{q}_{ab}, \hat{q}^a{}_a, \hat{q}_{ab} - \hat{q}_{ba}\}$

```
In[]:= IrrDecomp /@ %
```

Out[]= $\{0, 0, 0\}$

Work in progress: irreducible decomposition

1. Spatial part of tetrad perturbation:

```
In[]:= ExpandTau[CT[Tau][-T3a, -T3b]]
```

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```
In[]:= {BD[T3a][CT[TauSSt][-T3a, -T3b]], CT[TauSSt][T3a, -T3a],
    CT[TauSSt][-T3a, -T3b] - CT[TauSSt][-T3b, -T3a]}
```

Out[]= $\{\mathrm{d}^a\hat{q}_{ab}, \hat{q}^a{}_a, \hat{q}_{ab} - \hat{q}_{ba}\}$

```
In[]:= IrrDecomp /@ %
```

Out[]= $\{0, 0, 0\}$

3. Similar expansions for gravitational field and energy-momentum:

```
In[]:= ExpandGrav[CT[GravPert][-T3a, -LI[0]]]
```

Out[]= $\mathrm{d}_a\hat{Y} + \hat{V}_a$

```
In[]:= ExpandEnMom[CT[EnMomPert][-LI[0], -LI[0]]]
```

Out[]= $\hat{\mathcal{E}} + \rho\hat{\phi}$

Work in progress: gauge-invariant quantities

1. Gauge-invariant tetrad perturbation (zero gauge):

```
In[]:= ConvFromGaugeInvTau[CT[GinvTauSSva, "0"][T3a], "0"]
```

Out[]= $\hat{w}^a + \mathscr{A}\hat{c}^a$

```
In[]:= ConvToGaugeInvTau[%, "0"]
```

Out[]= $\underset{0}{\hat{\boldsymbol{w}}}^a$

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```
Out[]= $\hat{w}^a + \mathscr{A}\hat{c}^a$
```
In[]:= ConvToGaugeInvTau[%, "0"]
```
Out[]= $\underset{0}{\hat{\boldsymbol{w}}}^a$

2. Gauge-invariant gravitational field perturbation (Newton gauge):

```
In[]:= ConvFromGaugeInvGrav[CT[GinvGravPertSSsa, "N"][], "N"]
```
Out[]= $\hat{\Xi} + \mathscr{A}\mathfrak{H}\hat{\sigma}$
```
In[]:= ConvToGaugeInvGrav[%, "N"]
```
Out[]= $\underset{N}{\hat{\boldsymbol{\Xi}}}$

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```
Out[]= $\hat{w}^a + \mathscr{A}\hat{c}^a$
```
In[]:= ConvToGaugeInvTau[%, "0"]
```
Out[]= $\underset{0}{\hat{\boldsymbol{w}}}^a$

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```
In[]:= ConvFromGaugeInvGrav[CT[GinvGravPertSSsa, "N"][], "N"]
```
Out[]= $\hat{\Xi} + \mathscr{A}\mathfrak{H}\hat{\sigma}$
```
In[]:= ConvToGaugeInvGrav[%, "N"]
```
Out[]= $\underset{\mathrm{N}}{\hat{\boldsymbol{\Xi}}}$

3. Gauge-invariant time-time component of field equations (comoving gauge):

```
In[]:= CT[GinvGravPert, "C"][-LI[0], -LI[0]] -
   CT[GinvEnMomPert, "C"][-LI[0], -LI[0]];
In[]:= % // ExpandGrav // ExpandEnMom
```
Out[]= $\underset{\mathrm{C}}{\hat{\boldsymbol{\Phi}}} - \underset{\mathrm{C}}{\hat{\boldsymbol{\mathcal{E}}}} - \rho\underset{\mathrm{C}}{\hat{\boldsymbol{\phi}}}$

Work in progress: choice of time coordinate

1. Derivatives with respect to cosmological and conformal time:

```
In[]:= {DCosmTime[ScaleF[]], DConfTime[ScaleF[]]}
```

Out[]= $\left\{\frac{\partial_0 A}{N}, \frac{A\partial_0 A}{N}\right\}$

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```

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2. Hubble parameter:

```
In[]:= Hubble[]
```

Out[]= H

```
In[]:= HubbleToDScale[%]
```

Out[]= $\frac{\partial_0 A}{NA}$

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```

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2. Hubble parameter:

```
In[]:= Hubble[]
```

Out[]= H

```
In[]:= HubbleToDScale[%]
```

Out[]= $\frac{\partial_0 A}{NA}$

3. Conformal Hubble parameter:

```
In[]:= CHubble[]
```

Out[]= $\mathcal{H}$

```
In[]:= CHubbleToDScale[%]
```

Out[]= $\frac{\partial_0 A}{N}$

Outline

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Action and field equations

- Action for $f(T)$ class of gravity theories:

$$S = -\frac{1}{2\kappa^2}\int f(T)\,\theta\,\mathrm{d}^4x + S_{\mathrm{m}} \tag{75}$$

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- Terms in the action and field equations:
  - Torsion scalar:

$$T = \frac{1}{2}T^{\rho}{}_{\mu\nu}S_{\rho}{}^{\mu\nu} = \frac{1}{4}T^{\mu\nu\rho}T_{\mu\nu\rho} + \frac{1}{2}T^{\mu\nu\rho}T_{\rho\nu\mu} - T^{\mu}{}_{\mu\rho}T_{\nu}{}^{\nu\rho}\,. \tag{76}$$

  - Superpotential:

$$S_{\rho}{}^{\mu\nu} = K^{\mu\nu}{}_{\rho} - \delta^{\mu}_{\rho}T_{\sigma}{}^{\sigma\nu} + \delta^{\nu}_{\rho}T_{\sigma}{}^{\sigma\mu}\,. \tag{77}$$

  - Contortion:

$$K^{\mu}{}_{\nu\rho} = \Gamma^{\mu}{}_{\nu\rho} - \overset{\circ}{\Gamma}{}^{\mu}{}_{\nu\rho} = \frac{1}{2}\left(T_{\nu}{}^{\mu}{}_{\rho} + T_{\rho}{}^{\mu}{}_{\nu} - T^{\mu}{}_{\nu\rho}\right)\,, \tag{78}$$

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$$-\frac{1}{2}fg_{\mu\nu} + S^{\rho\sigma}{}_{\mu}(T_{\rho\sigma\nu} - K_{\rho\nu\sigma})f_T - \overset{\circ}{\nabla}_{\rho}(S_{\nu\mu}{}^{\rho}f_T) = \kappa^2\Theta_{\mu\nu} \tag{79}$$

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↯ Theory is strongly coupled around flat FLRW background: missing degrees of freedom.

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⇒ Gravitational field equations:

$$-\frac{1}{2}fg_{\mu\nu} + S^{\rho\sigma}{}_{\mu}(T_{\rho\sigma\nu} - K_{\rho\nu\sigma})f_T - \overset{\circ}{\nabla}_{\rho}(S_{\nu\mu}{}^{\rho}f_T) = \kappa^2\Theta_{\mu\nu} \tag{79}$$

↯ Theory is strongly coupled around flat FLRW background: missing degrees of freedom.

? Does strong coupling persist around non-flat FLRW background?

Background dynamics

- Cosmological background field equations:
  1. Vector branch:

$$f - 12\frac{f_T}{A^2}\mathcal{H}(\mathcal{H} + iu) = 2\kappa^2\bar{\rho}\,, \tag{80a}$$

$$-f + 4\frac{f_T}{A^2}(2\mathcal{H}^2 + 3iu\mathcal{H} - u^2 + \mathcal{H}') + 48\frac{f_{TT}}{A^4}(\mathcal{H} + iu)^2[\mathcal{H}' - \mathcal{H}(\mathcal{H} + iu)] = 2\kappa^2\bar{p}\,. \tag{80b}$$

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  3. Flat limiting case $u \to 0$:

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⇒ Dynamics qualitatively depends on choice of cosmological branch.

⇒ Dynamics approaches common flat limit for $u \to 0$.

Outline

Perturbations: tensor equations

- General form of tensor equation:

$$\frac{1}{2}\hat{Q}_{ab} = \hat{\mathcal{T}}_{ab} - \frac{1}{2}\bar{p}\hat{q}_{ab}\,. \tag{83}$$

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- Structure of the equations:
  1. Equations contain contribution from non-vanishing spatial curvature.
  2. f_T part of equations is identical to general relativity.

Outline

Antisymmetric field equation - axial branch

- Irreducible decomposition of antisymmetric field equations:

$$f_{TT}(\upsilon_{abc}\mathrm{d}^b\underset{0}{\hat{\boldsymbol{w}}}{}^c - 2u\underset{0}{\hat{\boldsymbol{w}}}{}_a) = 0\,, \quad f_{TT}(\mathrm{d}_{[a}\underset{0}{\hat{\boldsymbol{b}}}{}_{b]} - u\upsilon_{abc}\underset{0}{\hat{\boldsymbol{b}}}{}^c) = 0\,. \tag{87}$$

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⇒ Single mode in harmonic expansion which can be absorbed: $\underset{0}{\hat{\boldsymbol{w}}}{}_a = \underset{0}{\hat{\boldsymbol{b}}}{}_a = 0$.

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Symmetric field equation

1. Vector branch:

$$a^2 f_T \left[\triangle(\underset{0}{\hat{\boldsymbol{v}}}_a + \underset{0}{\hat{\boldsymbol{b}}}_a) - 2(2\mathcal{H}^2 + u^2 - 2\mathcal{H}')(\underset{0}{\hat{\boldsymbol{v}}}_a + \underset{0}{\hat{\boldsymbol{b}}}_a) - 4(\mathcal{H}^2 + u^2 - \mathcal{H}')\underset{0}{\hat{\boldsymbol{\mathcal{X}}}}_a\right]$$
$$-12f_{TT}(\mathcal{H}+iu)[\mathcal{H}(\mathcal{H}+iu)-\mathcal{H}']\left[\upsilon_{abc}\mathrm{d}^b\underset{0}{\hat{\boldsymbol{w}}}^c + 4(\mathcal{H}+iu)(\underset{0}{\hat{\boldsymbol{\mathcal{X}}}}_a + \underset{0}{\hat{\boldsymbol{v}}}_a) + 2(2\mathcal{H}+iu)\underset{0}{\hat{\boldsymbol{b}}}_a\right] = 0 \tag{93}$$

2. Axial branch:

$$a^2 f_T \left[\triangle(\underset{0}{\hat{\boldsymbol{v}}}_a + \underset{0}{\hat{\boldsymbol{b}}}_a) - 2(2\mathcal{H}^2 + u^2 - 2\mathcal{H}')(\underset{0}{\hat{\boldsymbol{v}}}_a + \underset{0}{\hat{\boldsymbol{b}}}_a) - 4(\mathcal{H}^2 + u^2 - \mathcal{H}')\underset{0}{\hat{\boldsymbol{\mathcal{X}}}}_a\right]$$
$$-12f_{TT}\mathcal{H}(\mathcal{H}^2 - u^2 - \mathcal{H}')\left[\upsilon_{abc}\mathrm{d}^b\underset{0}{\hat{\boldsymbol{w}}}^c - 2u\underset{0}{\hat{\boldsymbol{w}}}_a + 4\mathcal{H}(\underset{0}{\hat{\boldsymbol{\mathcal{X}}}}_a + \underset{0}{\hat{\boldsymbol{v}}}_a + \underset{0}{\hat{\boldsymbol{b}}}_a)\right] = 0 \tag{94}$$

3. Flat case:

$$a^2 f_T\triangle(\underset{0}{\hat{\boldsymbol{v}}}_a+\underset{0}{\hat{\boldsymbol{b}}}_a)+4(\mathcal{H}'-\mathcal{H}^2)\left[3f_{TT}\mathcal{H}\upsilon_{abc}\mathrm{d}^b\underset{0}{\hat{\boldsymbol{w}}}^c + (a^2 f_T + 12f_{TT}\mathcal{H}^2)(\underset{0}{\hat{\boldsymbol{\mathcal{X}}}}_a + \underset{0}{\hat{\boldsymbol{v}}}_a + \underset{0}{\hat{\boldsymbol{b}}}_a)\right] = 0\,. \tag{95}$$

Symmetric field equation

1. Vector branch:

$$a^2 f_T \left[\triangle(\underset{0}{\hat{\boldsymbol{v}}}_a + \underset{0}{\hat{\boldsymbol{b}}}_a) - 2(2\mathcal{H}^2 + u^2 - 2\mathcal{H}')(\underset{0}{\hat{\boldsymbol{v}}}_a + \underset{0}{\hat{\boldsymbol{b}}}_a) - 4(\mathcal{H}^2 + u^2 - \mathcal{H}')\underset{0}{\hat{\boldsymbol{\mathcal{X}}}}_a\right] \\ -12 f_{TT}(\mathcal{H}+iu)[\mathcal{H}(\mathcal{H}+iu)-\mathcal{H}']\left[\upsilon_{abc}\mathrm{d}^b\underset{0}{\hat{\boldsymbol{w}}}^c + 4(\mathcal{H}+iu)(\underset{0}{\hat{\boldsymbol{\mathcal{X}}}}_a + \underset{0}{\hat{\boldsymbol{v}}}_a) + 2(2\mathcal{H}+iu)\underset{0}{\hat{\boldsymbol{b}}}_a\right] = 0 \tag{93}$$

2. Axial branch:

$$a^2 f_T \left[\triangle(\underset{0}{\hat{\boldsymbol{v}}}_a + \underset{0}{\hat{\boldsymbol{b}}}_a) - 2(2\mathcal{H}^2 + u^2 - 2\mathcal{H}')(\underset{0}{\hat{\boldsymbol{v}}}_a + \underset{0}{\hat{\boldsymbol{b}}}_a) - 4(\mathcal{H}^2 + u^2 - \mathcal{H}')\underset{0}{\hat{\boldsymbol{\mathcal{X}}}}_a\right] \\ - 12 f_{TT}\mathcal{H}(\mathcal{H}^2 - u^2 - \mathcal{H}')\left[\upsilon_{abc}\mathrm{d}^b\underset{0}{\hat{\boldsymbol{w}}}^c - 2u\underset{0}{\hat{\boldsymbol{w}}}_a + 4\mathcal{H}(\underset{0}{\hat{\boldsymbol{\mathcal{X}}}}_a + \underset{0}{\hat{\boldsymbol{v}}}_a + \underset{0}{\hat{\boldsymbol{b}}}_a)\right] = 0 \tag{94}$$

3. Flat case:

$$a^2 f_T \triangle(\underset{0}{\hat{\boldsymbol{v}}}_a + \underset{0}{\hat{\boldsymbol{b}}}_a) + 4(\mathcal{H}' - \mathcal{H}^2)\left[3 f_{TT}\mathcal{H}\upsilon_{abc}\mathrm{d}^b\underset{0}{\hat{\boldsymbol{w}}}^c + (a^2 f_T + 12 f_{TT}\mathcal{H}^2)(\underset{0}{\hat{\boldsymbol{\mathcal{X}}}}_a + \underset{0}{\hat{\boldsymbol{v}}}_a + \underset{0}{\hat{\boldsymbol{b}}}_a)\right] = 0\,. \tag{95}$$

⇒ Screened Poisson equation for $\underset{0}{\hat{\boldsymbol{v}}}$.

Symmetric field equation

1. Vector branch:

$$a^2 f_T \left[\triangle(\underset{0}{\hat{v}}_a + \underset{0}{\hat{b}}_a) - 2(2\mathcal{H}^2 + u^2 - 2\mathcal{H}')(\underset{0}{\hat{v}}_a + \underset{0}{\hat{b}}_a) - 4(\mathcal{H}^2 + u^2 - \mathcal{H}')\underset{0}{\hat{\mathcal{X}}}_a\right]$$
$$-12f_{TT}(\mathcal{H}+iu)[\mathcal{H}(\mathcal{H}+iu)-\mathcal{H}'] \left[\upsilon_{abc}\mathrm{d}^b\underset{0}{\hat{w}}^c + 4(\mathcal{H} + iu)(\underset{0}{\hat{\mathcal{X}}}_a + \underset{0}{\hat{v}}_a) + 2(2\mathcal{H} + iu)\underset{0}{\hat{b}}_a\right] = 0 \quad (93)$$

2. Axial branch:

$$a^2 f_T \left[\triangle(\underset{0}{\hat{v}}_a + \underset{0}{\hat{b}}_a) - 2(2\mathcal{H}^2 + u^2 - 2\mathcal{H}')(\underset{0}{\hat{v}}_a + \underset{0}{\hat{b}}_a) - 4(\mathcal{H}^2 + u^2 - \mathcal{H}')\underset{0}{\hat{\mathcal{X}}}_a\right]$$
$$- 12f_{TT}\mathcal{H}(\mathcal{H}^2 - u^2 - \mathcal{H}') \left[\upsilon_{abc}\mathrm{d}^b\underset{0}{\hat{w}}^c - 2u\underset{0}{\hat{w}}_a + 4\mathcal{H}(\underset{0}{\hat{\mathcal{X}}}_a + \underset{0}{\hat{v}}_a + \underset{0}{\hat{b}}_a)\right] = 0 \quad (94)$$

3. Flat case:

$$a^2 f_T \triangle(\underset{0}{\hat{v}}_a + \underset{0}{\hat{b}}_a) + 4(\mathcal{H}' - \mathcal{H}^2) \left[3f_{TT}\mathcal{H}\upsilon_{abc}\mathrm{d}^b\underset{0}{\hat{w}}^c + (a^2 f_T + 12f_{TT}\mathcal{H}^2)(\underset{0}{\hat{\mathcal{X}}}_a + \underset{0}{\hat{v}}_a + \underset{0}{\hat{b}}_a)\right] = 0\,. \quad (95)$$

⇒ Screened Poisson equation for $\underset{0}{\hat{v}}$.

⇝ Dynamics for velocity perturbation follows from momentum conservation.

Outline

Mixed part of antisymmetric equations

- Field equations can be solved for $\underset{\mathrm{N}}{\hat{\boldsymbol{y}}}$ in Newton gauge:
  1. Vector branch:
  $$(\mathcal{H} + iu)\triangle\underset{\mathrm{N}}{\hat{\boldsymbol{y}}} + 3iu[\mathcal{H}' - \mathcal{H}(\mathcal{H} + iu)]\underset{\mathrm{N}}{\hat{\boldsymbol{y}}} = 3(\mathcal{H} + iu)(\mathcal{H}\underset{\mathrm{N}}{\hat{\phi}} + \mathcal{H}\underset{\mathrm{N}}{\hat{\psi}} - iu\underset{\mathrm{N}}{\hat{\psi}} + \underset{\mathrm{N}}{\hat{\psi}}') - 3\mathcal{H}'\underset{\mathrm{N}}{\hat{\psi}} . \quad (96)$$
  2. Axial branch:
  $$\mathcal{H}\triangle\underset{\mathrm{N}}{\hat{\boldsymbol{y}}} = 3\mathcal{H}(\mathcal{H}\underset{\mathrm{N}}{\hat{\phi}} + \mathcal{H}\underset{\mathrm{N}}{\hat{\psi}} + \underset{\mathrm{N}}{\hat{\psi}}' - u\mathcal{H}\underset{\mathrm{N}}{\hat{\boldsymbol{\xi}}}) - 3\mathcal{H}'(\underset{\mathrm{N}}{\hat{\psi}} - u\underset{\mathrm{N}}{\hat{\boldsymbol{\xi}}}) + 3u^3\underset{\mathrm{N}}{\hat{\boldsymbol{\xi}}} + u\triangle\underset{\mathrm{N}}{\hat{\boldsymbol{\xi}}} . \quad (97)$$
  3. Flat limiting case:
  $$\mathcal{H}\triangle\underset{\mathrm{N}}{\hat{\boldsymbol{y}}} = 3\mathcal{H}(\mathcal{H}\underset{\mathrm{N}}{\hat{\phi}} + \mathcal{H}\underset{\mathrm{N}}{\hat{\psi}} + \underset{\mathrm{N}}{\hat{\psi}}') - 3\mathcal{H}'\underset{\mathrm{N}}{\hat{\psi}} . \quad (98)$$

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  2. Axial branch:
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  3. Flat limiting case:
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⇒ Poisson equation is screened only in the vector branch.

Mixed part of antisymmetric equations

- Field equations can be solved for $\underset{\mathrm{N}}{\hat{\boldsymbol{y}}}$ in Newton gauge:
  1. Vector branch:
  $$(\mathcal{H}+iu)\triangle\underset{\mathrm{N}}{\hat{\boldsymbol{y}}}+3iu[\mathcal{H}'-\mathcal{H}(\mathcal{H}+iu)]\underset{\mathrm{N}}{\hat{\boldsymbol{y}}}=3(\mathcal{H}+iu)(\mathcal{H}\underset{\mathrm{N}}{\hat{\phi}}+\mathcal{H}\underset{\mathrm{N}}{\hat{\psi}}-iu\underset{\mathrm{N}}{\hat{\psi}}+\underset{\mathrm{N}}{\hat{\psi}}')-3\mathcal{H}'\underset{\mathrm{N}}{\hat{\psi}}\,. \tag{96}$$
  2. Axial branch:
  $$\mathcal{H}\triangle\underset{\mathrm{N}}{\hat{\boldsymbol{y}}}=3\mathcal{H}(\mathcal{H}\underset{\mathrm{N}}{\hat{\phi}}+\mathcal{H}\underset{\mathrm{N}}{\hat{\psi}}+\underset{\mathrm{N}}{\hat{\psi}}'-u\mathcal{H}\underset{\mathrm{N}}{\hat{\xi}})-3\mathcal{H}'(\underset{\mathrm{N}}{\hat{\psi}}-u\underset{\mathrm{N}}{\hat{\xi}})+3u^3\underset{\mathrm{N}}{\hat{\xi}}+u\triangle\underset{\mathrm{N}}{\hat{\xi}}\,. \tag{97}$$
  3. Flat limiting case:
  $$\mathcal{H}\triangle\underset{\mathrm{N}}{\hat{\boldsymbol{y}}}=3\mathcal{H}(\mathcal{H}\underset{\mathrm{N}}{\hat{\phi}}+\mathcal{H}\underset{\mathrm{N}}{\hat{\psi}}+\underset{\mathrm{N}}{\hat{\psi}}')-3\mathcal{H}'\underset{\mathrm{N}}{\hat{\psi}}\,. \tag{98}$$

⇒ Poisson equation is screened only in the vector branch.

⇒ Pseudo-scalar $\underset{\mathrm{N}}{\hat{\xi}}$ enters only in the axial branch.

Spatial part of antisymmetric equations (pseudo-scalar)

1. Vector branch: $\underset{\mathrm{N}}{\hat{\xi}}$ decouples from other fields and must vanish: $\underset{\mathrm{N}}{\hat{\xi}} = 0$.

Spatial part of antisymmetric equations (pseudo-scalar)

1. Vector branch: $\underset{\mathrm{N}}{\hat{\boldsymbol{\xi}}}$ decouples from other fields and must vanish: $\underset{\mathrm{N}}{\hat{\boldsymbol{\xi}}} = 0$.
2. Axial branch:
   - Screened Poisson equation involving $\underset{\mathrm{N}}{\hat{\boldsymbol{\xi}}}$ and $\underset{\mathrm{N}}{\hat{\boldsymbol{y}}}$:

$$u\triangle\underset{\mathrm{N}}{\hat{\boldsymbol{\xi}}} - \mathcal{H}\triangle\underset{\mathrm{N}}{\hat{\boldsymbol{y}}} + 3\mathcal{H}(\mathcal{H}^2 - \mathcal{H}' - u^2)\underset{\mathrm{N}}{\hat{\boldsymbol{y}}} + 3u^2\underset{\mathrm{N}}{\hat{\psi}} + 3\mathcal{H}(\mathcal{H}\underset{\mathrm{N}}{\hat{\phi}} + \underset{\mathrm{N}}{\hat{\psi}}') = 0\,. \tag{99}$$

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1. Vector branch: $\underset{\mathrm{N}}{\hat{\boldsymbol{\xi}}}$ decouples from other fields and must vanish: $\underset{\mathrm{N}}{\hat{\boldsymbol{\xi}}} = 0$.
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   - Screened Poisson equation involving $\underset{\mathrm{N}}{\hat{\boldsymbol{\xi}}}$ and $\underset{\mathrm{N}}{\hat{\boldsymbol{y}}}$:

$$u\triangle\underset{\mathrm{N}}{\hat{\boldsymbol{\xi}}} - \mathcal{H}\triangle\underset{\mathrm{N}}{\hat{\boldsymbol{y}}} + 3\mathcal{H}(\mathcal{H}^2 - \mathcal{H}' - u^2)\underset{\mathrm{N}}{\hat{\boldsymbol{y}}} + 3u^2\underset{\mathrm{N}}{\hat{\psi}} + 3\mathcal{H}(\mathcal{H}\underset{\mathrm{N}}{\hat{\phi}} + \underset{\mathrm{N}}{\hat{\psi}}') = 0\,. \tag{99}$$

   ⇝ Eliminate derivatives using previously found equation for $\underset{\mathrm{N}}{\hat{\boldsymbol{y}}}$:

$$\underset{\mathrm{N}}{\hat{\psi}} - u\underset{\mathrm{N}}{\hat{\boldsymbol{\xi}}} - \mathcal{H}\underset{\mathrm{N}}{\hat{\boldsymbol{y}}} = 0\,, \tag{100}$$

Spatial part of antisymmetric equations (pseudo-scalar)

1. Vector branch: $\underset{\mathrm{N}}{\hat{\xi}}$ decouples from other fields and must vanish: $\underset{\mathrm{N}}{\hat{\xi}} = 0$.
2. Axial branch:
   - Screened Poisson equation involving $\underset{\mathrm{N}}{\hat{\xi}}$ and $\underset{\mathrm{N}}{\hat{\boldsymbol{y}}}$:

$$u\triangle\underset{\mathrm{N}}{\hat{\xi}} - \mathcal{H}\triangle\underset{\mathrm{N}}{\hat{\boldsymbol{y}}} + 3\mathcal{H}(\mathcal{H}^2 - \mathcal{H}' - u^2)\underset{\mathrm{N}}{\hat{\boldsymbol{y}}} + 3u^2\underset{\mathrm{N}}{\hat{\psi}} + 3\mathcal{H}(\mathcal{H}\underset{\mathrm{N}}{\hat{\phi}} + \underset{\mathrm{N}}{\hat{\psi}}') = 0\,. \tag{99}$$

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$$\underset{\mathrm{N}}{\hat{\psi}} - u\underset{\mathrm{N}}{\hat{\xi}} - \mathcal{H}\underset{\mathrm{N}}{\hat{\boldsymbol{y}}} = 0\,, \tag{100}$$

   ⇝ Substituting $\underset{\mathrm{N}}{\hat{\xi}}$ in previous equation gives screened Poisson equation:

$$2\mathcal{H}\triangle\underset{\mathrm{N}}{\hat{\boldsymbol{y}}} + 3\mathcal{H}(\mathcal{H}' - \mathcal{H}^2 + u^2)\underset{\mathrm{N}}{\hat{\boldsymbol{y}}} = \triangle\underset{\mathrm{N}}{\hat{\psi}} + 3u^2\underset{\mathrm{N}}{\hat{\psi}} + 3\mathcal{H}\underset{\mathrm{N}}{\hat{\psi}}' + 3\mathcal{H}^2\hat{\phi}. \tag{101}$$

Spatial part of antisymmetric equations (pseudo-scalar)

1. Vector branch: $\underset{\mathrm{N}}{\hat{\xi}}$ decouples from other fields and must vanish: $\underset{\mathrm{N}}{\hat{\xi}} = 0$.
2. Axial branch:
   - Screened Poisson equation involving $\underset{\mathrm{N}}{\hat{\xi}}$ and $\underset{\mathrm{N}}{\hat{y}}$:

$$u\triangle\underset{\mathrm{N}}{\hat{\xi}} - \mathcal{H}\triangle\underset{\mathrm{N}}{\hat{y}} + 3\mathcal{H}(\mathcal{H}^2 - \mathcal{H}' - u^2)\underset{\mathrm{N}}{\hat{y}} + 3u^2\underset{\mathrm{N}}{\hat{\psi}} + 3\mathcal{H}(\mathcal{H}\underset{\mathrm{N}}{\hat{\phi}} + \underset{\mathrm{N}}{\hat{\psi}}') = 0\,. \tag{99}$$

   ⇝ Eliminate derivatives using previously found equation for $\underset{\mathrm{N}}{\hat{y}}$:

$$\underset{\mathrm{N}}{\hat{\psi}} - u\underset{\mathrm{N}}{\hat{\xi}} - \mathcal{H}\underset{\mathrm{N}}{\hat{y}} = 0\,, \tag{100}$$

   ⇝ Substituting $\underset{\mathrm{N}}{\hat{\xi}}$ in previous equation gives screened Poisson equation:

$$2\mathcal{H}\triangle\underset{\mathrm{N}}{\hat{y}} + 3\mathcal{H}(\mathcal{H}' - \mathcal{H}^2 + u^2)\underset{\mathrm{N}}{\hat{y}} = \triangle\underset{\mathrm{N}}{\hat{\psi}} + 3u^2\underset{\mathrm{N}}{\hat{\psi}} + 3\mathcal{H}\underset{\mathrm{N}}{\hat{\psi}}' + 3\mathcal{H}^2\hat{\phi}. \tag{101}$$

3. Flat case: equation solved identically and $\underset{\mathrm{N}}{\hat{\xi}}$ undetermined ⇒ strong coupling ↯

Time part

- Perturbed field equation in Newton gauge:
  1. Vector branch:

$$\frac{1}{2}\kappa^2 a^2 \underset{\mathrm{N}}{\hat{\boldsymbol{\mathcal{E}}}} = f_T(\triangle\underset{\mathrm{N}}{\hat{\psi}} - 3\mathcal{H}^2\underset{\mathrm{N}}{\hat{\phi}} - 3\mathcal{H}\underset{\mathrm{N}}{\hat{\psi}}' + 3u^2\underset{\mathrm{N}}{\hat{\psi}}) + 12\frac{f_{TT}}{a^2}\mathcal{H}(\mathcal{H} + iu)^2(\triangle\underset{\mathrm{N}}{\hat{\mathbf{y}}} - 3\mathcal{H}\underset{\mathrm{N}}{\hat{\phi}} - 3\underset{\mathrm{N}}{\hat{\psi}}' + 3iu\underset{\mathrm{N}}{\hat{\psi}}) \,. \tag{102}$$

  2. Axial branch:

$$\frac{1}{2}\kappa^2 a^2 \underset{\mathrm{N}}{\hat{\boldsymbol{\mathcal{E}}}} = f_T(\triangle\underset{\mathrm{N}}{\hat{\psi}} - 3\mathcal{H}^2\underset{\mathrm{N}}{\hat{\phi}} - 3\mathcal{H}\underset{\mathrm{N}}{\hat{\psi}}' + 3u^2\underset{\mathrm{N}}{\hat{\psi}}) + 12\frac{f_{TT}}{a^2}\mathcal{H}^2(\mathcal{H}\triangle\underset{\mathrm{N}}{\hat{\mathbf{y}}} - u\triangle\underset{\mathrm{N}}{\hat{\boldsymbol{\xi}}} - 3\mathcal{H}^2\underset{\mathrm{N}}{\hat{\phi}} - 3\mathcal{H}\underset{\mathrm{N}}{\hat{\psi}}' - 3u^2\underset{\mathrm{N}}{\hat{\psi}}) \,. \tag{103}$$

  3. Flat limiting case:

$$\frac{1}{2}\kappa^2 a^2 \underset{\mathrm{N}}{\hat{\boldsymbol{\mathcal{E}}}} = f_T\triangle\underset{\mathrm{N}}{\hat{\psi}} - 3\left(f_T + 12\mathcal{H}^2\frac{f_{TT}}{a^2}\right)\mathcal{H}(\mathcal{H}\underset{\mathrm{N}}{\hat{\phi}} + \underset{\mathrm{N}}{\hat{\psi}}') + 12\frac{f_{TT}}{a^2}\mathcal{H}^3\triangle\underset{\mathrm{N}}{\hat{\mathbf{y}}} \,. \tag{104}$$

Time part

- Perturbed field equation in Newton gauge:
  1. Vector branch:

$$\frac{1}{2}\kappa^2 a^2 \underset{\mathrm{N}}{\hat{\mathcal{E}}} = f_T(\triangle\underset{\mathrm{N}}{\hat{\psi}} - 3\mathcal{H}^2\underset{\mathrm{N}}{\hat{\phi}} - 3\mathcal{H}\underset{\mathrm{N}}{\hat{\psi}}' + 3u^2\underset{\mathrm{N}}{\hat{\psi}}) + 12\frac{f_{TT}}{a^2}\mathcal{H}(\mathcal{H} + iu)^2(\triangle\underset{\mathrm{N}}{\hat{\boldsymbol{y}}} - 3\mathcal{H}\underset{\mathrm{N}}{\hat{\phi}} - 3\underset{\mathrm{N}}{\hat{\psi}}' + 3iu\underset{\mathrm{N}}{\hat{\psi}}) . \tag{102}$$

  2. Axial branch:

$$\frac{1}{2}\kappa^2 a^2 \underset{\mathrm{N}}{\hat{\mathcal{E}}} = f_T(\triangle\underset{\mathrm{N}}{\hat{\psi}} - 3\mathcal{H}^2\underset{\mathrm{N}}{\hat{\phi}} - 3\mathcal{H}\underset{\mathrm{N}}{\hat{\psi}}' + 3u^2\underset{\mathrm{N}}{\hat{\psi}}) + 12\frac{f_{TT}}{a^2}\mathcal{H}^2(\mathcal{H}\triangle\underset{\mathrm{N}}{\hat{\boldsymbol{y}}} - u\triangle\underset{\mathrm{N}}{\hat{\boldsymbol{\xi}}} - 3\mathcal{H}^2\underset{\mathrm{N}}{\hat{\phi}} - 3\mathcal{H}\underset{\mathrm{N}}{\hat{\psi}}' - 3u^2\underset{\mathrm{N}}{\hat{\psi}}) . \tag{103}$$

  3. Flat limiting case:

$$\frac{1}{2}\kappa^2 a^2 \underset{\mathrm{N}}{\hat{\mathcal{E}}} = f_T\triangle\underset{\mathrm{N}}{\hat{\psi}} - 3\left(f_T + 12\mathcal{H}^2\frac{f_{TT}}{a^2}\right)\mathcal{H}(\mathcal{H}\underset{\mathrm{N}}{\hat{\phi}} + \underset{\mathrm{N}}{\hat{\psi}}') + 12\frac{f_{TT}}{a^2}\mathcal{H}^3\triangle\underset{\mathrm{N}}{\hat{\boldsymbol{y}}} . \tag{104}$$

$\rightsquigarrow$ Substitute $\underset{\mathrm{N}}{\hat{\boldsymbol{y}}}$ and $\underset{\mathrm{N}}{\hat{\boldsymbol{\xi}}}$ from previous equations.

Time part

- Perturbed field equation in Newton gauge:
  1. Vector branch:

$$\frac{1}{2}\kappa^2 a^2 \underset{\mathrm{N}}{\hat{\boldsymbol{\mathcal{E}}}} = f_T(\triangle\underset{\mathrm{N}}{\hat{\psi}} - 3\mathcal{H}^2\underset{\mathrm{N}}{\hat{\phi}} - 3\mathcal{H}\underset{\mathrm{N}}{\hat{\psi}}' + 3u^2\underset{\mathrm{N}}{\hat{\psi}}) + 12\frac{f_{TT}}{a^2}\mathcal{H}(\mathcal{H} + iu)^2(\triangle\underset{\mathrm{N}}{\hat{\boldsymbol{y}}} - 3\mathcal{H}\underset{\mathrm{N}}{\hat{\phi}} - 3\underset{\mathrm{N}}{\hat{\psi}}' + 3iu\underset{\mathrm{N}}{\hat{\psi}})\,. \tag{102}$$

  2. Axial branch:

$$\frac{1}{2}\kappa^2 a^2 \underset{\mathrm{N}}{\hat{\boldsymbol{\mathcal{E}}}} = f_T(\triangle\underset{\mathrm{N}}{\hat{\psi}} - 3\mathcal{H}^2\underset{\mathrm{N}}{\hat{\phi}} - 3\mathcal{H}\underset{\mathrm{N}}{\hat{\psi}}' + 3u^2\underset{\mathrm{N}}{\hat{\psi}}) + 12\frac{f_{TT}}{a^2}\mathcal{H}^2(\mathcal{H}\triangle\underset{\mathrm{N}}{\hat{\boldsymbol{y}}} - u\triangle\underset{\mathrm{N}}{\hat{\boldsymbol{\xi}}} - 3\mathcal{H}^2\underset{\mathrm{N}}{\hat{\phi}} - 3\mathcal{H}\underset{\mathrm{N}}{\hat{\psi}}' - 3u^2\underset{\mathrm{N}}{\hat{\psi}})\,. \tag{103}$$

  3. Flat limiting case:

$$\frac{1}{2}\kappa^2 a^2 \underset{\mathrm{N}}{\hat{\boldsymbol{\mathcal{E}}}} = f_T\triangle\underset{\mathrm{N}}{\hat{\psi}} - 3\left(f_T + 12\mathcal{H}^2\frac{f_{TT}}{a^2}\right)\mathcal{H}(\mathcal{H}\underset{\mathrm{N}}{\hat{\phi}} + \underset{\mathrm{N}}{\hat{\psi}}') + 12\frac{f_{TT}}{a^2}\mathcal{H}^3\triangle\underset{\mathrm{N}}{\hat{\boldsymbol{y}}}\,. \tag{104}$$

⇝ Substitute $\underset{\mathrm{N}}{\hat{\boldsymbol{y}}}$ and $\underset{\mathrm{N}}{\hat{\boldsymbol{\xi}}}$ from previous equations.

⇒ Resulting equation gives screened Poisson equation for $\underset{\mathrm{N}}{\hat{\psi}}$.

Remaining mixed part

- Perturbed field equation in Newton gauge:
  1. Vector branch:
  $$-\frac{1}{2}\kappa^2 a^2(\bar{\rho}+\bar{p})\underset{\mathrm{N}}{\hat{\boldsymbol{\mathcal{L}}}} = f_T(\mathcal{H}\underset{\mathrm{N}}{\hat{\phi}} + \underset{\mathrm{N}}{\hat{\psi}}') + 12(\mathcal{H}+iu)[\mathcal{H}' - \mathcal{H}(\mathcal{H}+iu)]\frac{f_{TT}}{a^2}(\underset{\mathrm{N}}{\hat{\psi}} + iu\underset{\mathrm{N}}{\hat{\mathbf{y}}})\,. \quad (105)$$
  2. Axial branch:
  $$-\frac{1}{2}\kappa^2 a^2(\bar{\rho}+\bar{p})\underset{\mathrm{N}}{\hat{\boldsymbol{\mathcal{L}}}} = f_T(\mathcal{H}\underset{\mathrm{N}}{\hat{\phi}} + \underset{\mathrm{N}}{\hat{\psi}}') + 12\mathcal{H}(\mathcal{H}' - \mathcal{H}^2 + u^2)\frac{f_{TT}}{a^2}(\underset{\mathrm{N}}{\hat{\psi}} - u\underset{\mathrm{N}}{\hat{\boldsymbol{\xi}}})\,. \quad (106)$$
  3. Flat limiting case:
  $$-\frac{1}{2}\kappa^2 a^2(\bar{\rho}+\bar{p})\underset{\mathrm{N}}{\hat{\boldsymbol{\mathcal{L}}}} = f_T(\mathcal{H}\underset{\mathrm{N}}{\hat{\phi}} + \underset{\mathrm{N}}{\hat{\psi}}') + 12\mathcal{H}(\mathcal{H}' - \mathcal{H}^2)\frac{f_{TT}}{a^2}\underset{\mathrm{N}}{\hat{\psi}}\,. \quad (107)$$

Remaining mixed part

- Perturbed field equation in Newton gauge:
  1. Vector branch:

$$-\frac{1}{2}\kappa^2 a^2(\bar{\rho}+\bar{p})\underset{\mathrm{N}}{\hat{\boldsymbol{\mathcal{L}}}} = f_T(\mathcal{H}\underset{\mathrm{N}}{\hat{\phi}} + \underset{\mathrm{N}}{\hat{\psi}}') + 12(\mathcal{H}+iu)[\mathcal{H}' - \mathcal{H}(\mathcal{H}+iu)]\frac{f_{TT}}{a^2}(\underset{\mathrm{N}}{\hat{\psi}} + iu\underset{\mathrm{N}}{\hat{\boldsymbol{y}}})\,. \tag{105}$$

  2. Axial branch:

$$-\frac{1}{2}\kappa^2 a^2(\bar{\rho}+\bar{p})\underset{\mathrm{N}}{\hat{\boldsymbol{\mathcal{L}}}} = f_T(\mathcal{H}\underset{\mathrm{N}}{\hat{\phi}} + \underset{\mathrm{N}}{\hat{\psi}}') + 12\mathcal{H}(\mathcal{H}' - \mathcal{H}^2 + u^2)\frac{f_{TT}}{a^2}(\underset{\mathrm{N}}{\hat{\psi}} - u\underset{\mathrm{N}}{\hat{\boldsymbol{\xi}}})\,. \tag{106}$$

  3. Flat limiting case:

$$-\frac{1}{2}\kappa^2 a^2(\bar{\rho}+\bar{p})\underset{\mathrm{N}}{\hat{\boldsymbol{\mathcal{L}}}} = f_T(\mathcal{H}\underset{\mathrm{N}}{\hat{\phi}} + \underset{\mathrm{N}}{\hat{\psi}}') + 12\mathcal{H}(\mathcal{H}' - \mathcal{H}^2)\frac{f_{TT}}{a^2}\underset{\mathrm{N}}{\hat{\psi}}\,. \tag{107}$$

⇝ Substituting previous equations leads to same result for all branches:

$$\frac{1}{2}\kappa^2 a^2[\underset{\mathrm{N}}{\hat{\boldsymbol{\mathcal{E}}}} - 3\mathcal{H}(\bar{\rho}+\bar{p})\underset{\mathrm{N}}{\hat{\boldsymbol{\mathcal{L}}}}] = f_T(\triangle\underset{\mathrm{N}}{\hat{\psi}} + 3u^2\underset{\mathrm{N}}{\hat{\psi}})\,. \tag{108}$$

Remaining mixed part

- Perturbed field equation in Newton gauge:
  1. Vector branch:
$$-\frac{1}{2}\kappa^2 a^2(\bar{\rho}+\bar{p})\underset{\mathrm{N}}{\hat{\boldsymbol{\mathcal{L}}}} = f_T(\mathcal{H}\underset{\mathrm{N}}{\hat{\phi}} + \underset{\mathrm{N}}{\hat{\psi}}') + 12(\mathcal{H}+iu)[\mathcal{H}' - \mathcal{H}(\mathcal{H}+iu)]\frac{f_{TT}}{a^2}(\underset{\mathrm{N}}{\hat{\psi}} + iu\underset{\mathrm{N}}{\hat{\boldsymbol{y}}})\,. \tag{105}$$
  2. Axial branch:
$$-\frac{1}{2}\kappa^2 a^2(\bar{\rho}+\bar{p})\underset{\mathrm{N}}{\hat{\boldsymbol{\mathcal{L}}}} = f_T(\mathcal{H}\underset{\mathrm{N}}{\hat{\phi}} + \underset{\mathrm{N}}{\hat{\psi}}') + 12\mathcal{H}(\mathcal{H}' - \mathcal{H}^2 + u^2)\frac{f_{TT}}{a^2}(\underset{\mathrm{N}}{\hat{\psi}} - u\underset{\mathrm{N}}{\hat{\boldsymbol{\xi}}})\,. \tag{106}$$
  3. Flat limiting case:
$$-\frac{1}{2}\kappa^2 a^2(\bar{\rho}+\bar{p})\underset{\mathrm{N}}{\hat{\boldsymbol{\mathcal{L}}}} = f_T(\mathcal{H}\underset{\mathrm{N}}{\hat{\phi}} + \underset{\mathrm{N}}{\hat{\psi}}') + 12\mathcal{H}(\mathcal{H}' - \mathcal{H}^2)\frac{f_{TT}}{a^2}\underset{\mathrm{N}}{\hat{\psi}}\,. \tag{107}$$

⇝ Substituting previous equations leads to same result for all branches:
$$\frac{1}{2}\kappa^2 a^2[\underset{\mathrm{N}}{\hat{\boldsymbol{\mathcal{E}}}} - 3\mathcal{H}(\bar{\rho}+\bar{p})\underset{\mathrm{N}}{\hat{\boldsymbol{\mathcal{L}}}}] = f_T(\triangle\underset{\mathrm{N}}{\hat{\psi}} + 3u^2\underset{\mathrm{N}}{\hat{\psi}})\,. \tag{108}$$

⇝ Express matter variables in fluid comoving gauge:
$$\frac{1}{2}\kappa^2 a^2\underset{\mathrm{C}}{\hat{\boldsymbol{\mathcal{E}}}} = f_T(\triangle\underset{\mathrm{N}}{\hat{\psi}} + 3u^2\underset{\mathrm{N}}{\hat{\psi}})\,. \tag{109}$$

Off-diagonal part

- Perturbed field equation in Newton gauge:
  1. Vector branch:

$$\kappa^2 a^2 \underset{\mathrm{N}}{\hat{\boldsymbol{\mathcal{S}}}} = f_T(\underset{\mathrm{N}}{\hat{\psi}} - \underset{\mathrm{N}}{\hat{\phi}}) - 12\frac{f_{TT}}{a^2}(\mathcal{H} + iu)[\mathcal{H}(\mathcal{H} + iu) - \mathcal{H}']\underset{\mathrm{N}}{\hat{\boldsymbol{y}}} \tag{110}$$

  2. Axial branch:

$$\kappa^2 a^2 \underset{\mathrm{N}}{\hat{\boldsymbol{\mathcal{S}}}} = f_T(\underset{\mathrm{N}}{\hat{\psi}} - \underset{\mathrm{N}}{\hat{\phi}}) - 12\frac{f_{TT}}{a^2}\mathcal{H}(\mathcal{H}^2 - \mathcal{H}' - u^2)\underset{\mathrm{N}}{\hat{\boldsymbol{y}}} \tag{111}$$

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$\Rightarrow$ Equations determine gravitational slip $\underset{\mathrm{N}}{\hat{\psi}} - \underset{\mathrm{N}}{\hat{\phi}}$.

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⇒ Equations determine gravitational slip $\underset{\mathrm{N}}{\hat{\psi}} - \underset{\mathrm{N}}{\hat{\phi}}$.

⇝ Substitute $\underset{\mathrm{N}}{\hat{\boldsymbol{y}}}$ from preceding equations.

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⇝ Substitute $\underset{\mathrm{N}}{\hat{\boldsymbol{y}}}$ from preceding equations.

- Dynamics follow after combining with trace equation (lengthy, not shown here).

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  - Geometric description using Lorentzian metric and affine connection.
  - Alternative description in terms of tetrad and spin connection.
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  - Teleparallel case: perturbation described by tensor field $\tau_{\mu\nu}$.

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⇝ Use newly developed tools to further study cosmological perturbations.