Classification of metric-affine geometries by spacetime symmetries

Manuel Hohmann

Laboratory of Theoretical Physics - Institute of Physics - University of Tartu
Center of Excellence “The Dark Side of the Universe”

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Motivation

- Open questions in gravity theory:
  - Accelerating expansion of the universe.
  - Homogeneity of cosmic microwave background.
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- Metric-affine class of geometries:
  - Consider metric $g_{\mu\nu}$ and connection $\Gamma^{\mu}{}_{\nu\rho}$ as independent fields.
  - Impose relations between $g_{\mu\nu}$ and $\Gamma^{\mu}{}_{\nu\rho}$ by Lagrange multipliers.
  - Large range of possible dynamics.
  - Possible to relate to gauge theory - other forces?
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  - Large range of possible dynamics.
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  - Geometry is special case of Cartan geometry.
- Consider solutions with particular spacetime symmetries:
  - Field equations greatly simplify after imposing symmetry.
  - Possible to classify all metric-affine geometries by their symmetries.

Metric-affine geometries and properties

- Objects defining the geometry:
  - Pseudo-Riemannian metric $g_{\mu\nu}$.
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- Properties of metric-affine geometries:
  - Curvature:
$$R^\mu{}_{\nu\rho\sigma} = \partial_\rho \Gamma^\mu{}_{\nu\sigma} - \partial_\sigma \Gamma^\mu{}_{\nu\rho} + \Gamma^\mu{}_{\omega\rho}\Gamma^\omega{}_{\nu\sigma} - \Gamma^\mu{}_{\omega\sigma}\Gamma^\omega{}_{\nu\rho} \,.$$
  - Torsion:
$$T^\rho{}_{\mu\nu} = \Gamma^\rho{}_{\nu\mu} - \Gamma^\rho{}_{\mu\nu} \,.$$
  - Nonmetricity:
$$Q_{\mu\nu\rho} = \nabla_\mu g_{\nu\rho} \,.$$

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$\Rightarrow$ 8 types of geometries based on (non-)vanishing of R, T, Q.

Decomposition of affine connection

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- Terms in the decomposition:
  - Levi-Civita connection coefficients:

$$\left\{{}^{\mu}_{\nu\rho}\right\} = \frac{1}{2} g^{\mu\sigma} \left(\partial_{\nu} g_{\sigma\rho} + \partial_{\rho} g_{\nu\sigma} - \partial_{\sigma} g_{\nu\rho}\right) \,.$$

  - Contortion tensor:

$$K^{\rho}{}_{\mu\nu} = \frac{1}{2} \left(T_{\mu}{}^{\rho}{}_{\nu} + T_{\nu}{}^{\rho}{}_{\mu} - T^{\rho}{}_{\mu\nu}\right) \,.$$

  - Disformation tensor:

$$L^{\rho}{}_{\mu\nu} = \frac{1}{2} \left(Q^{\rho}{}_{\mu\nu} - Q_{\mu\nu}{}^{\rho} - Q_{\nu\mu}{}^{\rho}\right) \,.$$

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$\Rightarrow$ Decomposition is unique if both metric and affine connection are given.

Brief detour: symmetries in Cartan geometry

- Ingredients of a Cartan geometry:
  - Lie group G with closed Lie subgroup $H \subset G \Rightarrow$ homogeneous space G/H.
  - Principal H-bundle $\pi : P \to M$ over manifold M.
  - Cartan connection is Lie algebra valued 1-form $A \in \Omega^1(P, \mathfrak{g})$, where:
    - For all $p \in P$, $A_p : T_pP \to \mathfrak{g}$ is a linear isomorphism.
    - A is H-equivariant: $(R_h)^*A = \mathrm{Ad}(h^{-1}) \circ A$ for all $h \in H$.
    - $A(\tilde{h}) = h$ for all $h \in \mathfrak{h}$, where $\tilde{h}$ is the fundamental vector field of h.

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- Further constraints:
  - First order: quotient representation of adjoint representations of H on $\mathfrak{g}/\mathfrak{h}$ is faithful.
  - Reductive: Lie algebra of G is direct sum $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{z}$ of subrepresentations of H.

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$\Rightarrow$ Notion of symmetry under generating vector field ξ on M:
  - P canonically identified with subbundle of frame bundle of M.
  - Canonical decomposition $A = \omega + e$:
    - $\omega \in \Omega^1(P, \mathfrak{h})$ is affine (Ehresmann) connection.
    - $e \in \Omega^1(P, \mathfrak{z})$ is tautological (solder) form.
  - Vector field ξ canonically lifted to vector field Ξ on frame bundle (functorial lift).
  - Symmetry of Cartan connection defined by lifted vector field Ξ: [MH '15]
    - Lifted vector field Ξ is tangent to P.
    - Lie derivative $\mathcal{L}_\Xi\omega$ vanishes (sufficient since $\mathcal{L}_\Xi e$ always vanishes).

Finite and infinitesimal transformations of metric-affine geometry

- Finite spacetime transformation:
  - Generated by 1-parameter diffeomorphism group $\varphi_t : M \to M$ with $x' = \varphi(x)$ and $t \in \mathbb{R}$.
  - Transformations of fundamental geometric objects:
    - Metric:

$$(\varphi_t^* g)_{\mu\nu}(x) = g_{\tau\omega}(x') \frac{\partial x'^\tau}{\partial x^\mu} \frac{\partial x'^\omega}{\partial x^\nu} \,.$$

    - Connection coefficients:

$$(\varphi_t^* \Gamma)^\mu{}_{\nu\rho}(x) = \Gamma^\sigma{}_{\tau\omega}(x') \frac{\partial x^\mu}{\partial x'^\sigma} \frac{\partial x'^\tau}{\partial x^\nu} \frac{\partial x'^\omega}{\partial x^\rho} + \frac{\partial x^\mu}{\partial x'^\sigma} \frac{\partial^2 x'^\sigma}{\partial x^\nu \partial x^\rho} \,.$$

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- Infinitesimal spacetime transformation:
  - Generated by vector field ξ on M.
  - Lie derivatives of fundamental geometric objects are tensor fields:
    - Metric:
$$(\mathcal{L}_\xi g)_{\mu\nu} = \xi^\rho\partial_\rho g_{\mu\nu} + \partial_\mu\xi^\rho g_{\rho\nu} + \partial_\nu\xi^\rho g_{\mu\rho}\,.$$
    - Connection coefficients:
$$\begin{aligned}(\mathcal{L}_\xi\Gamma)^\mu{}_{\nu\rho} &= \xi^\sigma\partial_\sigma\Gamma^\mu{}_{\nu\rho} - \partial_\sigma\xi^\mu\Gamma^\sigma{}_{\nu\rho} + \partial_\nu\xi^\sigma\Gamma^\mu{}_{\sigma\rho} + \partial_\rho\xi^\sigma\Gamma^\mu{}_{\nu\sigma} + \partial_\nu\partial_\rho\xi^\mu\\ &= \nabla_\rho\nabla_\nu\xi^\mu - \xi^\sigma R^\mu{}_{\nu\rho\sigma} - \nabla_\rho(\xi^\sigma T^\mu{}_{\nu\sigma})\,.\end{aligned}$$

Symmetries and properties of metric-affine geometry

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  - Converse is not true: $\mathcal{L}_\xi \{\} = 0 \nRightarrow \mathcal{L}_\xi g = 0$.
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- Assume symmetric metric-affine geometry: $\mathcal{L}_\xi g = 0$ and $\mathcal{L}_\xi \Gamma = 0$.
  - $\Rightarrow$ Constituents of connection: $\mathcal{L}_\xi K = 0$, $\mathcal{L}_\xi L = 0$ and $\mathcal{L}_\xi \{\} = 0$.
  - $\Rightarrow$ Tensorial properties: $\mathcal{L}_\xi T = 0$, $\mathcal{L}_\xi Q = 0$ and $\mathcal{L}_\xi R = 0$.
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  - $\Rightarrow$ Covariant derivatives of any tensor field U: $\mathcal{L}_\xi (\nabla U) = 0$.
- Special case: symmetric teleparallel geometry $T = 0$ and $R = 0$.
  - $\Rightarrow$ Connection takes the form $\Gamma^\mu{}_{\nu\rho} = \frac{\partial x^\mu}{\partial x'^\sigma} \frac{\partial^2 x'^\sigma}{\partial x^\nu \partial x^\rho}$.
  - $\Rightarrow$ Choose coordinates such that $\Gamma^\mu{}_{\nu\rho} = 0$ in open neighborhood.
  - $\Rightarrow$ Lie derivative simplifies to $(\mathcal{L}_\xi \Gamma)^\mu{}_{\nu\rho} = \partial_\rho \partial_\nu \xi^\mu$.
  - $\Rightarrow$ Every vector field linear in coordinates generates symmetry.

Example: metric-affine cosmological spacetime

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- Most general cosmologically symmetric connection:

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From symmetric geometry to gravity theory

- Gravitational part of the field equations:
  - Tensorial expression (follows from diffeomorphism invariance).
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- Example: fully general teleparallel gravity $R = 0$ and $Q = 0$:
  - Field equations are of the form $E_{\mu\nu} = \Theta_{\mu\nu}$ (right hand side is energy-momentum tensor).
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- Impose cosmological symmetry (homogeneity and isotropy):

$\Rightarrow$ Most general geometry defined by two free functions of time.
$\Rightarrow$ One free function can be eliminated by time redefinition.
$\Rightarrow$ Remaining free function takes role of scale factor.
$\Rightarrow$ Antisymmetric field equations $E_{[\mu\nu]} = 0$ solved identically.

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⇒ Possible to classify teleparallel geometries by symmetry: [MH, Järv, Krššák, Pfeifer '19]
  - Express metric and connection through tetrad and (flat) spin connection.
  - Derive symmetry conditions on tetrad and spin connection.
  - ⇒ Symmetric geometries can be labelled by Lie group homomorphisms $\Lambda : G \to SO(1,3)$.

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  - MH, "Spacetime and observer space symmetries in the language of Cartan geometry", J. Math. Phys. **57** (2016) 082502 [arXiv:1505.07809 [math-ph]].
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