Propagation of gravitational waves in teleparallel gravity

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E(2) classes

Perturbation around flat metric:

$$g_{\mu\nu} = \eta_{\mu\nu} + \epsilon h_{\mu\nu}\,. \tag{1}$$

Plane wave in z direction:

$$h_{\mu\nu} = H_{\mu\nu}e^{i\omega(t-z)}\,. \tag{2}$$

Possible modes:

$$\Psi_2 = -\frac{1}{6}R_{nlnl}\,, \tag{3a}$$
$$\Psi_3 = -\frac{1}{2}R_{nln\bar{m}} = -\frac{1}{2}\overline{R_{nlnm}}\,, \tag{3b}$$
$$\Psi_4 = -R_{n\bar{m}n\bar{m}} = -\overline{R_{nmnm}}\,, \tag{3c}$$
$$\Phi_{22} = -R_{nmn\bar{m}}\,. \tag{3d}$$

E(2) classes are combinations:

- ■ II_6: 6 polarizations, all modes are allowed.
- ■ III_5: 5 polarizations, $\Psi_2 = 0$, all other modes are allowed.
- □ N_3: 3 polarizations, $\Psi_2 = \Psi_3 = 0$, tensor and breathing modes are allowed.
- ■ N_2: 2 polarizations, $\Psi_2 = \Psi_3 = \Phi_{22} = 0$, only tensor modes are allowed.
- ■ O_1: 1 polarization, $\Psi_2 = \Psi_3 = \Psi_4 = 0$, only breathing mode is allowed.
- ■ O_0: no gravitational waves.

References

[1] M. Hohmann, “Polarization of gravitational waves in general teleparallel theories of gravity,” Astron. Rep. **62** (2018) no.12, 890 [arXiv:1806.10429 [gr-qc]].

[2] M. Hohmann, M. Krššák, C. Pfeifer and U. Ualikhanova, “Propagation of gravitational waves in teleparallel gravity theories,” Phys. Rev. D **98** (2018) no.12, 124004 [arXiv:1807.04580 [gr-qc]].

[3] M. Hohmann, C. Pfeifer, J. L. Said and U. Ualikhanova, “Propagation of gravitational waves in symmetric teleparallel gravity theories,” Phys. Rev. D **99** (2019) no.2, 024009 [arXiv:1808.02894 [gr-qc]].

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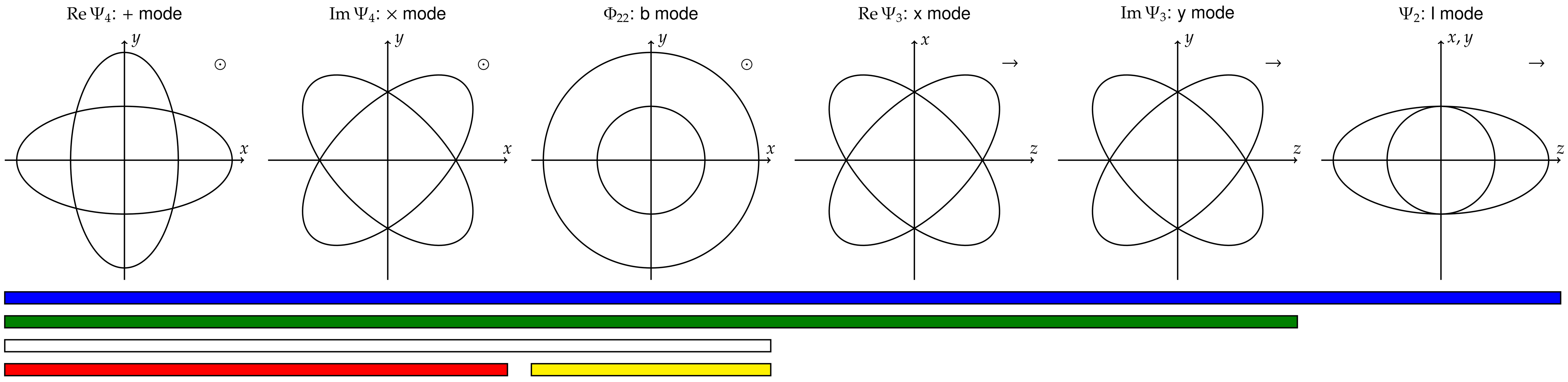
Polarizations in the Newman-Penrose formalism



Torsion teleparallel gravity [1, 2]

The fundamental fields are a *tetrad* $\theta^a{}_\mu$ and a flat Lorentz *spin connection* $\omega^a{}_{b\mu}$. We denote by $e_a{}^\mu$ the *inverse tetrad*, which satisfies $\theta^a{}_\mu e_a{}^\nu = \delta^\nu_\mu$ and $\theta^a{}_\mu e_b{}^\mu = \delta^a_b$. We further use the *metric* $g_{\mu\nu} = \eta_{ab}\theta^a{}_\mu\theta^b{}_\nu$ and the *torsion* $T^\rho{}_{\mu\nu} = 2e_a{}^\rho\left(\partial_{[\mu}\theta^a{}_{\nu]} + \omega^a{}_{b[\mu}\theta^b{}_{\nu]}\right)$. The *action* we consider is of the form

$$S[\theta,\omega,\chi] = S_g[\theta,\omega] + S_m[\theta,\chi]\,, \tag{4}$$

where χ denotes *matter fields*. The *gravitational part* takes the form

$$S_g[\theta,\omega] = \frac{1}{2\kappa^2}\int_M \mathcal{F}(T^{\mu\nu\rho}T_{\mu\nu\rho}, T^{\mu\nu\rho}T_{\rho\nu\mu}, T^\mu{}_{\mu\rho}T_\nu{}^{\nu\rho})\,\det\theta\,d^4x\,, \tag{5}$$

with a free function $\mathcal{F}$. The relevant *field equations* obtained from this action are

$$\begin{aligned}\kappa^2\Theta_{\mu\nu} = {}&\frac{1}{2}\mathcal{F}g_{\mu\nu} + 2\overset{\circ}{\nabla}{}^\rho\left(\mathcal{F}_{,1}T_{\nu\mu\rho} + \mathcal{F}_{,2}T_{[\rho\mu]\nu} + \mathcal{F}_{,3}T^\sigma{}_{\sigma[\rho}g_{\mu]\nu}\right) + \mathcal{F}_{,1}T^{\rho\sigma}{}_\mu\left(T_{\nu\rho\sigma} - 2T_{[\rho\sigma]\nu}\right)\\ &+ \frac{1}{2}\mathcal{F}_{,2}\left[T_\mu{}^{\rho\sigma}\left(2T_{\rho\sigma\nu} - T_{\nu\rho\sigma}\right) + T^{\rho\sigma}{}_\mu\left(2T_{[\rho\sigma]\nu} - T_{\nu\rho\sigma}\right)\right] - \frac{1}{2}\mathcal{F}_{,3}T^\sigma{}_{\sigma\rho}\left(T^\rho{}_{\mu\nu} + 2T_{(\mu\nu)}{}^\rho\right)\,,\end{aligned} \tag{6}$$

where $\Theta_{\mu\nu}$ is the *energy-momentum tensor*. Using the *linear perturbation*

$$\theta^a{}_\mu = \Delta^a{}_\mu + \epsilon\tau^a{}_\mu\,, \quad e_a{}^\mu = (\Delta^{-1})_a{}^\mu - \epsilon\tau_a{}^\mu\,, \quad \omega^a{}_{b\mu} = \epsilon\partial_\mu\lambda^a{}_b\,, \tag{7}$$

around the *diagonal tetrad* $\Delta^a{}_\mu = \mathrm{diag}(1,1,1,1)$ and the *Taylor expansion*

$$\mathcal{F} = \mathcal{F}|_{T^\rho{}_{\mu\nu}=0} + \mathcal{O}(\epsilon^2) = F + \mathcal{O}(\epsilon^2)\,, \quad \mathcal{F}_{,i} = \mathcal{F}_{,i}|_{T^\rho{}_{\mu\nu}=0} + \mathcal{O}(\epsilon^2) = F_{,i} + \mathcal{O}(\epsilon^2)\,, \tag{8}$$

as well as the symmetric-antisymmetric decomposition $s_{\mu\nu} = \tau_{(\mu\nu)}$ and $a_{\mu\nu} = \tau_{[\mu\nu]} - \lambda_{\mu\nu}$, and assuming $F = 0$, one finds the *linearized vacuum equations*

$$\begin{aligned}0 = E_{\mu\nu} = {}&\partial^\rho\left[2(2F_{,1} - F_{,2})\partial_{[\nu}a_{\rho]\mu} + (2F_{,2} + F_{,3})\partial_\mu a_{\rho\nu}\right]\\ &+ 2\partial^\rho\left[(2F_{,1} + F_{,2})\partial_{[\rho}s_{\nu]\mu} + F_{,3}\left(\eta_{\mu[\nu}\partial_{\rho]}s^\sigma{}_\sigma - \partial^\sigma s_{\sigma[\rho}\eta_{\nu]\mu}\right)\right]\,,\end{aligned} \tag{9}$$

For the *plane wave* $s_{\mu\nu} = S_{\mu\nu}e^{i\omega(t-z)}$ and $a_{\mu\nu} = A_{\mu\nu}e^{i\omega(t-z)}$ one finds in the Newman-Penrose basis the components

$$0 = E_{nn} = (2F_{,1} + F_{,2} + F_{,3})\ddot{s}_{nl} + 2F_{,3}\ddot{s}_{m\bar{m}} + (2F_{,1} + F_{,2} + F_{,3})\ddot{a}_{nl}\,, \tag{10a}$$
$$0 = E_{mn} = \overline{E_{\bar{m}n}} = (2F_{,1} + F_{,2})\ddot{s}_{ml} + (2F_{,1} - F_{,2})\ddot{a}_{ml}\,, \tag{10b}$$
$$0 = E_{nm} = \overline{E_{n\bar{m}}} = -F_{,3}\ddot{s}_{ml} + (2F_{,2} + F_{,3})\ddot{a}_{ml}\,, \tag{10c}$$
$$0 = E_{m\bar{m}} = E_{\bar{m}m} = -F_{,3}\ddot{s}_{ll}\,, \tag{10d}$$
$$0 = E_{ln} = (2F_{,1} + F_{,2})\ddot{s}_{ll}\,. \tag{10e}$$

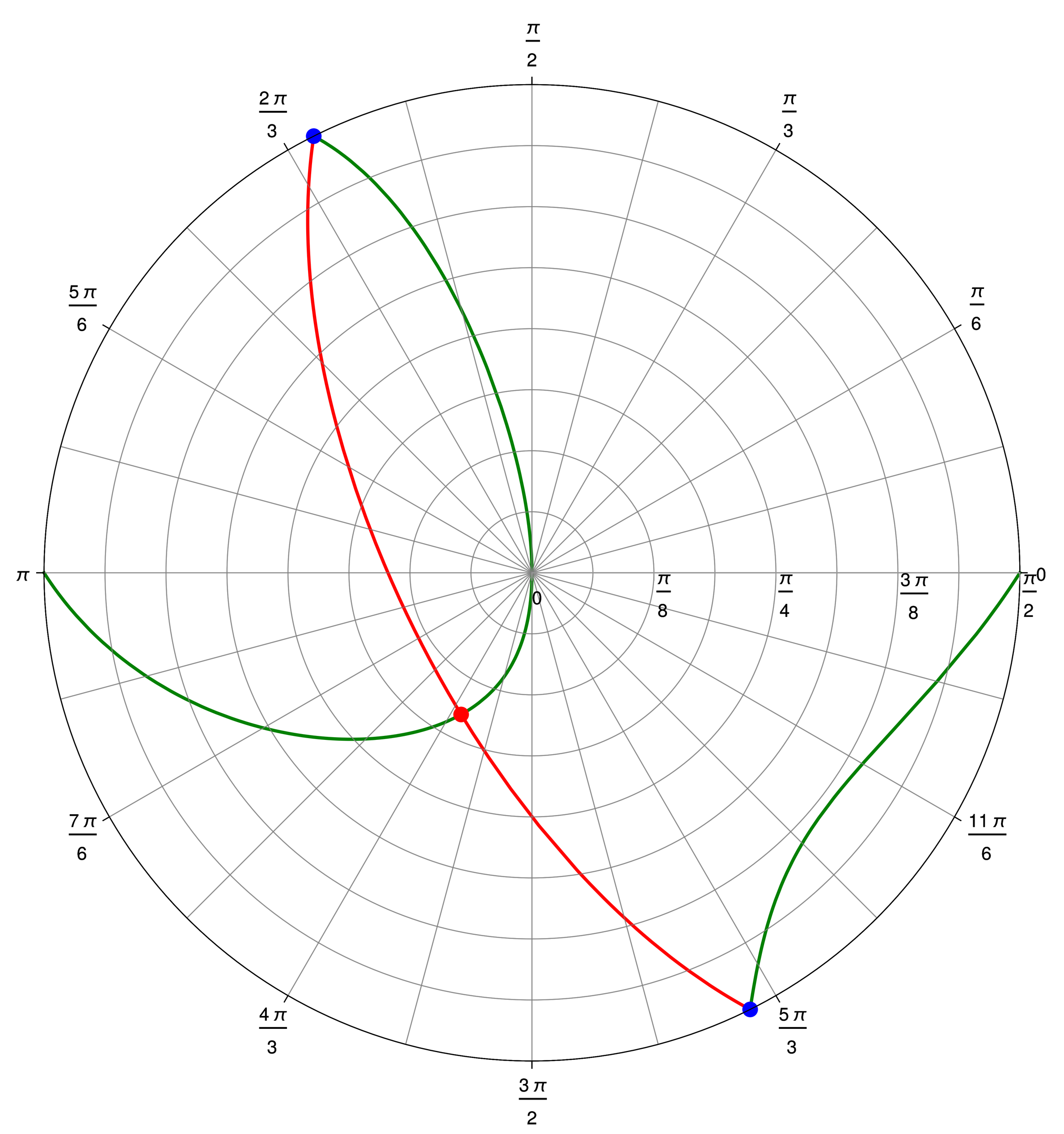
This yields the following polarizations:

- ■ $2F_{,1} + F_{,2} = F_{,3} = 0$: None of the six possible modes is restricted by the linearized field equations. Theories satisfying these conditions belong to the $\mathrm{E}(2)$ class II_6, shown in blue in the figure.
- ■ $2F_{,1}(F_{,2}+F_{,3})+F_{,2}^2 = 0$ and $2F_{,1}+F_{,2}+F_{,3} \neq 0$: In this case the field equations enforce $\Psi_2 = 0$, so that there is no longitudinal mode. All other modes are unrestricted. Theories of this type belong to the $\mathrm{E}(2)$ class III_5. This case is represented by the green line in the figure.
- □ $2F_{,1}(F_{,2}+F_{,3})+F_{,2}^2 \neq 0$ and $2F_{,1}+F_{,2}+F_{,3} \neq 0$: From the field equations follows $\Psi_2 = \Psi_3 = 0$, while the breathing mode Φ_{22} and tensor modes Ψ_4 are unrestricted. This wave has the $\mathrm{E}(2)$ class N_3. Almost all points of the parameter space, shown in white in the figure, belong to this class.
- ■ $2F_{,1} + F_{,2} + F_{,3} = 0$ and $F_{,3} \neq 0$: The only mode which is allowed to be nonzero is given by the two tensor polarizations Ψ_4. The $\mathrm{E}(2)$ class of this wave is N_2. This case is shown as a red line in the figure. Also TEGR, marked as a red point, belongs to this class.

Introducing polar coordinates

$$F_{,1} = C\sin\vartheta\cos\varphi\,, \quad F_{,2} = C\sin\vartheta\sin\varphi\,, \quad F_{,3} = C\cos\vartheta\,, \tag{11}$$

with $C = \sqrt{F_{,1}^2 + F_{,2}^2 + F_{,3}^2} \neq 0$, one can visualize the modes:



Non-metricity teleparallel gravity [1, 3]

The fundamental fields are a *metric* $g_{\mu\nu}$ and a flat, symmetric *affine connection* $\Gamma^\rho{}_{\mu\nu}$. They define the *non-metricity* $Q_{\rho\mu\nu} = \nabla_\rho g_{\mu\nu}$. The *action* we consider is of the form

$$S[g,\Gamma,\chi] = S_g[g,\Gamma] + S_m[g,\chi]\,, \tag{12}$$

where χ denotes *matter fields*. The *gravitational part* takes the form

$$S_g[g,\Gamma] = \frac{1}{2\kappa^2}\int_M \mathcal{F}(Q^{\mu\nu\rho}Q_{\mu\nu\rho}, Q^{\mu\nu\rho}Q_{\rho\mu\nu}, Q^{\rho\mu}{}_\mu Q_{\rho\nu}{}^\nu, Q^\mu{}_{\mu\rho}Q_\nu{}^{\nu\rho}, Q^\mu{}_{\mu\rho}Q^{\rho\nu}{}_\nu)\sqrt{-\det g}\,d^4x\,, \tag{13}$$

with a free function $\mathcal{F}$. The relevant *field equations* obtained from this action are

$$\begin{aligned}\kappa^2\Theta_{\mu\nu} = {}&-2\overset{\circ}{\nabla}{}_\rho\left[\mathcal{F}_{,1}Q^\rho{}_{\mu\nu} + \mathcal{F}_{,2}Q_{(\mu\nu)}{}^\rho + \mathcal{F}_{,3}Q^{\rho\sigma}{}_\sigma g_{\mu\nu} + \mathcal{F}_{,4}Q^\sigma{}_{\sigma(\mu}\delta^\rho_{\nu)} + \frac{\mathcal{F}_{,5}}{2}\left(Q_\sigma{}^{\sigma\rho}g_{\mu\nu} + \delta^\rho_{(\mu}Q_{\nu)\sigma}{}^\sigma\right)\right]\\ &+ \frac{1}{2}\mathcal{F}g_{\mu\nu} - \mathcal{F}_{,3}Q_{\mu\rho}{}^\rho Q_{\nu\sigma}{}^\sigma + \mathcal{F}_{,1}\left(2Q^{\rho\sigma}{}_\mu Q_{\sigma\rho\nu} - Q_\mu{}^{\rho\sigma}Q_{\nu\rho\sigma} - 2Q^{\rho\sigma}{}_{(\mu}Q_{\nu)\rho\sigma}\right)\\ &+ \mathcal{F}_{,2}\left(Q^{\rho\sigma}{}_\mu Q_{\rho\sigma\nu} - Q_\mu{}^{\rho\sigma}Q_{\nu\rho\sigma} - Q^{\rho\sigma}{}_{(\mu}Q_{\nu)\rho\sigma}\right) + \frac{1}{2}\mathcal{F}_{,5}\left[Q^{\rho\sigma}{}_\sigma\left(Q_{\rho\mu\nu} - 2Q_{(\mu\nu)\rho}\right) - Q_{\mu\rho}{}^\rho Q_{\nu\sigma}{}^\sigma\right]\\ &+ \mathcal{F}_{,4}\left[Q_\rho{}^{\rho\sigma}\left(Q_{\sigma\mu\nu} - 2Q_{(\mu\nu)\sigma}\right) + Q^\rho{}_{\rho\mu}Q^\sigma{}_{\sigma\nu} - Q^\rho{}_{\rho(\mu}Q_{\nu)\sigma}{}^\sigma\right]\,,\end{aligned} \tag{14}$$

where $\Theta_{\mu\nu}$ is the *energy-momentum tensor*. Using the *linear perturbation*

$$g_{\mu\nu} = \eta_{\mu\nu} + \epsilon h_{\mu\nu}\,, \quad \Gamma^\rho{}_{\mu\nu} = \epsilon\partial_\mu\partial_\nu\xi^\rho \tag{15}$$

around the *Minkowski metric* $\eta_{\mu\nu} = \mathrm{diag}(-1,1,1,1)$ and the *Taylor expansion*

$$\mathcal{F} = \mathcal{F}|_{Q_{\rho\mu\nu}=0} + \mathcal{O}(\epsilon^2) = F + \mathcal{O}(\epsilon^2)\,, \quad \mathcal{F}_{,i} = \mathcal{F}_{,i}|_{Q_{\rho\mu\nu}=0} + \mathcal{O}(\epsilon^2) = F_{,i} + \mathcal{O}(\epsilon^2)\,, \tag{16}$$

as well as the gauge-invariant perturbation $b_{\mu\nu} = h_{\mu\nu} - 2\partial_{(\mu}\xi_{\nu)}$, and assuming $F = 0$, one finds the *linearized vacuum equations*

$$\begin{aligned}0 = E_{\mu\nu} = {}&2F_{,1}\Box b_{\mu\nu} + (F_{,2} + F_{,4})\eta^{\alpha\sigma}\left(\partial_\alpha\partial_\mu b_{\sigma\nu} + \partial_\alpha\partial_\nu b_{\sigma\mu}\right)\\ &+ 2F_{,3}\eta_{\mu\nu}\eta^{\tau\omega}\Box b_{\tau\omega} + F_{,5}\eta_{\mu\nu}\eta^{\omega\gamma}\eta^{\alpha\sigma}\partial_\alpha\partial_\omega b_{\sigma\gamma} + F_{,5}\eta^{\omega\sigma}\partial_\mu\partial_\nu b_{\omega\sigma}\,.\end{aligned} \tag{17}$$

For the *plane wave* $b_{\mu\nu} = B_{\mu\nu}e^{i\omega(t-z)}$ one finds in the Newman-Penrose basis the components

$$0 = E_{nn} = 2F_{,5}\ddot{b}_{m\bar{m}} - 2(F_{,2} + F_{,4} + F_{,5})\ddot{b}_{ln}\,, \tag{18a}$$
$$0 = E_{nm} = \overline{E_{n\bar{m}}} = -(F_{,2} + F_{,4})\ddot{b}_{lm}\,, \tag{18b}$$
$$0 = E_{m\bar{m}} = F_{,5}\ddot{b}_{ll}\,, \tag{18c}$$
$$0 = E_{ln} = -(F_{,2} + F_{,4})\ddot{b}_{ll}\,. \tag{18d}$$

This yields the following polarizations:

- ■ $F_{,2} + F_{,4} = F_{,5} = 0$: In this case the linearized field equations are satisfied identically for an arbitrary plane null wave and there are no restrictions on the allowed polarizations. Theories of this type belong to the $\mathrm{E}(2)$ class II_6, shown in blue in the figure.
- ■ $F_{,2} + F_{,4} = 0$ and $F_{,5} \neq 0$: The longitudinal mode Ψ_2 is prohibited in this case, while the remaining modes are unrestricted. This corresponds to the $\mathrm{E}(2)$ class III_5, shown as a green line in the figure.
- □ $F_{,2} + F_{,4} \neq 0$ and $F_{,2} + F_{,4} + F_{,5} \neq 0$: The only allowed modes in this case are the breathing mode Φ_{22} and the two tensor modes Ψ_4, while the longitudinal mode Ψ_2 and the two vector modes Ψ_3 are prohibited. These theories have the $\mathrm{E}(2)$ class N_3, occupying most of the parameter space shown in the figure in white.
- ■ $F_{,2} + F_{,4} + F_{,5} = 0$ and $F_{,5} \neq 0$: The only allowed polarizations are the two tensor modes Ψ_2. These theories belong to the $\mathrm{E}(2)$ class N_2, shown as a red line in the figure. This includes STEGR, marked as a red point.

Introducing polar coordinates

$$F_{,2} = C\sin\vartheta\cos\varphi\,, \quad F_{,4} = C\sin\vartheta\sin\varphi\,, \quad F_{,5} = C\cos\vartheta\,, \tag{19}$$

with $C = \sqrt{F_{,2}^2 + F_{,4}^2 + F_{,5}^2} \neq 0$, one can visualize the modes: