Scalar-torsion theories of gravity

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Outline

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Motivation

- Open questions in cosmology and gravity:
  - Accelerating phases in the history of the Universe?
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  - Based on tetrad and flat spin connection.
  - Describes gravity as gauge theory of the translation group.
  - Gravitational field strength is torsion.
  - First order action, second order field equations.
  - Spin connection as Lorentz gauge degree of freedom.

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  - Possibly arises from more fundamental theory.
  - Differs from non-minimal coupling to curvature.
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- Scalar field non-minimally coupled to torsion [Geng '11]:
  - Possibly arises from more fundamental theory.
  - Differs from non-minimal coupling to curvature.
  - Possible model for so far unexplained observations.
- Arising questions:
  - Most general class of scalar-torsion gravity theories?
  - Behavior under conformal transformations?

Ingredients of scalar-torsion gravity

- Fundamental fields:
  - Coframe field $\theta^a = \theta^a{}_\mu \mathrm{d}x^\mu$.
  - Flat spin connection $\overset{\bullet}{\omega}{}^a{}_b = \overset{\bullet}{\omega}{}^a{}_{b\mu} \mathrm{d}x^\mu$.
  - N scalar fields $\phi = (\phi^A; A = 1, \ldots, N)$.
  - Arbitrary matter fields χ^I.

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- Derived quantities:
  - Frame field $e_a = e_a{}^\mu \partial_\mu$ with $\iota_{e_a} \theta^b = \delta^b_a$.
  - Metric $g_{\mu\nu} = \eta_{ab} \theta^a{}_\mu \theta^b{}_\nu$.
  - Volume form $\theta \mathrm{d}^4 x = \theta^0 \wedge \theta^1 \wedge \theta^2 \wedge \theta^3$.
  - Levi-Civita connection

$$\overset{\circ}{\omega}_{ab} = -\frac{1}{2}(\iota_{e_b} \iota_{e_c} \mathrm{d}\theta_a + \iota_{e_c} \iota_{e_a} \mathrm{d}\theta_b - \iota_{e_a} \iota_{e_b} \mathrm{d}\theta_c)\theta^c \,.$$

  - Torsion $T^a = \mathrm{d}\theta^a + \overset{\bullet}{\omega}{}^a{}_b \wedge \theta^b$.

Overview

$$S_g\left[\theta^a, \overset{\bullet}{\omega}{}^a{}_b, \phi^A\right] + S_m\left[\theta^a, \phi^A, \chi^I\right]$$

$$L(T, X, Y, \phi)$$

$$\mathcal{A}(\phi)T + \mathcal{B}(\phi)\partial_\mu\phi\partial^\mu\phi + \mathcal{C}(\phi)T^\mu\partial_\mu\phi + \mathcal{V}(\phi)$$

$$\mathcal{A}(\phi)T + \mathcal{B}(\phi)\partial_\mu\phi\partial^\mu\phi + \mathcal{V}(\phi)$$

$$f(T, \phi) + Z(\phi)\partial_\mu\phi\partial^\mu\phi$$

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- Structure of the action [MH '18]:

$$S\left[\theta^a, \overset{\bullet}{\omega}{}^a{}_b, \phi^A, \chi^I\right] = S_g\left[\theta^a, \overset{\bullet}{\omega}{}^a{}_b, \phi^A\right] + S_m\left[\theta^a, \phi^A, \chi^I\right] .$$

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- Variation of the action:
  - Gravitational part:

$$\begin{aligned}\delta S_g &= \int_M \left(\Delta_a \wedge \delta\theta^a + \frac{1}{2}\Xi_a{}^b \wedge \delta\overset{\bullet}{\omega}{}^a{}_b + \Phi_A \wedge \delta\phi^A\right)\\ &= \int_M \left(\Upsilon_a \wedge \delta\theta^a + \Pi_a \wedge \delta T^a + \Phi_A \wedge \delta\phi^A\right)\,.\end{aligned}$$

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  - Matter part:

$$\delta S_m = \int_M \left(\Sigma_a \wedge \delta\theta^a + \Psi_A \wedge \delta\phi^A + \Omega_I \wedge \delta\chi^I\right) .$$

- Relation between different terms used to write field equations:

$$\Delta_a = \Upsilon_a - \overset{\bullet}{\mathrm{D}}\Pi_a\,, \quad \Xi^{ab} = -2\Pi^{[a} \wedge \theta^{b]}\,,$$

$$\Pi^a = \frac{1}{4}\iota_{e_c}\iota_{e_b}\Xi^{bc} \wedge \theta^a - \iota_{e_b}\Xi^{ab}\,, \quad \Upsilon^a = \Delta^a + \overset{\bullet}{\mathrm{D}}\left(\frac{1}{4}\iota_{e_c}\iota_{e_b}\Xi^{bc} \wedge \theta^a - \iota_{e_b}\Xi^{ab}\right)\,.$$

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- Field equations:
  - Tetrad field equations:

$$\Delta_a + \Sigma_a = 0 \quad \Leftrightarrow \quad \Upsilon_a - \overset{\bullet}{\mathrm{D}}\Pi_a + \Sigma_a = 0\,.$$

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  - Scalar field equations: $\Phi_A + \Psi_A = 0$.
  - Matter field equations: $\Omega_I = 0$.

General scalar-torsion gravity - local Lorentz invariance

- Local Lorentz transformation of the fundamental fields:

$$\delta_\lambda \theta^a = \lambda^a{}_b \theta^b \,, \quad \delta_\lambda \overset{\bullet}{\omega}{}^a{}_b = \lambda^a{}_c \overset{\bullet}{\omega}{}^c{}_b - \overset{\bullet}{\omega}{}^a{}_c \lambda^c{}_b - \mathrm{d}\lambda^a{}_b = -\overset{\bullet}{\mathrm{D}} \lambda^a{}_b \,.$$

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$$\delta_\lambda S_g = \int_M \left[\Upsilon_a \wedge (\lambda^a{}_b \theta^b) + \Pi_a \wedge \left(\lambda^a{}_b T^b\right)\right] = \int_M \left(\Upsilon^{[a} \wedge \theta^{b]} + \Pi^{[a} \wedge T^{b]}\right) \lambda_{ab}$$

$$= \int_M \left[\Delta_a \wedge (\lambda^a{}_b \theta^b) - \frac{1}{2} \Xi_a{}^b \wedge \overset{\bullet}{\mathrm{D}}\lambda^a{}_b\right] = \int_M \left(\Delta^{[a} \wedge \theta^{b]} - \frac{1}{2} \overset{\bullet}{\mathrm{D}}\Xi^{ab}\right) \lambda_{ab} \,.$$

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  - Symmetry of the energy-momentum tensor:

$$\Sigma^{[a} \wedge \theta^{b]} = 0\,.$$

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  - Connection equations $\equiv$ antisymmetric part of tetrad equations:

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- Variation of the matter action under infinitesimal diffeomorphisms ξ:

$$\delta_\xi S_m = \int_M \left(\Sigma_a \wedge \mathcal{L}_\xi \theta^a + \Psi_A \wedge \mathcal{L}_\xi \phi^A + \Omega_I \wedge \mathcal{L}_\xi \chi^I\right) .$$

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- Energy-momentum conservation:

$$\overset{\circ}{\mathrm{D}}\Sigma_a + \Psi_A \wedge \iota_{e_a} \mathrm{d}\phi^A = 0 .$$

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- Disformal transformation:
  - Change of fundamental fields:

$$\bar{\theta}^a = C(\phi, X)\theta^a + D(\phi, X)\eta^{ab}(\iota_{e_b}\mathrm{d}\phi)\mathrm{d}\phi\,, \quad X = -\frac{1}{2}\eta^{ab}(\iota_{e_a}\mathrm{d}\phi)(\iota_{e_b}\mathrm{d}\phi)$$

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  - Leads to more lengthy relation between original and transformed variables.

General scalar-torsion gravity - cosmology

- Condition for symmetry under ξ of fundamental fields [MH '15]:

$$\mathcal{L}_\xi\theta^a = -\lambda^a{}_b\theta^b\,, \quad \mathcal{L}_\xi\overset{\bullet}{\omega}{}^a{}_b = \overset{\bullet}{D}\lambda^a{}_b\,, \quad \mathcal{L}_\xi\phi = 0\,.$$

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- Six generating vector fields $\xi_1, \ldots, \xi_6$ of cosmological symmetry.
  - Translation generators:

$$\xi_1 = \chi \sin\vartheta \cos\varphi \partial_r + \frac{\chi}{r} \cos\vartheta \cos\varphi \partial_\vartheta - \frac{\chi \sin\varphi}{r \sin\vartheta} \,,$$
$$\xi_2 = \chi \sin\vartheta \sin\varphi \partial_r + \frac{\chi}{r} \cos\vartheta \sin\varphi \partial_\vartheta + \frac{\chi \cos\varphi}{r \sin\vartheta} \,,$$
$$\xi_3 = \chi \cos\vartheta \partial_r - \frac{\chi}{r} \sin\vartheta \partial_\vartheta \,.$$

  - Rotation generators:

$$\xi_4 = \sin\varphi \partial_\vartheta + \frac{\cos\varphi}{\tan\vartheta} \partial_\varphi \,,$$
$$\xi_5 = -\cos\varphi \partial_\vartheta + \frac{\sin\varphi}{\tan\vartheta} \partial_\varphi \,,$$
$$\xi_6 = \partial_\varphi \,.$$

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$$\xi_3 = \chi\cos\vartheta\partial_r - \frac{\chi}{r}\sin\vartheta\partial_\vartheta\,.$$

  - Rotation generators:

$$\xi_4 = \sin\varphi\partial_\vartheta + \frac{\cos\varphi}{\tan\vartheta}\partial_\varphi\,,$$

$$\xi_5 = -\cos\varphi\partial_\vartheta + \frac{\sin\varphi}{\tan\vartheta}\partial_\varphi\,,$$

$$\xi_6 = \partial_\varphi\,.$$

- Any 2-form constructed from $\theta, \overset{\bullet}{\omega}, \phi$ with cosmological symmetry vanishes.

General scalar-torsion gravity - cosmology

- Condition for symmetry under ξ of fundamental fields [MH '15]:

$$\mathcal{L}_\xi \theta^a = -\lambda^a{}_b \theta^b\,, \quad \mathcal{L}_\xi \overset{\bullet}{\omega}{}^a{}_b = \overset{\bullet}{D}\lambda^a{}_b\,, \quad \mathcal{L}_\xi \phi = 0\,.$$

- Six generating vector fields $\xi_1, \ldots, \xi_6$ of cosmological symmetry.
  - Translation generators:

$$\xi_1 = \chi \sin\vartheta \cos\varphi \partial_r + \frac{\chi}{r}\cos\vartheta\cos\varphi\partial_\vartheta - \frac{\chi\sin\varphi}{r\sin\vartheta}\,,$$
$$\xi_2 = \chi \sin\vartheta \sin\varphi \partial_r + \frac{\chi}{r}\cos\vartheta\sin\varphi\partial_\vartheta + \frac{\chi\cos\varphi}{r\sin\vartheta}\,,$$
$$\xi_3 = \chi\cos\vartheta\partial_r - \frac{\chi}{r}\sin\vartheta\partial_\vartheta\,.$$

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- Any 2-form constructed from $\theta, \overset{\bullet}{\omega}, \phi$ with cosmological symmetry vanishes.
- Connection field equation solved identically [MH, L. Järv, M. Kršśák, C. Pfeifer '18 to appear].

General scalar-torsion gravity - cosmological spin connections

- Diagonal tetrad in spherical coordinates:

$$\theta^a{}_\mu = \operatorname{diag}\left(n(t), \frac{a(t)}{\sqrt{1-kr^2}}, a(t)r, a(t)r\sin\vartheta\right),$$

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- Cosmological spin connections [MH, L. Järv, U. Ualikhanova '18]:
  - Spatially flat spacetime $k = 0$:

$$\overset{\bullet}{\omega}{}^1{}_{2\vartheta} = -\overset{\bullet}{\omega}{}^2{}_{1\vartheta} = -1\,, \quad \overset{\bullet}{\omega}{}^1{}_{3\varphi} = -\overset{\bullet}{\omega}{}^3{}_{1\varphi} = -\sin\vartheta\,, \quad \overset{\bullet}{\omega}{}^2{}_{3\varphi} = -\overset{\bullet}{\omega}{}^3{}_{2\varphi} = -\cos\vartheta\,.$$

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  - Spatially closed spacetime $k = 1$:

$$\overset{\bullet}{\omega}{}^1{}_{2\vartheta} = -\overset{\bullet}{\omega}{}^2{}_{1\vartheta} = -\sqrt{1-r^2}\,, \quad \overset{\bullet}{\omega}{}^1{}_{2\varphi} = -\overset{\bullet}{\omega}{}^2{}_{1\varphi} = -r\sin\vartheta\,, \quad \overset{\bullet}{\omega}{}^1{}_{3\vartheta} = -\overset{\bullet}{\omega}{}^3{}_{1\vartheta} = r\,,$$

$$\overset{\bullet}{\omega}{}^1{}_{3\varphi} = -\overset{\bullet}{\omega}{}^3{}_{1\varphi} = -\sqrt{1-r^2}\sin\vartheta\,, \quad \overset{\bullet}{\omega}{}^2{}_{3r} = -\overset{\bullet}{\omega}{}^3{}_{2r} = -\frac{1}{\sqrt{1-r^2}}\,, \quad \overset{\bullet}{\omega}{}^2{}_{3\varphi} = -\overset{\bullet}{\omega}{}^3{}_{2\varphi} = -\cos\vartheta\,.$$

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  - Spatially open spacetime $k = -1$:

$$\overset{\bullet}{\omega}{}^0{}_{1r} = \overset{\bullet}{\omega}{}^1{}_{0r} = \frac{1}{\sqrt{1+r^2}}\,, \quad \overset{\bullet}{\omega}{}^0{}_{2\vartheta} = \overset{\bullet}{\omega}{}^2{}_{0\vartheta} = r\,, \quad \overset{\bullet}{\omega}{}^0{}_{3\varphi} = \overset{\bullet}{\omega}{}^3{}_{0\varphi} = r\sin\vartheta\,,$$

$$\overset{\bullet}{\omega}{}^1{}_{2\vartheta} = -\overset{\bullet}{\omega}{}^2{}_{1\vartheta} = -\sqrt{1+r^2}\,, \quad \overset{\bullet}{\omega}{}^1{}_{3\varphi} = -\overset{\bullet}{\omega}{}^3{}_{1\varphi} = -\sqrt{1+r^2}\sin\vartheta\,, \quad \overset{\bullet}{\omega}{}^2{}_{3\varphi} = -\overset{\bullet}{\omega}{}^3{}_{2\varphi} = -\cos\vartheta$$

Outline

$L(T, X, Y, \phi)$ theory - action

- Gravitational part of the action [MH, C. Pfeifer '18]:

$$S_g\left[\theta^a, \overset{\bullet}{\omega}{}^a{}_b, \phi^A\right] = \int_M L\left(T, X^{AB}, Y^A, \phi^A\right)\theta \mathrm{d}^4x\,.$$

  - Torsion scalar: $T = \frac{1}{2}T^\rho{}_{\mu\nu}S_\rho{}^{\mu\nu}$.
  - Superpotential:

$$S_{\rho\mu\nu} = \frac{1}{2}\left(T_{\nu\mu\rho} + T_{\rho\mu\nu} - T_{\mu\nu\rho}\right) - g_{\rho\mu}T^\sigma{}_{\sigma\nu} + g_{\rho\nu}T^\sigma{}_{\sigma\mu}\,.$$

  - Scalar field kinetic term: $X^{AB} = -\frac{1}{2}g^{\mu\nu}\phi^A_{,\mu}\phi^B_{,\nu}$.
  - Kinetic coupling term: $Y^A = T_\mu{}^{\mu\nu}\phi^A_{,\nu}$.

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  - Kinetic coupling term: $Y^A = T_\mu{}^{\mu\nu}\phi^A_{,\nu}$.
- Matter action variation expressed in components:

$$\delta S_m[\theta^a, \phi^A, \chi^I] = \int_M \left(\Theta_a{}^\mu\delta\theta^a{}_\mu + \vartheta_A\delta\phi^A + \varpi_I\delta\chi^I\right)\theta \mathrm{d}^4x\,.$$

$L(T, X, Y, \phi)$ theory - field equations

- Symmetric part of tetrad equations:

$$\overset{\circ}{\nabla}_{(\mu}\left(L_{Y^A}\phi^A_{,\nu)}\right) - \overset{\circ}{\nabla}_{\sigma}\left(L_{Y^A}\phi^A_{,\rho}\right)g^{\rho\sigma}g_{\mu\nu} + L_{Y^A}\left(T_{(\mu\nu)}{}^{\rho}\phi^A_{,\rho} + T^{\rho}{}_{\rho(\mu}\phi^A_{,\nu)}\right)$$
$$- Lg_{\mu\nu} - 2\overset{\circ}{\nabla}_{\rho}\left(L_T S_{(\mu\nu)}{}^{\rho}\right) + L_T S_{(\mu}{}^{\rho\sigma}T_{\nu)\rho\sigma} - L_{X^{AB}}\phi^A_{,\mu}\phi^B_{,\nu} = \Theta_{\mu\nu}\,.$$

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- Antisymmetric part of tetrad equations $\equiv$ connection equations:

$$3\partial_{[\rho}L_T T^\rho{}_{\mu\nu]} + \partial_{[\mu}L_{Y^A}\phi^A_{,\nu]} - \frac{3}{2}L_{Y^A}T^\rho{}_{[\mu\nu}\phi^A_{,\rho]} = 0\,.$$

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- Scalar field equations:

$$g^{\mu\nu}\mathring{\nabla}_{\mu}\left(L_{Y^A}T^{\rho}{}_{\rho\nu} - L_{X^{AB}}\phi^B_{,\nu}\right) - L_{\phi^A} = \vartheta_A\,.$$

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- Matter field equations: $\varpi_I = 0$.

Outline

“Scalar-curvature”-like class - action

- Action [MH '18]:
  - Gravitational part:

$$S_g\left[\theta^a, \overset{\bullet}{\omega}{}^a{}_b, \phi^A\right] = \frac{1}{2\kappa^2}\int_M \left[-\mathcal{A}(\phi)T + 2\mathcal{B}_{AB}(\phi)X^{AB} + 2\mathcal{C}_A(\phi)Y^A - 2\kappa^2\mathcal{V}(\phi)\right]\theta\mathrm{d}^4x\,.$$

  - Matter part:

$$S_m[\theta^a, \phi^A, \chi^I] = S_m^{\mathfrak{J}}\left[e^{\alpha(\phi)}\theta^a, \chi^I\right]\,.$$

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- Free functions $\mathcal{A}, \mathcal{B}_{AB}, \mathcal{C}_A, \mathcal{V}, \alpha$ of scalar fields.

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- Free functions $\mathcal{A}, \mathcal{B}_{AB}, \mathcal{C}_A, \mathcal{V}, \alpha$ of scalar fields.
- $\mathcal{C}_A \equiv -\mathcal{A}_{,A} \Leftrightarrow$ theory reduces to scalar-curvature gravity.
- Special subclass of $L(T, X, Y, \phi)$ class of theories.

"Scalar-curvature"-like class - field equation

- Symmetric part of the tetrad equations:

$$\left(\mathcal{A}_{,A} + \mathcal{C}_A\right) S_{(\mu\nu)}{}^{\rho}\phi^A_{,\rho} + \mathcal{A}\left(\overset{\circ}{R}_{\mu\nu} - \frac{1}{2}\overset{\circ}{R}g_{\mu\nu}\right) + \left(\frac{1}{2}\mathcal{B}_{AB} - \mathcal{C}_{(A,B)}\right)\phi^A_{,\rho}\phi^B_{,\sigma}g^{\rho\sigma}g_{\mu\nu}$$
$$- \left(\mathcal{B}_{AB} - \mathcal{C}_{(A,B)}\right)\phi^A_{,\mu}\phi^B_{,\nu} + \mathcal{C}_A\left(\overset{\circ}{\nabla}_\mu\overset{\circ}{\nabla}_\nu\phi^A - \overset{\circ}{\Box}\phi^A g_{\mu\nu}\right) + \kappa^2\mathcal{V}g_{\mu\nu} = \kappa^2\Theta_{\mu\nu}\,,$$

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$$-\left(\mathcal{B}_{AB}-\mathcal{C}_{(A,B)}\right)\phi^A_{,\mu}\phi^B_{,\nu}+\mathcal{C}_A\left(\overset{\circ}{\nabla}_\mu\overset{\circ}{\nabla}_\nu\phi^A-\overset{\circ}{\square}\phi^A g_{\mu\nu}\right)+\kappa^2\mathcal{V}g_{\mu\nu}=\kappa^2\Theta_{\mu\nu}\,,$$

- Antisymmetric part of the tetrad field equations:

$$3(\mathcal{A}_{,A}+\mathcal{C}_A)T^{\rho}{}_{[\mu\nu}\phi^A_{,\rho]}+2\mathcal{C}_{[A,B]}\phi^A_{,\mu}\phi^B_{,\nu}=0\,.$$

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- Scalar field equation:

$$\frac{1}{2}\mathcal{A}_{,A}T - \mathcal{B}_{AB}\overset{\circ}{\square}\phi^B - \left(\mathcal{B}_{AB,C} - \frac{1}{2}\mathcal{B}_{BC,A}\right)g^{\mu\nu}\phi^B_{,\mu}\phi^C_{,\nu}$$
$$+ \mathcal{C}_A\overset{\circ}{\nabla}_\mu T_\nu{}^{\nu\mu} + 2\mathcal{C}_{[A,B]}T_\mu{}^{\mu\nu}\phi^B_{,\nu} + \kappa^2\mathcal{V}_{,A} = \kappa^2\alpha_{,A}\Theta\,.$$

"Scalar-curvature"-like class - conformal transf.

- Conformal transformation and scalar field redefinition:

$$\bar{\theta}^a = e^{\gamma(\phi)}\theta^a\,, \quad \bar{e}_a = e^{-\gamma(\phi)}e_a\,, \quad \bar{\phi}^A = f^A(\phi)\,.$$

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- Transformation of geometry:

$$\bar{T} = e^{-2\gamma}\left(T + 4\gamma_{,A}Y^A + 12\gamma_{,A}\gamma_{,B}X^{AB}\right)\,, \quad \bar{\phi}^A = f^A\,,$$

$$\bar{X}^{AB} = e^{-2\gamma}\frac{\partial\bar{\phi}^A}{\partial\phi^C}\frac{\partial\bar{\phi}^B}{\partial\phi^D}X^{CD}\,, \quad \bar{Y}^A = e^{-2\gamma}\frac{\partial\bar{\phi}^A}{\partial\phi^B}\left(Y^B + 6\gamma_{,C}X^{BC}\right)\,,$$

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- Transformation of parameter functions to preserve action:

$$\begin{aligned}
\mathcal{A} &= e^{2\gamma}\bar{\mathcal{A}}\,,\\
\mathcal{B} &= e^{2\gamma}\left(\bar{\mathcal{B}}f'^2 - 6\bar{\mathcal{A}}\gamma'^2 + 6\bar{\mathcal{C}}f'\gamma'\right)\,,\\
\mathcal{C} &= e^{2\gamma}\left(\bar{\mathcal{C}}f' - 2\bar{\mathcal{A}}\gamma'\right)\,,\\
\mathcal{V} &= e^{4\gamma}\bar{\mathcal{V}}\,,\\
\alpha &= \bar{\alpha} + \gamma\,.
\end{aligned}$$

"Scalar-curvature"-like class - invariants

- Quantities invariant under conformal transformations γ:
  - "Scalar" quantities:

$$\mathcal{I}_1 = \frac{e^{2\alpha}}{\mathcal{A}}\,, \quad \mathcal{I}_2 = \frac{\mathcal{V}}{\mathcal{A}^2}\,.$$

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  - “Covector” quantities:

$$\mathcal{H}_A = \frac{\mathcal{C}_A + \mathcal{A}_{,A}}{2\mathcal{A}}\,, \quad \mathcal{K}_A = \frac{\mathcal{C}_A + 2\alpha_{,A}\mathcal{A}}{2e^{2\alpha}}\,.$$

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  - “Metric” quantities:

$$\mathcal{F}_{AB} = \frac{2\mathcal{A}\mathcal{B}_{AB} - 6\mathcal{A}_{,(A}\mathcal{C}_{B)} - 3\mathcal{A}_{,A}\mathcal{A}_{,B}}{4\mathcal{A}^2}\,,$$

$$\mathcal{G}_{AB} = \frac{\mathcal{B}_{AB} - 6\alpha_{,(A}\mathcal{C}_{B)} - 6\alpha_{,A}\alpha_{,B}\mathcal{A}}{2e^{2\alpha}}\,.$$

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$$\mathcal{G}_{AB} = \frac{\mathcal{B}_{AB} - 6\alpha_{,(A}\mathcal{C}_{B)} - 6\alpha_{,A}\alpha_{,B}\mathcal{A}}{2e^{2\alpha}}.$$
- Covariance under scalar field redefinitions:
$$\bar{\mathcal{I}}_{1,2} = \mathcal{I}_{1,2}, \quad (\bar{\mathcal{H}}, \bar{\mathcal{K}})_A = \frac{\partial \phi^B}{\partial \bar{\phi}^A}(\mathcal{H}, \mathcal{K})_B, \quad (\bar{\mathcal{F}}, \bar{\mathcal{G}})_{AB} = \frac{\partial \phi^C}{\partial \bar{\phi}^A}\frac{\partial \phi^D}{\partial \bar{\phi}^B}(\mathcal{F}, \mathcal{G})_{CD}.$$

“Scalar-curvature”-like class - special frames

- Jordan frame: minimal coupling to matter.

$$\mathcal{A}^{\tilde{\mathfrak{J}}} = \frac{1}{\mathcal{I}_1}\,, \quad \mathcal{B}^{\tilde{\mathfrak{J}}}_{AB} = 2\mathcal{G}_{AB}\,, \quad \mathcal{C}^{\tilde{\mathfrak{J}}}_{A} = 2\mathcal{K}_{A}\,, \quad \mathcal{V}^{\tilde{\mathfrak{J}}} = \frac{\mathcal{I}_2}{\mathcal{I}_1^2}\,, \quad \alpha^{\tilde{\mathfrak{J}}} = 0\,.$$

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- “Debraiding frame” (for $\mathcal{H}_{[A,B]} \equiv 0$): minimal coupling to torsion.

$$\left(\ln\mathcal{A}^{\mathfrak{D}}\right)_{,A} = 2\mathcal{H}_A\,, \quad \left(\ln\mathcal{B}^{\mathfrak{D}}\right)^{A}{}_{B,C} = [\ln(\mathcal{F} + 3\mathcal{H}\otimes\mathcal{H})]^{A}{}_{B,C} + 2\delta^A_B\mathcal{H}_C\,,$$

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Outline

Scalar-torsion gravity without derivative coupling

- Gravitational action [MH, L. Järv, U. Ualikhanova '18]:

$$S = \frac{1}{2\kappa^2}\int_M \left[f(T,\phi) + Z_{AB}(\phi)g^{\mu\nu}\phi^A_{,\mu}\phi^B_{,\nu}\right]\theta d^4x + S_m[\theta^a,\chi^I]\,.$$

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- Field equations:
  - Symmetric part of the tetrad field equations:

$$\frac{1}{2} f g_{\mu\nu} + \overset{\circ}{\nabla}_\rho \left(f_T S_{(\mu\nu)}{}^\rho \right) - \frac{1}{2} f_T S_{(\mu}{}^{\rho\sigma} T_{\nu)\rho\sigma} - Z_{AB} \phi^A_{,\mu} \phi^B_{,\nu} + \frac{1}{2} Z_{AB} \phi^A_{,\rho} \phi^B_{,\sigma} g^{\rho\sigma} g_{\mu\nu} = \kappa^2 \Theta_{\mu\nu} \,,$$

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  - Scalar field equation:

$$f_{\phi^A} - \left(2Z_{AB,\phi^C} - Z_{BC,\phi^A} \right) g^{\mu\nu} \phi^B_{,\mu} \phi^C_{,\nu} - 2Z_{AB} \overset{\circ}{\square} \phi^B = 0 \,.$$

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- Contains various interesting examples: $f(T)$ equivalent, teleparallel dark energy, ...

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Summary

- General scalar-torsion theories:
  - Most general class of theories based on tetrad, flat spin connection, scalar field(s).
  - No direct coupling between matter and spin connection.
  - Local Lorentz invariance: symmetric energy-momentum, dependence of equations.
  - Energy-momentum conservation up to exchange between matter and scalar field(s).
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- Theory without derivative couplings:
  - Simple, yet interesting class of scalar-torsion theories.
  - Includes $f(T)$, teleparallel dark energy, other studied models.
  - Good test case for application and development of formalisms.

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  - Dynamical systems analysis and stable fixed points
  - Dark energy?
  - Inflation and related parameters

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- Hamiltonian formulation:
  - Degrees of freedom in fully dynamical theory.
  - Potential appearance of ghosts?
  - Work towards numerical simulations.

Literature

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