Quantum manifolds

. . . with classical limit

Manuel Hohmann

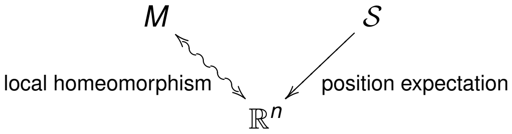
II. Institut für theoretische Physik

Universität Hamburg
DER FORSCHUNG | DER LEHRE | DER BILDUNG

Foundations of Probability and Physics - 6
Växjö - 13 June 2011

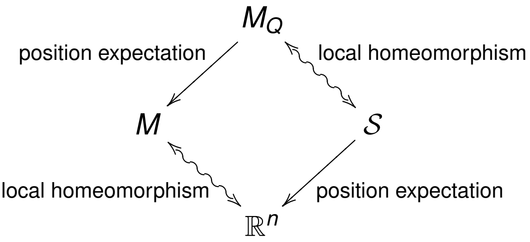
Basic idea

- Classical mechanics: Euclidean space $\mathbb{R}^n$
- General relativity: Riemannian manifold M
- Quantum mechanics: Schwartz space $\mathcal{S}$



Basic idea

- Classical mechanics: Euclidean space $\mathbb{R}^n$
- General relativity: Riemannian manifold M
- Quantum mechanics: Schwartz space $\mathcal{S}$
- Quantum gravity: quantum manifold M_Q?



Schwartz space

- Schwartz space $\mathcal{S}(\mathbb{R}^n)$ of fast-decreasing functions:

$$\mathcal{S}(\mathbb{R}^n) = \left\{ f \in C^\infty(\mathbb{R}^n) \,|\, \forall\, \boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{N}^n : \sup_{\boldsymbol{x} \in \mathbb{R}^n} |x^{\boldsymbol{\alpha}} D_{\boldsymbol{\beta}} f(\boldsymbol{x})| < \infty \right\}$$

- Infinite family of seminorms generates natural topology:

$$\|f\|_{\boldsymbol{\alpha},\boldsymbol{\beta}} = \sup_{\boldsymbol{x} \in \mathbb{R}^n} |x^{\boldsymbol{\alpha}} D_{\boldsymbol{\beta}} f(\boldsymbol{x})|$$

- Scalar product inherited from $L^2(\mathbb{R}^n)$:

$$\langle f, g \rangle = \int_{\mathbb{R}^n} d\boldsymbol{x}\, f(\boldsymbol{x})^* g(\boldsymbol{x})$$

- For convenience, drop argument $\mathbb{R}^n$ and define $\mathcal{S}^{\neq 0} = \mathcal{S} \setminus \{0\}$.

Position expectation value

- Position operator $\boldsymbol{Q} : f \mapsto (\boldsymbol{x} \mapsto f(\boldsymbol{x})\boldsymbol{x})$
- Position expectation value:

$$\bar{\boldsymbol{Q}}(f) = \frac{\langle f, \boldsymbol{Q}f \rangle}{\langle f, f \rangle}$$

- Expectation value topology: open subsets of $\mathcal{S}^{\neq 0}$ are pre-images of open sets $W \subset \mathbb{R}^n$:

$$\bar{\boldsymbol{Q}}^{-1}(W) = \left\{ f \in \mathcal{S}^{\neq 0} | \bar{\boldsymbol{Q}}(f) \in W \right\}$$

- For later use, define $\mathcal{S}_0 = \bar{\boldsymbol{Q}}^{-1}(0)$.

Quantum manifold

- Set M_Q of points
- Quantum atlas: Collection of pairs (U_i, ϕ_i) with the following properties:
  - Each U_i is a subset of M_Q and the U_i cover M_Q
  - Each ϕ_i is a bijection of U_i onto a set $\phi_i(U_i) \subset \mathcal{S}^{\neq 0}$
  - For each i, j, the set $\phi_i(U_i \cap U_j)$ is open in the expectation value topology
  - For each i, j, the transition map $\phi_{ji} = \phi_j \circ \phi_i^{-1}$ on the overlap of any two charts, $\phi_{ji} : \phi_i(U_i \cap U_j) \to \phi_j(U_i \cap U_j)$ is continuous in the expectation value topology and differentiable in the natural topology

Quantum manifold

- Set M_Q of points
- Quantum atlas: Collection of pairs (U_i, ϕ_i) with the following properties:
  - Each U_i is a subset of M_Q and the U_i cover M_Q
  - Each ϕ_i is a bijection of U_i onto a set $\phi_i(U_i) \subset \mathcal{S}^{\neq 0}$
  - For each i, j, the set $\phi_i(U_i \cap U_j)$ is open in the expectation value topology
  - For each i, j, the transition map $\phi_{ji} = \phi_j \circ \phi_i^{-1}$ on the overlap of any two charts, $\phi_{ji} : \phi_i(U_i \cap U_j) \to \phi_j(U_i \cap U_j)$ is continuous in the expectation value topology and differentiable in the natural topology
- A *quantum manifold* is a set together with a quantum atlas.

Classical limit

- Expectation value topology lifts to M_Q
- Identify topologically indistinguishable elements of M_Q: Kolmogorov quotient $M_Q \xrightarrow{\mathcal{Q}} M$

Classical limit

- Expectation value topology lifts to M_Q
- Identify topologically indistinguishable elements of M_Q: Kolmogorov quotient $M_Q \xrightarrow{\mathcal{Q}} M$
- Construct "chart" of M:

$$\begin{array}{ccccc} M_Q \supset & U_i & \xrightarrow{\phi_i} & V_i & \subset \mathcal{S}^{\neq 0} \\ & \downarrow \mathcal{Q} & & \downarrow \bar{Q} & \\ M \supset & X_i & \xrightarrow[\chi_i]{} & W_i & \subset \mathbb{R}^n \end{array}$$

$\Rightarrow$ χ_i is homeomorphism w.r.t. the quotient topology on X_i and the standard topology on W_i.

Classical limit

- Expectation value topology lifts to M_Q
- Identify topologically indistinguishable elements of M_Q: Kolmogorov quotient $M_Q \stackrel{\mathcal{Q}}{\to} M$
- Construct "chart" of M:

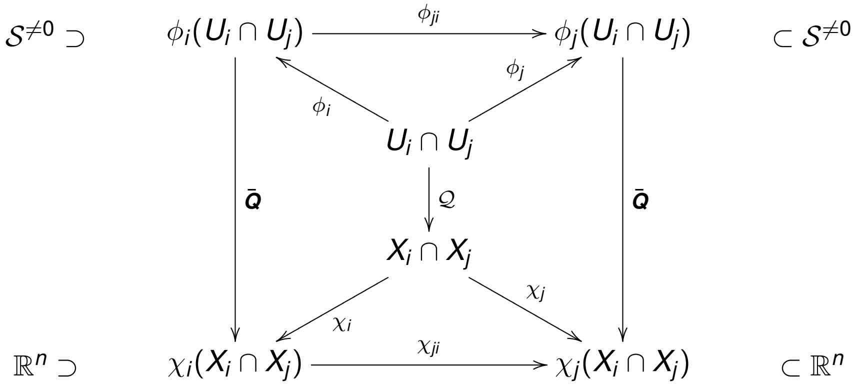
$$
\begin{array}{ccccc}
M_Q \supset & U_i & \xrightarrow{\phi_i} & V_i & \subset \mathcal{S}^{\neq 0} \\
 & \mathcal{Q}\downarrow & & \downarrow \bar{\boldsymbol{Q}} & \\
M \supset & X_i & \xrightarrow[\chi_i]{} & W_i & \subset \mathbb{R}^n
\end{array}
$$

$\Rightarrow$ χ_i is homeomorphism w.r.t. the quotient topology on X_i and the standard topology on W_i.

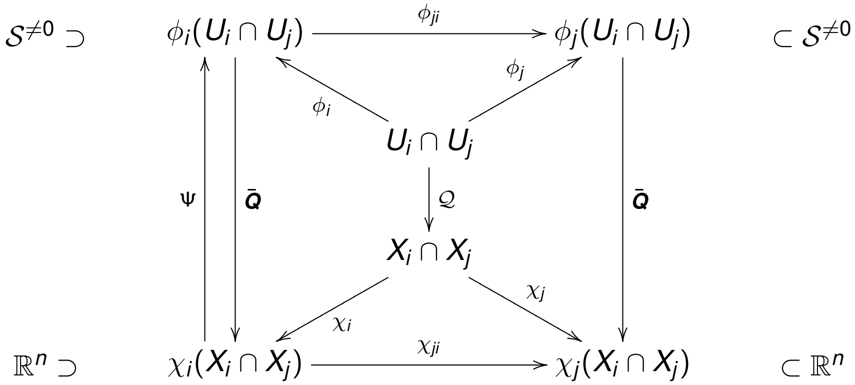
- Charts can be glued together to form an atlas.

$\Rightarrow$ Classical topological limit manifold exists.

Classical *differentiable* limit



Classical *differentiable* limit



- Define:

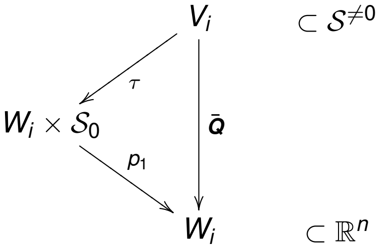
$$\Psi : \boldsymbol{x} \mapsto \left(\boldsymbol{y} \mapsto e^{-(\boldsymbol{y}-\boldsymbol{x})^2} \right)$$

$\Rightarrow$ $\chi_{ji} = \bar{\boldsymbol{Q}} \circ \phi_{ji} \circ \Psi$ is differentiable.

$\Rightarrow$ Classical limit M is a differentiable manifold.

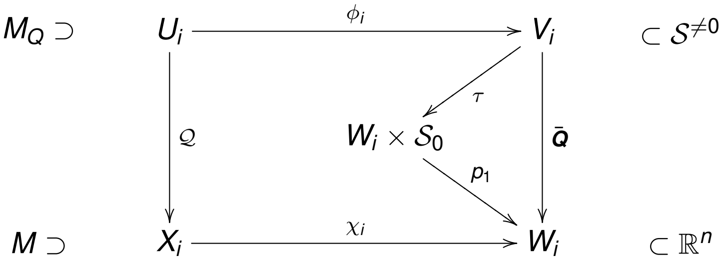
Fiber bundle structure

- Fact: $(\mathcal{S}^{\neq 0}, \mathbb{R}^n, \bar{\boldsymbol{Q}}, \mathcal{S}_0)$ is a trivial fiber bundle.



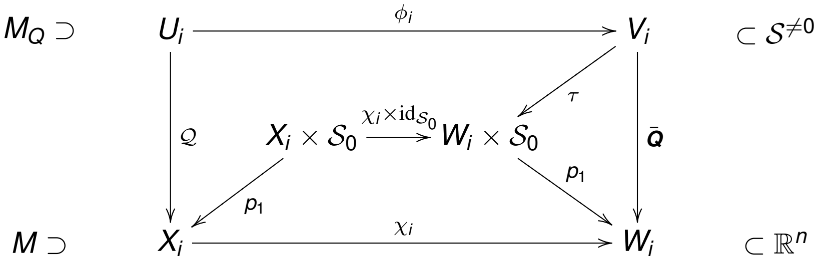
Fiber bundle structure

- Fact: $(\mathcal{S}^{\neq 0}, \mathbb{R}^n, \bar{\boldsymbol{Q}}, \mathcal{S}_0)$ is a trivial fiber bundle.
- Consider a quantum atlas of M_Q.



Fiber bundle structure

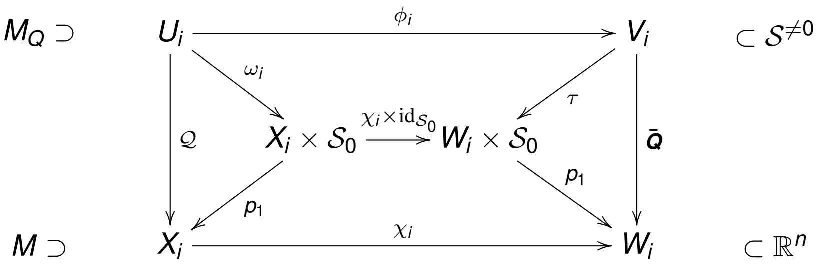
- Fact: $(\mathcal{S}^{\neq 0}, \mathbb{R}^n, \bar{\boldsymbol{Q}}, \mathcal{S}_0)$ is a trivial fiber bundle.
- Consider a quantum atlas of M_Q.
- Homeomorphism χ_i can be lifted.



Fiber bundle structure

- Fact: $(\mathcal{S}^{\neq 0}, \mathbb{R}^n, \bar{\boldsymbol{Q}}, \mathcal{S}_0)$ is a trivial fiber bundle.
- Consider a quantum atlas of M_Q.
- Homeomorphism χ_i can be lifted.
- Homeomorphism ω_i exists.

$\Rightarrow$ $(M_Q, M, \mathcal{Q}, \mathcal{S}_0)$ is a fiber bundle.



Trivial quantization

- Take some arbitrary classical manifold M with charts (X_i, χ_i)
- Define $M_Q = M \times \mathcal{S}_0$, $U_i = X_i \times \mathcal{S}_0$ and

$$\phi_i : (\xi, g) \mapsto (\boldsymbol{x} \mapsto g(\boldsymbol{x} - \chi_i(\xi)))$$

Trivial quantization

- Take some arbitrary classical manifold M with charts (X_i, χ_i)
- Define $M_Q = M \times \mathcal{S}_0$, $U_i = X_i \times \mathcal{S}_0$ and

$$\phi_i : (\xi, g) \mapsto (\boldsymbol{x} \mapsto g(\boldsymbol{x} - \chi_i(\xi)))$$

$\Rightarrow$ (U_i, ϕ_i) are charts of a quantum atlas.

$\Rightarrow$ M_Q is a quantum manifold.

$\Rightarrow$ Classical limit of M_Q is M.

$\Rightarrow$ $(M_Q, M, \mathcal{Q}, \mathcal{S}_0)$ is a trivial fiber bundle - hence the name “trivial quantization”.

Results

- Quantum manifold M_Q
- Locally homeomorphic to Schwarz space $\mathcal{S}$
- Classical differentiable limit manifold M
- M_Q is fiber bundle over M
- Trivial quantization of every classical manifold

Outlook

- External vs. internal time?
- Quantization of momentum?
- Quantum algebra?
- Quantization of (tensor) fields?
- Quantum gravity?

Outlook

- External vs. internal time?
- Quantization of momentum?
- Quantum algebra?
- Quantization of (tensor) fields?
- Quantum gravity?

MH, R. Punzi and M. N. R. Wohlfarth,
"Quantum manifolds with classical limit",
arXiv:0809.3111