The gauge-invariant parametrized post-Newtonian formalism

arXiv:1910.09245

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Gravitational waves, black holes and fundamental physics - 15. 1. 2020

Outline

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Motivation

- Experimental tests of modified gravity theories:
  - Cosmological observations (CMB, supernovae, . . .).
  - Gravitational waves.
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- Parametrized post-Newtonian formalism:
  - Weak-field approximation of metric gravity theories.
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  - Characterizes gravity theories by 10 (constant) parameters.
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  - More fundamental fields constituting the metric.
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$\rightsquigarrow$ Use gauge-invariant higher order perturbation theory.

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Concept and use of gauge

- Reference spacetime:
  - Manifold M_0 with metric $g^{(0)}$ and coordinates (x^μ).
  - Usually some highly symmetric standard spacetime:
    - maximally symmetric spacetime: Minkowski, (anti-)de Sitter
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$\rightsquigarrow$ Introduce a *gauge*: diffeomorphism $\mathcal{X} : M_0 \to M$.

1. Identification of (coordinated) points on M and M_0.
2. Comparison between reference metric $g^{(0)}$ and ${}^{\mathcal{X}}g = \mathcal{X}^*g$ on M_0.

Gauge and perturbations

- Parameter dependent physical metric:
  - Assume physical metric $g \equiv g_\epsilon$ depends on parameter $\epsilon \in \mathbb{R}$.
  - Assume every g_ϵ is defined on its own M_ϵ.
  - Assume $g_0 = g^{(0)}$ is the reference metric on M_0.
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$$ {}^{\mathcal{X}}g_\epsilon = \sum_{k=0}^{\infty} \frac{\epsilon^k}{k!} \left. \frac{\partial^k\, {}^{\mathcal{X}}g_\epsilon}{\partial \epsilon^k} \right|_{\epsilon=0} = \sum_{k=0}^{\infty} \frac{\epsilon^k}{k!}\, {}^{\mathcal{X}}g^{(k)} \,. $$

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  - Series coefficients ${}^{\mathcal{X}}g^{(k)}$ depend on gauge choice $\mathcal{X}$.

Gauge invariant perturbations

- Choose a fixed "distinguished" gauge $\mathcal{S}_\epsilon : M_0 \to M_\epsilon$:
  - E.g., impose gauge conditions on the metric.
  - Examples: harmonic gauge used for GWs, standard PPN gauge.

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- Metric in arbitrary gauge $\mathcal{X}$:

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{}^{\mathcal{X}}g_\epsilon = \sum_{l_1=0}^{\infty} \cdots \sum_{l_k=0}^{\infty} \cdots \frac{\epsilon^{l_1+\cdots+kl_k+\cdots}}{(1!)^{l_1}\cdots(k!)^{l_k}\cdots l_1!\cdots l_k!\cdots} \pounds_{X_{(1)}}^{l_1} \cdots \pounds_{X_{(k)}}^{l_k} \cdots \mathbf{g}_\epsilon \,.
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- Number # of independent components:

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\#({}^{\mathcal{X}}g_\epsilon) = \#(\mathbf{g}_\epsilon) + \#(X_{(k)}) \,.
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Post-Newtonian matter and velocity orders

- Energy-momentum tensor of a perfect fluid:

$$T^{\mu\nu} = (\rho + \rho\Pi + p)u^{\mu}u^{\nu} + pg^{\mu\nu}\,.$$

  - Rest mass density ρ.
  - Specific internal energy Π.
  - Pressure p.
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- Universe rest frame and slow-moving source matter:
  - Consider some gauge $\mathcal{X} : M_0 \to M$ ("universe rest frame").
  - Pullback of metric and matter variables along $\mathcal{X}$.
  - Velocity of the source matter: ${}^{\mathcal{X}}v^i = {}^{\mathcal{X}}u^i / {}^{\mathcal{X}}u^0$.
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- Quasi-static: assign additional $\mathcal{O}(1)$ to time derivatives ∂_0.

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- Metric in standard PPN gauge:

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{}^{\mathcal{P}}\overset{2}{g}_{00} &= 2\,{}^{\mathcal{P}}U\,,\\
{}^{\mathcal{P}}\overset{2}{g}_{ij} &= 2\gamma\,{}^{\mathcal{P}}U\delta_{ij}\,,\\
{}^{\mathcal{P}}\overset{3}{g}_{0i} &= -\frac{1}{2}(3+4\gamma+\alpha_1-\alpha_2+\zeta_1-2\xi)\,{}^{\mathcal{P}}V_i-\frac{1}{2}(1+\alpha_2-\zeta_1+2\xi)\,{}^{\mathcal{P}}W_i\,,\\
{}^{\mathcal{P}}\overset{4}{g}_{00} &= -2\beta\,{}^{\mathcal{P}}U^2+(2+2\gamma+\alpha_3+\zeta_1-2\xi)\,{}^{\mathcal{P}}\Phi_1+2(1+3\gamma-2\beta+\zeta_2+\xi)\,{}^{\mathcal{P}}\Phi_2\\
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- Metric contains PPN parameters and PPN potentials.
  - PPN potentials describe source matter distribution.
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$\rightsquigarrow$ Decompose metric into gauge-invariant and pure gauge parts.

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Gauge-invariant metric

- Definition of gauge-invariant metric components:

$$\mathbf{g}_{00} = \mathbf{g}^{\star}\,, \quad \mathbf{g}_{0i} = \mathbf{g}^{\diamond}_{i}\,, \quad \mathbf{g}_{ij} = \mathbf{g}^{\bullet}\delta_{ij} + \mathbf{g}^{\dagger}_{ij}\,.$$

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- Conditions imposed on components:

$$\partial^i\mathbf{g}^{\diamond}_{i} = 0\,, \quad \partial^i\mathbf{g}^{\dagger}_{ij} = 0\,, \quad \mathbf{g}^{\dagger}_{[ij]} = 0\,, \quad \mathbf{g}^{\dagger}_{ii} = 0\,.$$

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- Relation to arbitrary gauge $\mathcal{X}$:

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- Gauge defining vector fields:

$$X_i = \partial_i X^{\blacklozenge} + X^{\diamond}_{i}\,, \quad X_0 = X^{\star}\,, \quad \partial^i X^{\diamond}_{i} = 0\,.$$

Decomposition of metric components

- Count number of independent components at each order:

total		invariant		pure gauge	
$\mathcal{X}\overset{2}{g}_{00}$	1	$\overset{2}{\mathbf{g}}^{\star}$	1	-	0
$\mathcal{X}\overset{2}{g}_{ij}$	6	$\overset{2}{\mathbf{g}}^{\star}, \overset{2}{\mathbf{g}}^{\dagger}_{ij}$	$1+2$	$\overset{2}{X}^{\blacklozenge}, \overset{2}{X}^{\diamond}_{i}$	$1+2$
$\mathcal{X}\overset{3}{g}_{0i}$	3	$\overset{3}{\mathbf{g}}^{\diamond}_{i}$	2	$\overset{3}{X}^{\star}$	1
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- Count number of independent components at each order:

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$\Rightarrow$ Components split into invariant and gauge parts.

$\Rightarrow$ Possible to separate physical information from coordinate choice.

Relation to standard PPN gauge

- Use relation between expansion coefficients:

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{}^{\mathcal{P}}\overset{k}{g} = \sum_{0 \leq l_1 + 2l_2 + \ldots \leq k} \frac{1}{l_1!\,l_2!\ldots} \mathcal{L}^{l_1}_{\overset{1}{P}} \cdots \mathcal{L}^{l_k}_{\overset{k}{P}} \cdots \overset{k-l_1-2l_2-\ldots}{\mathbf{g}} .
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$$\overset{2}{P}{}^{\blacklozenge} = 0\,, \quad \overset{2}{P}{}^{\diamond}_{i} = 0\,, \quad \overset{3}{P}{}^{\star} = -\frac{1}{4}(2 + 4\gamma + \alpha_1 - 2\alpha_2 + 2\zeta_1 - 4\xi)\chi_{,0}\,.$$

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$$-2\xi\mathbf{\Phi}_W - 2\beta\mathbf{U}^2 + \frac{1}{2}(2 + 4\gamma + \alpha_1 - 2\alpha_2)\mathfrak{A} + \frac{1}{2}(2 + 4\gamma + \alpha_1 - 2\alpha_2 + 2\zeta_1 - 4\xi)\mathfrak{B}\,.$$

Gauge-invariant field equations

- Perform similar decomposition of energy-momentum tensor:

$$\begin{aligned} \mathbf{T}^{\star} &= \mathbf{T}_{00} = \rho\left(1 - \overset{2}{\mathbf{g}}_{00} + \mathbf{v}^2 + \mathbf{\Pi}\right) + \mathcal{O}(6)\,, \\ \mathbf{T}^{\diamond}_{i} + \partial_i \mathbf{T}^{\blacklozenge} &= \mathbf{T}_{0i} = -\rho \mathbf{v}_i + \mathcal{O}(5)\,, \\ \mathbf{T}^{\bullet}\delta_{ij} + \triangle_{ij}\mathbf{T}^{\blacktriangle} + 2\partial_{(i}\mathbf{T}^{\triangle}_{j)} + \mathbf{T}^{\dagger}_{ij} &= \mathbf{T}_{ij} = \rho \mathbf{v}_i \mathbf{v}_j + \mathbf{p}\delta_{ij} + \mathcal{O}(6)\,. \end{aligned}$$

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⇒ Find PPN parameters by comparing coefficients on both sides.

Outline

Action and field equations

- Action of scalar-tensor gravity with massless scalar field: [Nordtvedt '70]

$$S = \frac{1}{2\kappa^2}\int_M d^4x\sqrt{-g}\left(\psi R - \frac{\omega(\psi)}{\psi}\partial_\rho\psi\partial^\rho\psi\right) + S_m[g_{\mu\nu},\chi]\,.$$

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⇝ Decompose into gauge-invariant field equations.

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⇒ Remaining equations determine gauge-invariant metric components.

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⇒ PPN parameters reproduce well-known result: [Nordtvedt '70]

$$\gamma = \frac{\omega_0 + 1}{\omega_0 + 2}\,, \quad \beta = 1 + \frac{\omega_1\Psi}{4(2\omega_0 + 3)(\omega_0 + 2)^2}\,, \quad \alpha_1 = \alpha_2 = \alpha_3 = \zeta_1 = \zeta_2 = \zeta_3 = \zeta_4 = \xi = 0\,.$$

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Summary

- Gauge-invariant perturbation theory:
  - Distinguish between physical and background spacetime.
  - Gauge pulls physical metric to background spacetime.
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  - Decompose perturbations into physical data and gauge data.

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- Post-Newtonian limit of scalar-tensor gravity:
  - Perturbative field equations simplify in gauge-invariant formulation.
  - Consistency check: obtain well-known PPN parameters.
  - Also possible to use tetrad formulation to calculate solution.

Outlook

- Extend formalism by including higher perturbation orders:
  - General covariant expansion instead of space-time split.
  - Allow also for fast-moving source masses.
  - Consider inspiral phase of black hole merger event.
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- Further possible extensions and modifications:
  - Apply to formalism based on modified density $\rho^* = \rho\sqrt{-g}u^0$. [Will '18]
  - Allow for multiple dynamical metrics / tetrads.
  - Take into account Vainshtein or other screening mechanisms.
  - Consider cosmological background and time variability of PPN parameters.
  - Include further PPN potentials appearing in higher derivative theories.

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- Extend formalism by including higher perturbation orders:
  - General covariant expansion instead of space-time split.
  - Allow also for fast-moving source masses.
  - Consider inspiral phase of black hole merger event.
  - Devise method for calculating gravitational waves.
- Further possible extensions and modifications:
  - Apply to formalism based on modified density $\rho^* = \rho\sqrt{-g}u^0$. [Will '18]
  - Allow for multiple dynamical metrics / tetrads.
  - Take into account Vainshtein or other screening mechanisms.
  - Consider cosmological background and time variability of PPN parameters.
  - Include further PPN potentials appearing in higher derivative theories.
- Apply formalism to complicated gravity theories:
  - Bimetric and multimetric gravity theories.
  - Multi-scalar Horndeski generalizations.
  - Theories involving generalized Proca fields.
  - Extensions based on metric-affine geometry.
  - Extensions of teleparallel and symmetric teleparallel gravity.

Literature

Further reading

MH,
"Gauge invariant approach to the parametrized post-Newtonian formalism",
arXiv:1910.09245 [gr-qc] (to appear in Phys. Rev. D).