The gauge-invariant parametrized post-Newtonian formalism

arXiv:1910.09245

Manuel Hohmann

Laboratory of Theoretical Physics - Institute of Physics - University of Tartu
Center of Excellence "The Dark Side of the Universe"

European Union
European Regional
Development Fund

Investing
in your future

Theoretical Physics Seminar, 22. October 2019

Outline

Outline

Motivation

- Open questions in gravity theory and cosmology:
  - Accelerating expansion of the universe.
  - Homogeneity of cosmic microwave background.
  - Unification with quantum theory and other fundamental forces.

Motivation

- Open questions in gravity theory and cosmology:
  - Accelerating expansion of the universe.
  - Homogeneity of cosmic microwave background.
  - Unification with quantum theory and other fundamental forces.
- Possible solutions addressing these questions:
  - Additional matter fields.
  - Modification of gravitational dynamics.
  - More general geometric structures to model gravity.

Motivation

- Open questions in gravity theory and cosmology:
  - Accelerating expansion of the universe.
  - Homogeneity of cosmic microwave background.
  - Unification with quantum theory and other fundamental forces.
- Possible solutions addressing these questions:
  - Additional matter fields.
  - Modification of gravitational dynamics.
  - More general geometric structures to model gravity.
- Things to (approximately) preserve in a modified theory:
  - Notions of observers, clocks, measurement rods.
  - Weak equivalence principle - universality of free fall.
  - Conservation of total energy-momentum.
  - Local position and Lorentz invariance.
  - Consistency with solar system observations.

Motivation

- Open questions in gravity theory and cosmology:
  - Accelerating expansion of the universe.
  - Homogeneity of cosmic microwave background.
  - Unification with quantum theory and other fundamental forces.
- Possible solutions addressing these questions:
  - Additional matter fields.
  - Modification of gravitational dynamics.
  - More general geometric structures to model gravity.
- Things to (approximately) preserve in a modified theory:
  - Notions of observers, clocks, measurement rods.
  - Weak equivalence principle - universality of free fall.
  - Conservation of total energy-momentum.
  - Local position and Lorentz invariance.
  - Consistency with solar system observations.

$\Rightarrow$ Conditions arising on viable theories:
  - Diffeomorphism invariance / background independence.
  - Well-defined causal structure.
  - Length of trajectories and geodesic motion.

Motivation

- Open questions in gravity theory and cosmology:
  - Accelerating expansion of the universe.
  - Homogeneity of cosmic microwave background.
  - Unification with quantum theory and other fundamental forces.
- Possible solutions addressing these questions:
  - Additional matter fields.
  - Modification of gravitational dynamics.
  - More general geometric structures to model gravity.
- Things to (approximately) preserve in a modified theory:
  - Notions of observers, clocks, measurement rods.
  - Weak equivalence principle - universality of free fall.
  - Conservation of total energy-momentum.
  - Local position and Lorentz invariance.
  - Consistency with solar system observations.

$\Rightarrow$ Conditions arising on viable theories:
  - Diffeomorphism invariance / background independence.
  - Well-defined causal structure.
  - Length of trajectories and geodesic motion.

$\Rightarrow$ Metric theories of gravity.

Idea

- Parametrized post-Newtonian formalism:
  - Weak-field approximation of metric gravity theories.
  - Assumes particular coordinate system (“universe rest frame”).
  - Characterizes gravity theories by 10 (constant) parameters.
  - Parameters closely related to solar system observations.

Idea

- Parametrized post-Newtonian formalism:
  - Weak-field approximation of metric gravity theories.
  - Assumes particular coordinate system (“universe rest frame”).
  - Characterizes gravity theories by 10 (constant) parameters.
  - Parameters closely related to solar system observations.

$\rightsquigarrow$ Extensions of the PPN formalism:
  - More fundamental fields constituting the metric.
  - More than one dynamical metric.
  - Diffeomorphism invariant / purely geometric formalism.

Idea

- Parametrized post-Newtonian formalism:
  - Weak-field approximation of metric gravity theories.
  - Assumes particular coordinate system ("universe rest frame").
  - Characterizes gravity theories by 10 (constant) parameters.
  - Parameters closely related to solar system observations.

$\leadsto$ Extensions of the PPN formalism:

  - More fundamental fields constituting the metric.
  - More than one dynamical metric.
  - Diffeomorphism invariant / purely geometric formalism.

$\leadsto$ Improvements presented here:

  - Use gauge-invariant higher order perturbation theory.
  - Allow for tetrad instead of metric as fundamental field.

Outline

Reference and physical spacetimes

- Reference spacetime:
  - Manifold M_0 with metric $g^{(0)}$ and coordinates (x^μ).
  - Usually some highly symmetric standard spacetime:
    - maximally symmetric spacetime: Minkowski, (anti-)de Sitter
    - cosmological (background) solution of a gravity theory

Reference and physical spacetimes

- Reference spacetime:
  - Manifold M_0 with metric $g^{(0)}$ and coordinates (x^μ).
  - Usually some highly symmetric standard spacetime:
    - maximally symmetric spacetime: Minkowski, (anti-)de Sitter
    - cosmological (background) solution of a gravity theory
- Physical spacetime:
  - Manifold M with metric g.
  - No preferred / canonical choice of coordinates.

Reference and physical spacetimes

- Reference spacetime:
  - Manifold M_0 with metric $g^{(0)}$ and coordinates (x^μ).
  - Usually some highly symmetric standard spacetime:
    - maximally symmetric spacetime: Minkowski, (anti-)de Sitter
    - cosmological (background) solution of a gravity theory
- Physical spacetime:
  - Manifold M with metric g.
  - No preferred / canonical choice of coordinates.

$\lightning$ No canonical relation between physical and reference spacetime.

Reference and physical spacetimes

- Reference spacetime:
  - Manifold M_0 with metric $g^{(0)}$ and coordinates (x^μ).
  - Usually some highly symmetric standard spacetime:
    - maximally symmetric spacetime: Minkowski, (anti-)de Sitter
    - cosmological (background) solution of a gravity theory
- Physical spacetime:
  - Manifold M with metric g.
  - No preferred / canonical choice of coordinates.

↯ No canonical relation between physical and reference spacetime.

1. No identification of points on M and M_0: no coordinates on M.

Reference and physical spacetimes

- Reference spacetime:
  - Manifold M_0 with metric $g^{(0)}$ and coordinates (x^μ).
  - Usually some highly symmetric standard spacetime:
    - maximally symmetric spacetime: Minkowski, (anti-)de Sitter
    - cosmological (background) solution of a gravity theory
- Physical spacetime:
  - Manifold M with metric g.
  - No preferred / canonical choice of coordinates.

↯ No canonical relation between physical and reference spacetime.

1. No identification of points on M and M_0: no coordinates on M.
2. No possibility to compare g and $g^{(0)}$: different manifolds.

Concept and use of gauge

Definition of gauge

A *gauge* is a diffeomorphism $\mathcal{X} : M_0 \to M$.

Concept and use of gauge

Definition of gauge

A *gauge* is a diffeomorphism $\mathcal{X} : M_0 \to M$.

1. Identification of points on M and M_0:
   - Recall that a diffeomorphism is a bijective mapping.
   - Coordinates ("point labels") can be carried from M_0 to M.

Definition of gauge

A *gauge* is a diffeomorphism $\mathcal{X} : M_0 \to M$.

1. Identification of points on M and M_0:
   - Recall that a diffeomorphism is a bijective mapping.
   - Coordinates ("point labels") can be carried from M_0 to M.
2. Comparison between reference and physical metric:
   - Define pullback ${}^{\mathcal{X}}g = \mathcal{X}^* g$ of the metric g to M_0.
   - ${}^{\mathcal{X}}g$ and $g^{(0)}$ are tensors on the same manifold M_0.

Gauge and perturbations

- Parameter dependent physical metric:
  - Assume physical metric $g \equiv g_\epsilon$ depends on parameter $\epsilon \in \mathbb{R}$.
  - Assume every g_ϵ is defined on its own M_ϵ.
  - Assume $g_0 = g^{(0)}$ is the reference metric on M_0.
  - $\Rightarrow$ Family of physical spacetimes (M_ϵ, g_ϵ).

Gauge and perturbations

- Parameter dependent physical metric:
  - Assume physical metric $g \equiv g_\epsilon$ depends on parameter $\epsilon \in \mathbb{R}$.
  - Assume every g_ϵ is defined on its own M_ϵ.
  - Assume $g_0 = g^{(0)}$ is the reference metric on M_0.

  $\Rightarrow$ Family of physical spacetimes (M_ϵ, g_ϵ).
- Aim: perturbative expansion of g_ϵ in ϵ around $g^{(0)}$.

Gauge and perturbations

- Parameter dependent physical metric:
  - Assume physical metric $g \equiv g_\epsilon$ depends on parameter $\epsilon \in \mathbb{R}$.
  - Assume every g_ϵ is defined on its own M_ϵ.
  - Assume $g_0 = g^{(0)}$ is the reference metric on M_0.

  $\Rightarrow$ Family of physical spacetimes (M_ϵ, g_ϵ).
- Aim: perturbative expansion of g_ϵ in ϵ around $g^{(0)}$.

$\lightning$ Metrics g_ϵ are defined on different manifolds for different ϵ.

Gauge and perturbations

- Parameter dependent physical metric:
  - Assume physical metric $g \equiv g_\epsilon$ depends on parameter $\epsilon \in \mathbb{R}$.
  - Assume every g_ϵ is defined on its own M_ϵ.
  - Assume $g_0 = g^{(0)}$ is the reference metric on M_0.
  - $\Rightarrow$ Family of physical spacetimes (M_ϵ, g_ϵ).
- Aim: perturbative expansion of g_ϵ in ϵ around $g^{(0)}$.

$\lightning$ Metrics g_ϵ are defined on different manifolds for different ϵ.

$\Rightarrow$ Use gauge to relate different manifolds:
  - Family of diffeomorphisms $\mathcal{X}_\epsilon : M_0 \to M_\epsilon$.
  - Assume $\mathcal{X}_0 = \mathrm{id}_{M_0}$ on the reference spacetime.

Gauge and perturbations

- Parameter dependent physical metric:
  - Assume physical metric $g \equiv g_\epsilon$ depends on parameter $\epsilon \in \mathbb{R}$.
  - Assume every g_ϵ is defined on its own M_ϵ.
  - Assume $g_0 = g^{(0)}$ is the reference metric on M_0.
  - ⇒ Family of physical spacetimes (M_ϵ, g_ϵ).
- Aim: perturbative expansion of g_ϵ in ϵ around $g^{(0)}$.
- ↯ Metrics g_ϵ are defined on different manifolds for different ϵ.
- ⇒ Use gauge to relate different manifolds:
  - Family of diffeomorphisms $\mathcal{X}_\epsilon : M_0 \to M_\epsilon$.
  - Assume $\mathcal{X}_0 = \mathrm{id}_{M_0}$ on the reference spacetime.
- Perturbative expansion:
  - Pullback ${}^{\mathcal{X}}g_\epsilon = \mathcal{X}_\epsilon^* g_\epsilon$ defined on M_0.

Gauge and perturbations

- Parameter dependent physical metric:
  - Assume physical metric $g \equiv g_\epsilon$ depends on parameter $\epsilon \in \mathbb{R}$.
  - Assume every g_ϵ is defined on its own M_ϵ.
  - Assume $g_0 = g^{(0)}$ is the reference metric on M_0.
  - $\Rightarrow$ Family of physical spacetimes (M_ϵ, g_ϵ).
- Aim: perturbative expansion of g_ϵ in ϵ around $g^{(0)}$.
- ↯ Metrics g_ϵ are defined on different manifolds for different ϵ.
- $\Rightarrow$ Use gauge to relate different manifolds:
  - Family of diffeomorphisms $\mathcal{X}_\epsilon : M_0 \to M_\epsilon$.
  - Assume $\mathcal{X}_0 = \mathrm{id}_{M_0}$ on the reference spacetime.
- Perturbative expansion:
  - Pullback ${}^{\mathcal{X}}g_\epsilon = \mathcal{X}_\epsilon^* g_\epsilon$ defined on M_0.
  - Introduce series expansion in ϵ:

$$ {}^{\mathcal{X}}g_\epsilon = \sum_{k=0}^{\infty} \frac{\epsilon^k}{k!} \left. \frac{\partial^k\, {}^{\mathcal{X}}g_\epsilon}{\partial \epsilon^k} \right|_{\epsilon=0} = \sum_{k=0}^{\infty} \frac{\epsilon^k}{k!}\, {}^{\mathcal{X}}g^{(k)} \,. $$

Gauge and perturbations

- Parameter dependent physical metric:
  - Assume physical metric $g \equiv g_\epsilon$ depends on parameter $\epsilon \in \mathbb{R}$.
  - Assume every g_ϵ is defined on its own M_ϵ.
  - Assume $g_0 = g^{(0)}$ is the reference metric on M_0.
  - ⇒ Family of physical spacetimes (M_ϵ, g_ϵ).
- Aim: perturbative expansion of g_ϵ in ϵ around $g^{(0)}$.
- ↯ Metrics g_ϵ are defined on different manifolds for different ϵ.
- ⇒ Use gauge to relate different manifolds:
  - Family of diffeomorphisms $\mathcal{X}_\epsilon : M_0 \to M_\epsilon$.
  - Assume $\mathcal{X}_0 = \mathrm{id}_{M_0}$ on the reference spacetime.
- Perturbative expansion:
  - Pullback ${}^{\mathcal{X}}g_\epsilon = \mathcal{X}_\epsilon^* g_\epsilon$ defined on M_0.
  - Introduce series expansion in ϵ:

$$
{}^{\mathcal{X}}g_\epsilon = \sum_{k=0}^{\infty} \frac{\epsilon^k}{k!} \left.\frac{\partial^k\, {}^{\mathcal{X}}g_\epsilon}{\partial \epsilon^k}\right|_{\epsilon=0} = \sum_{k=0}^{\infty} \frac{\epsilon^k}{k!}\, {}^{\mathcal{X}}g^{(k)}\,.
$$

  - Series coefficients ${}^{\mathcal{X}}g^{(k)}$ depend on gauge choice $\mathcal{X}$.

Gauge transformations

- Consider two different gauges $\mathcal{X}_\epsilon : M_0 \to M_\epsilon$ and $\mathcal{Y}_\epsilon : M_0 \to M_\epsilon$.

Gauge transformations

- Consider two different gauges $\mathcal{X}_\epsilon : M_0 \to M_\epsilon$ and $\mathcal{Y}_\epsilon : M_0 \to M_\epsilon$.

$\Rightarrow$ One-parameter family $\Phi_\epsilon = \mathcal{X}_\epsilon^{-1} \circ \mathcal{Y}_\epsilon$ of diffeomorphisms.

Gauge transformations

- Consider two different gauges $\mathcal{X}_\epsilon : M_0 \to M_\epsilon$ and $\mathcal{Y}_\epsilon : M_0 \to M_\epsilon$.

⇒ One-parameter family $\Phi_\epsilon = \mathcal{X}_\epsilon^{-1} \circ \mathcal{Y}_\epsilon$ of diffeomorphisms.

↯ $\Phi_\epsilon : M_0 \to M_0$ is not a one-parameter *group*:
  - In general has $\Phi_{\epsilon+\epsilon'} \neq \Phi_\epsilon \circ \Phi_{\epsilon'}$ and $\Phi_{-\epsilon} \neq \Phi_\epsilon^{-1}$.
  - Not generated by flow (integral curves) of a vector field.

Gauge transformations

- Consider two different gauges $\mathcal{X}_\epsilon : M_0 \to M_\epsilon$ and $\mathcal{Y}_\epsilon : M_0 \to M_\epsilon$.

⇒ One-parameter family $\Phi_\epsilon = \mathcal{X}_\epsilon^{-1} \circ \mathcal{Y}_\epsilon$ of diffeomorphisms.

↯ $\Phi_\epsilon : M_0 \to M_0$ is not a one-parameter *group*:
  - In general has $\Phi_{\epsilon+\epsilon'} \neq \Phi_\epsilon \circ \Phi_{\epsilon'}$ and $\Phi_{-\epsilon} \neq \Phi_\epsilon^{-1}$.
  - Not generated by flow (integral curves) of a vector field.

- But there exists series of one-parameter groups $\phi_\epsilon^{(k)}$ such that:

$$\Phi_\epsilon = \cdots \phi^{(k)}_{\epsilon^k/k!} \circ \cdots \circ \phi^{(2)}_{\epsilon^2/2} \circ \phi^{(1)}_\epsilon \,.$$

Gauge transformations

- Consider two different gauges $\mathcal{X}_\epsilon : M_0 \to M_\epsilon$ and $\mathcal{Y}_\epsilon : M_0 \to M_\epsilon$.

⇒ One-parameter family $\Phi_\epsilon = \mathcal{X}_\epsilon^{-1} \circ \mathcal{Y}_\epsilon$ of diffeomorphisms.

↯ $\Phi_\epsilon : M_0 \to M_0$ is not a one-parameter *group*:
  - In general has $\Phi_{\epsilon+\epsilon'} \neq \Phi_\epsilon \circ \Phi_{\epsilon'}$ and $\Phi_{-\epsilon} \neq \Phi_\epsilon^{-1}$.
  - Not generated by flow (integral curves) of a vector field.

- But there exists series of one-parameter groups $\phi_\epsilon^{(k)}$ such that:

$$\Phi_\epsilon = \cdots \phi^{(k)}_{\epsilon^k/k!} \circ \cdots \circ \phi^{(2)}_{\epsilon^2/2} \circ \phi^{(1)}_\epsilon \,.$$

⇒ Each one-parameter group $\phi_\epsilon^{(k)}$ generated by vector field $\xi^{(k)}$.

Gauge transformations

- Consider two different gauges $\mathcal{X}_\epsilon : M_0 \to M_\epsilon$ and $\mathcal{Y}_\epsilon : M_0 \to M_\epsilon$.

⇒ One-parameter family $\Phi_\epsilon = \mathcal{X}_\epsilon^{-1} \circ \mathcal{Y}_\epsilon$ of diffeomorphisms.

↯ $\Phi_\epsilon : M_0 \to M_0$ is not a one-parameter *group*:
  - In general has $\Phi_{\epsilon+\epsilon'} \neq \Phi_\epsilon \circ \Phi_{\epsilon'}$ and $\Phi_{-\epsilon} \neq \Phi_\epsilon^{-1}$.
  - Not generated by flow (integral curves) of a vector field.

- But there exists series of one-parameter groups $\phi_\epsilon^{(k)}$ such that:

$$\Phi_\epsilon = \cdots \phi^{(k)}_{\epsilon^k/k!} \circ \cdots \circ \phi^{(2)}_{\epsilon^2/2} \circ \phi^{(1)}_\epsilon \,.$$

⇒ Each one-parameter group $\phi_\epsilon^{(k)}$ generated by vector field $\xi^{(k)}$.

⇒ Vector fields $\xi^{(k)}$ are "Taylor expansion" coefficients of Φ_ϵ.

Gauge transformation of tensor fields

- Metrics in different gauges are related:

$$ {}^{\mathcal{Y}}g_{\epsilon} = \sum_{l_1=0}^{\infty} \cdots \sum_{l_k=0}^{\infty} \cdots \frac{\epsilon^{l_1+\cdots+kl_k+\cdots}}{(1!)^{l_1}\cdots(k!)^{l_k}\cdots l_1!\cdots l_k!\cdots} \mathfrak{L}^{l_1}_{\xi_{(1)}} \cdots \mathfrak{L}^{l_k}_{\xi_{(k)}} \cdots {}^{\mathcal{X}}g_{\epsilon} \,. $$

Gauge transformation of tensor fields

- Metrics in different gauges are related:

$$ {}^{\mathcal{Y}}g_\epsilon = \sum_{l_1=0}^{\infty}\cdots\sum_{l_k=0}^{\infty}\cdots\frac{\epsilon^{l_1+\cdots+kl_k+\cdots}}{(1!)^{l_1}\cdots(k!)^{l_k}\cdots l_1!\cdots l_k!\cdots}\mathfrak{L}_{\xi_{(1)}}^{l_1}\cdots\mathfrak{L}_{\xi_{(k)}}^{l_k}\cdots{}^{\mathcal{X}}g_\epsilon\,. $$

⇒ Relation of Taylor coefficients:

$$ {}^{\mathcal{Y}}g^{(k)} = \sum_{0\leq l_1+2l_2+\ldots\leq k}\frac{k!}{(k-l_1-2l_2-\ldots)!(1!)^{l_1}(2!)^{l_2}\cdots l_1!l_2!\cdots}\mathfrak{L}_{\xi_{(1)}}^{l_1}\cdots\mathfrak{L}_{\xi_{(k)}}^{l_k}\cdots{}^{\mathcal{X}}g^{(k-l_1-2l_2-\ldots)}\,. $$

Gauge invariant perturbations

- Choose a fixed "distinguished" gauge $\mathcal{S}_\epsilon : M_0 \to M_\epsilon$:
  - E.g., impose gauge conditions on the metric.
  - Examples: harmonic gauge used for GWs, standard PPN gauge.

Gauge invariant perturbations

- Choose a fixed "distinguished" gauge $\mathcal{S}_\epsilon : M_0 \to M_\epsilon$:
  - E.g., impose gauge conditions on the metric.
  - Examples: harmonic gauge used for GWs, standard PPN gauge.
- Define gauge-invariant metric $\mathbf{g}_\epsilon = {}^{\mathcal{S}}g_\epsilon = \mathcal{S}_\epsilon^* g_\epsilon$.

Gauge invariant perturbations

- Choose a fixed "distinguished" gauge $\mathcal{S}_\epsilon : M_0 \to M_\epsilon$:
  - E.g., impose gauge conditions on the metric.
  - Examples: harmonic gauge used for GWs, standard PPN gauge.
- Define gauge-invariant metric $\mathbf{g}_\epsilon = {}^{\mathcal{S}}g_\epsilon = \mathcal{S}_\epsilon^* g_\epsilon$.
- Metric in arbitrary gauge $\mathcal{X}$:

$$^{\mathcal{X}}g_\epsilon = \sum_{l_1=0}^{\infty} \cdots \sum_{l_k=0}^{\infty} \cdots \frac{\epsilon^{l_1+\cdots+kl_k+\cdots}}{(1!)^{l_1}\cdots(k!)^{l_k}\cdots l_1!\cdots l_k!\cdots} \pounds_{X_{(1)}}^{l_1} \cdots \pounds_{X_{(k)}}^{l_k} \cdots \mathbf{g}_\epsilon \,.$$

Gauge invariant perturbations

- Choose a fixed “distinguished” gauge $\mathcal{S}_\epsilon : M_0 \to M_\epsilon$:
  - E.g., impose gauge conditions on the metric.
  - Examples: harmonic gauge used for GWs, standard PPN gauge.
- Define gauge-invariant metric $\mathbf{g}_\epsilon = {}^{\mathcal{S}}g_\epsilon = \mathcal{S}_\epsilon^* g_\epsilon$.
- Metric in arbitrary gauge $\mathcal{X}$:

$$ {}^{\mathcal{X}}g_\epsilon = \sum_{l_1=0}^{\infty} \cdots \sum_{l_k=0}^{\infty} \cdots \frac{\epsilon^{l_1+\cdots+kl_k+\cdots}}{(1!)^{l_1}\cdots(k!)^{l_k}\cdots l_1!\cdots l_k!\cdots} \pounds_{X_{(1)}}^{l_1}\cdots\pounds_{X_{(k)}}^{l_k}\cdots\mathbf{g}_\epsilon\,. $$

- Metric components split into two parts:
  - Gauge-invariant part $\mathbf{g}_\epsilon$: physical content.
  - Gauge defining vector fields $X_{(k)}$: coordinate choice.

Gauge invariant perturbations

- Choose a fixed "distinguished" gauge $\mathcal{S}_\epsilon : M_0 \to M_\epsilon$:
  - E.g., impose gauge conditions on the metric.
  - Examples: harmonic gauge used for GWs, standard PPN gauge.
- Define gauge-invariant metric $\mathbf{g}_\epsilon = {}^{\mathcal{S}}g_\epsilon = \mathcal{S}_\epsilon^* g_\epsilon$.
- Metric in arbitrary gauge $\mathcal{X}$:

$$
{}^{\mathcal{X}}g_\epsilon = \sum_{l_1=0}^{\infty} \cdots \sum_{l_k=0}^{\infty} \cdots \frac{\epsilon^{l_1+\cdots+kl_k+\cdots}}{(1!)^{l_1}\cdots(k!)^{l_k}\cdots l_1!\cdots l_k!\cdots} \pounds_{X_{(1)}}^{l_1} \cdots \pounds_{X_{(k)}}^{l_k} \cdots \mathbf{g}_\epsilon \,.
$$

- Metric components split into two parts:
  - Gauge-invariant part $\mathbf{g}_\epsilon$: physical content.
  - Gauge defining vector fields $X_{(k)}$: coordinate choice.
- Number # of independent components:

$$
\#({}^{\mathcal{X}}g_\epsilon) = \#(\mathbf{g}_\epsilon) + \#(X_{(k)}) \,.
$$

Outline

Post-Newtonian matter and velocity orders

- Energy-momentum tensor of a perfect fluid:

$$T^{\mu\nu} = (\rho + \rho\Pi + p)u^{\mu}u^{\nu} + pg^{\mu\nu} .$$

  - Rest mass density ρ.
  - Specific internal energy Π.
  - Pressure p.
  - Four-velocity u^{μ}.

Post-Newtonian matter and velocity orders

- Energy-momentum tensor of a perfect fluid:

$$T^{\mu\nu} = (\rho + \rho\Pi + p)u^\mu u^\nu + pg^{\mu\nu} \,.$$

  - Rest mass density ρ.
  - Specific internal energy Π.
  - Pressure p.
  - Four-velocity u^μ.
- Universe rest frame and slow-moving source matter:
  - Consider some gauge $\mathcal{X} : M_0 \to M$ ("universe rest frame").
  - Pullback of metric and matter variables along $\mathcal{X}$.
  - Velocity of the source matter: ${}^{\mathcal{X}}v^i = {}^{\mathcal{X}}u^i / {}^{\mathcal{X}}u^0$.
  - Assume that source matter is slow-moving: $|{}^{\mathcal{X}}\vec{v}| \ll 1$.

Post-Newtonian matter and velocity orders

- Energy-momentum tensor of a perfect fluid:

$$T^{\mu\nu} = (\rho + \rho\Pi + p)u^\mu u^\nu + pg^{\mu\nu} .$$

  - Rest mass density ρ.
  - Specific internal energy Π.
  - Pressure p.
  - Four-velocity u^μ.
- Universe rest frame and slow-moving source matter:
  - Consider some gauge $\mathcal{X} : M_0 \to M$ ("universe rest frame").
  - Pullback of metric and matter variables along $\mathcal{X}$.
  - Velocity of the source matter: ${}^{\mathcal{X}}v^i = {}^{\mathcal{X}}u^i / {}^{\mathcal{X}}u^0$.
  - Assume that source matter is slow-moving: $|{}^{\mathcal{X}}\vec{v}| \ll 1$.
- Use $\epsilon = |{}^{\mathcal{X}}\vec{v}|$ as perturbation parameter.

Post-Newtonian matter and velocity orders

- Energy-momentum tensor of a perfect fluid:

$$T^{\mu\nu} = (\rho + \rho\Pi + p)u^{\mu}u^{\nu} + pg^{\mu\nu}\,.$$

  - Rest mass density $\rho \sim \mathcal{O}(2)$.
  - Specific internal energy $\Pi \sim \mathcal{O}(2)$.
  - Pressure $p \sim \mathcal{O}(4)$.
  - Four-velocity u^{μ}.
- Universe rest frame and slow-moving source matter:
  - Consider some gauge $\mathcal{X} : M_0 \to M$ ("universe rest frame").
  - Pullback of metric and matter variables along $\mathcal{X}$.
  - Velocity of the source matter: ${}^{\mathcal{X}}v^i = {}^{\mathcal{X}}u^i / {}^{\mathcal{X}}u^0$.
  - Assume that source matter is slow-moving: $|{}^{\mathcal{X}}\vec{v}| \ll 1$.
- Use $\epsilon = |{}^{\mathcal{X}}\vec{v}|$ as perturbation parameter.
- Assign velocity orders $\mathcal{O}(n) \sim \epsilon^n$ to all quantities.

Post-Newtonian matter and velocity orders

- Energy-momentum tensor of a perfect fluid:

$$T^{\mu\nu} = (\rho + \rho\Pi + p)u^\mu u^\nu + pg^{\mu\nu} .$$

  - Rest mass density $\rho \sim \mathcal{O}(2)$.
  - Specific internal energy $\Pi \sim \mathcal{O}(2)$.
  - Pressure $p \sim \mathcal{O}(4)$.
  - Four-velocity u^μ.
- Universe rest frame and slow-moving source matter:
  - Consider some gauge $\mathcal{X} : M_0 \to M$ ("universe rest frame").
  - Pullback of metric and matter variables along $\mathcal{X}$.
  - Velocity of the source matter: ${}^{\mathcal{X}}v^i = {}^{\mathcal{X}}u^i / {}^{\mathcal{X}}u^0$.
  - Assume that source matter is slow-moving: $|{}^{\mathcal{X}}\vec{v}| \ll 1$.
- Use $\epsilon = |{}^{\mathcal{X}}\vec{v}|$ as perturbation parameter.
- Assign velocity orders $\mathcal{O}(n) \sim \epsilon^n$ to all quantities.
- Quasi-static: assign additional $\mathcal{O}(1)$ to time derivatives ∂_0.

Post-Newtonian metric

- Standard post-Newtonian metric expansion:

$$\mathcal{X}g_{\mu\nu} = \mathcal{X}\overset{0}{g}_{\mu\nu} + \mathcal{X}\overset{1}{g}_{\mu\nu} + \mathcal{X}\overset{2}{g}_{\mu\nu} + \mathcal{X}\overset{3}{g}_{\mu\nu} + \mathcal{X}\overset{4}{g}_{\mu\nu} + \mathcal{O}(5)\,.$$

Post-Newtonian metric

- Standard post-Newtonian metric expansion:

$$^{\mathcal{X}}g_{\mu\nu} = {}^{\mathcal{X}}\overset{0}{g}_{\mu\nu} + {}^{\mathcal{X}}\overset{1}{g}_{\mu\nu} + {}^{\mathcal{X}}\overset{2}{g}_{\mu\nu} + {}^{\mathcal{X}}\overset{3}{g}_{\mu\nu} + {}^{\mathcal{X}}\overset{4}{g}_{\mu\nu} + \mathcal{O}(5)\,.$$

- Note difference in notation: ${}^{\mathcal{X}}\overset{k}{g} = {}^{\mathcal{X}}g^{(k)}\epsilon^k/k!$.

Post-Newtonian metric

- Standard post-Newtonian metric expansion:

$$^{\mathcal{X}}g_{\mu\nu} = {}^{\mathcal{X}}\overset{0}{g}_{\mu\nu} + {}^{\mathcal{X}}\overset{1}{g}_{\mu\nu} + {}^{\mathcal{X}}\overset{2}{g}_{\mu\nu} + {}^{\mathcal{X}}\overset{3}{g}_{\mu\nu} + {}^{\mathcal{X}}\overset{4}{g}_{\mu\nu} + \mathcal{O}(5)\,.$$

- Note difference in notation: ${}^{\mathcal{X}}\overset{k}{g} = {}^{\mathcal{X}}g^{(k)}\epsilon^k/k!$.
- Background metric given by Minkowski metric: ${}^{\mathcal{X}}\overset{0}{g}_{\mu\nu} = \eta_{\mu\nu}$.

Post-Newtonian metric

- Standard post-Newtonian metric expansion:

$$^{\mathcal{X}}g_{\mu\nu} = {}^{\mathcal{X}}\overset{0}{g}_{\mu\nu} + {}^{\mathcal{X}}\overset{1}{g}_{\mu\nu} + {}^{\mathcal{X}}\overset{2}{g}_{\mu\nu} + {}^{\mathcal{X}}\overset{3}{g}_{\mu\nu} + {}^{\mathcal{X}}\overset{4}{g}_{\mu\nu} + \mathcal{O}(5)\,.$$

- Note difference in notation: ${}^{\mathcal{X}}\overset{k}{g} = {}^{\mathcal{X}}g^{(k)}\epsilon^k/k!$.
- Background metric given by Minkowski metric: ${}^{\mathcal{X}}\overset{0}{g}_{\mu\nu} = \eta_{\mu\nu}$.
- Higher than fourth velocity order $\mathcal{O}(4)$ is not considered.

Post-Newtonian metric

- Standard post-Newtonian metric expansion:

$$ {}^{\mathcal{X}}g_{\mu\nu} = {}^{\mathcal{X}}\overset{0}{g}_{\mu\nu} + {}^{\mathcal{X}}\overset{1}{g}_{\mu\nu} + {}^{\mathcal{X}}\overset{2}{g}_{\mu\nu} + {}^{\mathcal{X}}\overset{3}{g}_{\mu\nu} + {}^{\mathcal{X}}\overset{4}{g}_{\mu\nu} + \mathcal{O}(5)\,. $$

- Note difference in notation: ${}^{\mathcal{X}}\overset{k}{g} = {}^{\mathcal{X}}g^{(k)}\epsilon^k/k!$.
- Background metric given by Minkowski metric: ${}^{\mathcal{X}}\overset{0}{g}_{\mu\nu} = \eta_{\mu\nu}$.
- Higher than fourth velocity order $\mathcal{O}(4)$ is not considered.
- Only certain components are relevant and non-vanishing:

$$ {}^{\mathcal{X}}\overset{2}{g}_{00}\,, \quad {}^{\mathcal{X}}\overset{2}{g}_{ij}\,, \quad {}^{\mathcal{X}}\overset{3}{g}_{0i}\,, \quad {}^{\mathcal{X}}\overset{4}{g}_{00}\,, \quad {}^{\mathcal{X}}\overset{4}{g}_{ij}\,. $$

Post-Newtonian metric

- Standard post-Newtonian metric expansion:

$$\mathcal{X}g_{\mu\nu} = \mathcal{X}\overset{0}{g}_{\mu\nu} + \mathcal{X}\overset{1}{g}_{\mu\nu} + \mathcal{X}\overset{2}{g}_{\mu\nu} + \mathcal{X}\overset{3}{g}_{\mu\nu} + \mathcal{X}\overset{4}{g}_{\mu\nu} + \mathcal{O}(5)\,.$$

- Note difference in notation: $\mathcal{X}\overset{k}{g} = \mathcal{X}g^{(k)}\epsilon^k/k!$.
- Background metric given by Minkowski metric: $\mathcal{X}\overset{0}{g}_{\mu\nu} = \eta_{\mu\nu}$.
- Higher than fourth velocity order $\mathcal{O}(4)$ is not considered.
- Only certain components are relevant and non-vanishing:

$$\mathcal{X}\overset{2}{g}_{00}\,,\quad \mathcal{X}\overset{2}{g}_{ij}\,,\quad \mathcal{X}\overset{3}{g}_{0i}\,,\quad \mathcal{X}\overset{4}{g}_{00}\,,\quad \mathcal{X}\overset{4}{g}_{ij}\,.$$

- $\mathcal{X}\overset{4}{g}_{ij}$ not used in standard PPN formalism, but may couple.

Standard post-Newtonian gauge

- PPN formalism assumes fixed standard gauge $\mathcal{P}$.

Standard post-Newtonian gauge

- PPN formalism assumes fixed standard gauge $\mathcal{P}$.
- Metric in standard PPN gauge:

$$
\begin{aligned}
{}^{\mathcal{P}}\overset{2}{g}_{00} &= 2\,{}^{\mathcal{P}}U\,, \\
{}^{\mathcal{P}}\overset{2}{g}_{ij} &= 2\gamma\,{}^{\mathcal{P}}U\delta_{ij}\,, \\
{}^{\mathcal{P}}\overset{3}{g}_{0i} &= -\frac{1}{2}(3 + 4\gamma + \alpha_1 - \alpha_2 + \zeta_1 - 2\xi)\,{}^{\mathcal{P}}V_i - \frac{1}{2}(1 + \alpha_2 - \zeta_1 + 2\xi)\,{}^{\mathcal{P}}W_i\,, \\
{}^{\mathcal{P}}\overset{4}{g}_{00} &= -2\beta\,{}^{\mathcal{P}}U^2 + (2 + 2\gamma + \alpha_3 + \zeta_1 - 2\xi)\,{}^{\mathcal{P}}\Phi_1 + 2(1 + 3\gamma - 2\beta + \zeta_2 + \xi)\,{}^{\mathcal{P}}\Phi_2 \\
&\quad + 2(1 + \zeta_3)\,{}^{\mathcal{P}}\Phi_3 + 2(3\gamma + 3\zeta_4 - 2\xi)\,{}^{\mathcal{P}}\Phi_4 - 2\xi\,{}^{\mathcal{P}}\Phi_W - (\zeta_1 - 2\xi)\,{}^{\mathcal{P}}\mathfrak{A}\,,
\end{aligned}
$$

Standard post-Newtonian gauge

- PPN formalism assumes fixed standard gauge $\mathcal{P}$.
- Metric in standard PPN gauge:

$$
\begin{aligned}
{}^{\mathcal{P}}\overset{2}{g}_{00} &= 2\,{}^{\mathcal{P}}U\,,\\
{}^{\mathcal{P}}\overset{2}{g}_{ij} &= 2\gamma\,{}^{\mathcal{P}}U\delta_{ij}\,,\\
{}^{\mathcal{P}}\overset{3}{g}_{0i} &= -\frac{1}{2}(3+4\gamma+\alpha_1-\alpha_2+\zeta_1-2\xi)\,{}^{\mathcal{P}}V_i - \frac{1}{2}(1+\alpha_2-\zeta_1+2\xi)\,{}^{\mathcal{P}}W_i\,,\\
{}^{\mathcal{P}}\overset{4}{g}_{00} &= -2\beta\,{}^{\mathcal{P}}U^2 + (2+2\gamma+\alpha_3+\zeta_1-2\xi)\,{}^{\mathcal{P}}\Phi_1 + 2(1+3\gamma-2\beta+\zeta_2+\xi)\,{}^{\mathcal{P}}\Phi_2\\
&\quad + 2(1+\zeta_3)\,{}^{\mathcal{P}}\Phi_3 + 2(3\gamma+3\zeta_4-2\xi)\,{}^{\mathcal{P}}\Phi_4 - 2\xi\,{}^{\mathcal{P}}\Phi_W - (\zeta_1-2\xi)\,{}^{\mathcal{P}}\mathfrak{A}\,,
\end{aligned}
$$

- Metric contains PPN parameters and PPN potentials.

Standard post-Newtonian gauge

- PPN formalism assumes fixed standard gauge $\mathcal{P}$.
- Metric in standard PPN gauge:

$$
\begin{aligned}
{}^{\mathcal{P}}\overset{2}{g}_{00} &= 2\,{}^{\mathcal{P}}U\,, \\
{}^{\mathcal{P}}\overset{2}{g}_{ij} &= 2\gamma\,{}^{\mathcal{P}}U\delta_{ij}\,, \\
{}^{\mathcal{P}}\overset{3}{g}_{0i} &= -\frac{1}{2}(3 + 4\gamma + \alpha_1 - \alpha_2 + \zeta_1 - 2\xi)\,{}^{\mathcal{P}}V_i - \frac{1}{2}(1 + \alpha_2 - \zeta_1 + 2\xi)\,{}^{\mathcal{P}}W_i\,, \\
{}^{\mathcal{P}}\overset{4}{g}_{00} &= -2\beta\,{}^{\mathcal{P}}U^2 + (2 + 2\gamma + \alpha_3 + \zeta_1 - 2\xi)\,{}^{\mathcal{P}}\Phi_1 + 2(1 + 3\gamma - 2\beta + \zeta_2 + \xi)\,{}^{\mathcal{P}}\Phi_2 \\
&\quad + 2(1 + \zeta_3)\,{}^{\mathcal{P}}\Phi_3 + 2(3\gamma + 3\zeta_4 - 2\xi)\,{}^{\mathcal{P}}\Phi_4 - 2\xi\,{}^{\mathcal{P}}\Phi_W - (\zeta_1 - 2\xi)\,{}^{\mathcal{P}}\mathfrak{A}\,,
\end{aligned}
$$

- Metric contains PPN parameters and PPN potentials.
- Properties of standard PPN metric:
  - Second-order spatial part ${}^{\mathcal{P}}\overset{2}{g}_{ij}$ is diagonal.

Standard post-Newtonian gauge

- PPN formalism assumes fixed standard gauge $\mathcal{P}$.
- Metric in standard PPN gauge:

$$
\begin{aligned}
{}^{\mathcal{P}}\overset{2}{g}_{00} &= 2\,{}^{\mathcal{P}}U\,,\\
{}^{\mathcal{P}}\overset{2}{g}_{ij} &= 2\gamma\,{}^{\mathcal{P}}U\delta_{ij}\,,\\
{}^{\mathcal{P}}\overset{3}{g}_{0i} &= -\frac{1}{2}(3+4\gamma+\alpha_1-\alpha_2+\zeta_1-2\xi)\,{}^{\mathcal{P}}V_i-\frac{1}{2}(1+\alpha_2-\zeta_1+2\xi)\,{}^{\mathcal{P}}W_i\,,\\
{}^{\mathcal{P}}\overset{4}{g}_{00} &= -2\beta\,{}^{\mathcal{P}}U^2+(2+2\gamma+\alpha_3+\zeta_1-2\xi)\,{}^{\mathcal{P}}\Phi_1+2(1+3\gamma-2\beta+\zeta_2+\xi)\,{}^{\mathcal{P}}\Phi_2\\
&\quad+2(1+\zeta_3)\,{}^{\mathcal{P}}\Phi_3+2(3\gamma+3\zeta_4-2\xi)\,{}^{\mathcal{P}}\Phi_4-2\xi\,{}^{\mathcal{P}}\Phi_W-(\zeta_1-2\xi)\,{}^{\mathcal{P}}\mathfrak{A}\,,
\end{aligned}
$$

- Metric contains PPN parameters and PPN potentials.
- Properties of standard PPN metric:
  - Second-order spatial part ${}^{\mathcal{P}}\overset{2}{g}_{ij}$ is diagonal.
  - Fourth-order temporal part ${}^{\mathcal{P}}\overset{4}{g}_{00}$ does not contain potential $\mathfrak{B}$.

PPN parameters

- PPN parameters are linked to physical properties:
  - γ: spatial curvature genrated by unit mass.
  - β: non-linearity in gravity superposition law.
  - $\alpha_1, \alpha_2, \alpha_3$: violation of local Lorentz invariance.
  - $\alpha_3, \zeta_1, \zeta_2, \zeta_3, \zeta_4$: violation of energy-momentum conservation.
  - ξ: violation of local position invariance.

PPN parameters

- PPN parameters are linked to physical properties:
  - γ: spatial curvature genrated by unit mass.
  - β: non-linearity in gravity superposition law.
  - $\alpha_1, \alpha_2, \alpha_3$: violation of local Lorentz invariance.
  - $\alpha_3, \zeta_1, \zeta_2, \zeta_3, \zeta_4$: violation of energy-momentum conservation.
  - ξ: violation of local position invariance.
- In GR $\gamma = \beta = 1$ and $\xi = \alpha_1 = \alpha_2 = \alpha_3 = \zeta_1 = \zeta_2 = \zeta_3 = \zeta_4 = 0$.

PPN parameters

- PPN parameters are linked to physical properties:
  - γ: spatial curvature genrated by unit mass.
  - β: non-linearity in gravity superposition law.
  - $\alpha_1, \alpha_2, \alpha_3$: violation of local Lorentz invariance.
  - $\alpha_3, \zeta_1, \zeta_2, \zeta_3, \zeta_4$: violation of energy-momentum conservation.
  - ξ: violation of local position invariance.
- In GR $\gamma = \beta = 1$ and $\xi = \alpha_1 = \alpha_2 = \alpha_3 = \zeta_1 = \zeta_2 = \zeta_3 = \zeta_4 = 0$.

$\Rightarrow$ Fully conservative gravity theory:
  - No preferred frame or preferred location effects.
  - Total energy-momentum is conserved.

PPN parameters

- PPN parameters are linked to physical properties:
  - γ: spatial curvature genrated by unit mass.
  - β: non-linearity in gravity superposition law.
  - $\alpha_1, \alpha_2, \alpha_3$: violation of local Lorentz invariance.
  - $\alpha_3, \zeta_1, \zeta_2, \zeta_3, \zeta_4$: violation of energy-momentum conservation.
  - ξ: violation of local position invariance.
- In GR $\gamma = \beta = 1$ and $\xi = \alpha_1 = \alpha_2 = \alpha_3 = \zeta_1 = \zeta_2 = \zeta_3 = \zeta_4 = 0$.

⇒ Fully conservative gravity theory:
  - No preferred frame or preferred location effects.
  - Total energy-momentum is conserved.

⇒ Other theories will receive bounds from experiments.

Experimental bounds

Par.	Bound	Effects	Experiment
$\gamma - 1$	$2.3 \cdot 10^{-5}$	Time delay, light deflection	Cassini tracking
$\beta - 1$	$8 \cdot 10^{-5}$	Perihelion shift	Perihelion shift
ξ	$4 \cdot 10^{-9}$	Spin precession	Millisecond pulsars
α_1	10^{-4}	Orbital polarization	Lunar laser ranging
α_1	$4 \cdot 10^{-5}$	Orbital polarization	PSR J1738+0333
α_2	$2 \cdot 10^{-9}$	Spin precession	Millisecond pulsars
α_3	$4 \cdot 10^{-20}$	Self-acceleration	Pulsar spin-down statistics
η_N[1]	$9 \cdot 10^{-4}$	Nordtvedt effect	Lunar Laser Ranging
ζ_1	0.02	Combined PPN bounds	—
ζ_2	$4 \cdot 10^{-5}$	Binary pulsar acceleration	PSR 1913+16
ζ_3	10^{-8}	Newton's 3rd law	Lunar acceleration
ζ_4	0.006	—	Kreuzer experiment

[1] $\eta_N = 4\beta - \gamma - 3 - \frac{10}{3}\xi - \alpha_1 + \frac{2}{3}\alpha_2 - \frac{2}{3}\zeta_1 - \frac{1}{3}\zeta_2$

PPN potentials

- Newtonian potential:

$$^{\mathcal{X}}\chi = -\int d^3x' \, {}^{\mathcal{X}}\rho' |\vec{x} - \vec{x}'| \,, \quad {}^{\mathcal{X}}U = \int d^3x' \frac{{}^{\mathcal{X}}\rho'}{|\vec{x} - \vec{x}'|} \,, \quad {}^{\mathcal{X}}\rho' \equiv {}^{\mathcal{X}}\rho(t, \vec{x}') \,.$$

PPN potentials

- Newtonian potential:

$$ {}^{\mathcal{X}}\chi = -\int \mathrm{d}^3x' \, {}^{\mathcal{X}}\rho' |\vec{x} - \vec{x}'| \,, \quad {}^{\mathcal{X}}U = \int \mathrm{d}^3x' \frac{{}^{\mathcal{X}}\rho'}{|\vec{x} - \vec{x}'|} \,, \quad {}^{\mathcal{X}}\rho' \equiv {}^{\mathcal{X}}\rho(t, \vec{x}') \,. $$

- Vector potentials:

$$ {}^{\mathcal{X}}V_i = \int d^3x' \frac{{}^{\mathcal{X}}\rho' \, {}^{\mathcal{X}}v'_i}{|\vec{x} - \vec{x}'|} \,, \quad {}^{\mathcal{X}}W_i = \int d^3x' \frac{{}^{\mathcal{X}}\rho' \, {}^{\mathcal{X}}v'_j (x_i - x'_i)(x_j - x'_j)}{|\vec{x} - \vec{x}'|^3} \,. $$

PPN potentials

- Newtonian potential:

$$\overset{\mathcal{X}}{\chi} = -\int \mathrm{d}^3x' \, \overset{\mathcal{X}}{\rho}{}' |\vec{x} - \vec{x}'| \,, \quad \overset{\mathcal{X}}{U} = \int \mathrm{d}^3x' \frac{\overset{\mathcal{X}}{\rho}{}'}{|\vec{x} - \vec{x}'|} \,, \quad \overset{\mathcal{X}}{\rho}{}' \equiv \overset{\mathcal{X}}{\rho}(t, \vec{x}') \,.$$

- Vector potentials:

$$\overset{\mathcal{X}}{V}_i = \int d^3x' \frac{\overset{\mathcal{X}}{\rho}{}' \overset{\mathcal{X}}{v}{}'_i}{|\vec{x} - \vec{x}'|} \,, \quad \overset{\mathcal{X}}{W}_i = \int d^3x' \frac{\overset{\mathcal{X}}{\rho}{}' \overset{\mathcal{X}}{v}{}'_j (x_i - x'_i)(x_j - x'_j)}{|\vec{x} - \vec{x}'|^3} \,.$$

- Fourth-order scalar potentials:

$$\overset{\mathcal{X}}{\Phi}_1 = \int d^3x' \frac{\overset{\mathcal{X}}{\rho}{}' \overset{\mathcal{X}}{v}{}'^2}{|\vec{x} - \vec{x}'|} \,, \quad \overset{\mathcal{X}}{\Phi}_4 = \int d^3x' \frac{\overset{\mathcal{X}}{p}{}'}{|\vec{x} - \vec{x}'|} \,,$$

$$\overset{\mathcal{X}}{\Phi}_2 = \int d^3x' \frac{\overset{\mathcal{X}}{\rho}{}' \overset{\mathcal{X}}{U}{}'}{|\vec{x} - \vec{x}'|} \,, \quad \overset{\mathcal{X}}{\mathfrak{A}} = \int d^3x' \frac{\overset{\mathcal{X}}{\rho}{}' \left[\overset{\mathcal{X}}{v}{}'_i (x_i - x'_i)\right]^2}{|\vec{x} - \vec{x}'|^3} \,,$$

$$\overset{\mathcal{X}}{\Phi}_3 = \int d^3x' \frac{\overset{\mathcal{X}}{\rho}{}' \overset{\mathcal{X}}{\Pi}{}'}{|\vec{x} - \vec{x}'|} \,, \quad \overset{\mathcal{X}}{\mathfrak{B}} = \int d^3x' \frac{\overset{\mathcal{X}}{\rho}{}'}{|\vec{x} - \vec{x}'|} (x_i - x'_i) \frac{d \overset{\mathcal{X}}{v}{}'_i}{dt} \,,$$

$$\overset{\mathcal{X}}{\Phi}_W = \int d^3x' d^3x'' \, \overset{\mathcal{X}}{\rho}{}' \overset{\mathcal{X}}{\rho}{}'' \frac{x_i - x'_i}{|\vec{x} - \vec{x}'|^3} \left(\frac{x'_i - x''_i}{|\vec{x} - \vec{x}''|} - \frac{x_i - x''_i}{|\vec{x}' - \vec{x}''|} \right) .$$

Post-Newtonian field equations

- Expand energy-momentum tensor in velocity orders:

$$\begin{aligned} {}^{\mathcal{X}}T_{00} &= {}^{\mathcal{X}}\rho\left(1 - {}^{\mathcal{X}}\overset{2}{g}_{00} + ({}^{\mathcal{X}}v)^2 + {}^{\mathcal{X}}\Pi\right) + \mathcal{O}(6)\,, \\ {}^{\mathcal{X}}T_{0i} &= -{}^{\mathcal{X}}\rho\,{}^{\mathcal{X}}v_i + \mathcal{O}(5)\,, \\ {}^{\mathcal{X}}T_{ij} &= {}^{\mathcal{X}}\rho\,{}^{\mathcal{X}}v_i\,{}^{\mathcal{X}}v_j + {}^{\mathcal{X}}p\delta_{ij} + \mathcal{O}(6) \end{aligned}$$

Post-Newtonian field equations

- Expand energy-momentum tensor in velocity orders:

$$
\begin{aligned}
{}^{\mathcal{X}}T_{00} &= {}^{\mathcal{X}}\rho\left(1 - {}^{\mathcal{X}}\overset{2}{g}_{00} + ({}^{\mathcal{X}}v)^2 + {}^{\mathcal{X}}\Pi\right) + \mathcal{O}(6)\,,\\
{}^{\mathcal{X}}T_{0i} &= -{}^{\mathcal{X}}\rho\,{}^{\mathcal{X}}v_i + \mathcal{O}(5)\,,\\
{}^{\mathcal{X}}T_{ij} &= {}^{\mathcal{X}}\rho\,{}^{\mathcal{X}}v_i\,{}^{\mathcal{X}}v_j + {}^{\mathcal{X}}p\delta_{ij} + \mathcal{O}(6)
\end{aligned}
$$

- Behavior under gauge transformations:

$$
{}^{\mathcal{Y}}\overset{2}{T}_{00} = {}^{\mathcal{X}}\overset{2}{T}_{00}\,, \quad {}^{\mathcal{Y}}\overset{2}{T}_{ij} = {}^{\mathcal{X}}\overset{2}{T}_{ij}\,, \quad {}^{\mathcal{Y}}\overset{3}{T}_{0i} = {}^{\mathcal{X}}\overset{3}{T}_{0i}\,,
$$

$$
{}^{\mathcal{Y}}\overset{4}{T}_{00} = {}^{\mathcal{X}}\overset{4}{T}_{00} + \overset{2}{\xi}_i\partial_i\,{}^{\mathcal{X}}\overset{2}{T}_{00}\,, \quad {}^{\mathcal{Y}}\overset{4}{T}_{ij} = {}^{\mathcal{X}}\overset{4}{T}_{ij}\,.
$$

Post-Newtonian field equations

- Expand energy-momentum tensor in velocity orders:

$$
\begin{aligned}
{}^{\mathcal{X}}T_{00} &= {}^{\mathcal{X}}\rho\left(1 - {}^{\mathcal{X}}\overset{2}{g}_{00} + ({}^{\mathcal{X}}v)^2 + {}^{\mathcal{X}}\Pi\right) + \mathcal{O}(6)\,,\\
{}^{\mathcal{X}}T_{0i} &= -{}^{\mathcal{X}}\rho\,{}^{\mathcal{X}}v_i + \mathcal{O}(5)\,,\\
{}^{\mathcal{X}}T_{ij} &= {}^{\mathcal{X}}\rho\,{}^{\mathcal{X}}v_i\,{}^{\mathcal{X}}v_j + {}^{\mathcal{X}}p\delta_{ij} + \mathcal{O}(6)
\end{aligned}
$$

- Behavior under gauge transformations:

$$
{}^{\mathcal{Y}}\overset{2}{T}_{00} = {}^{\mathcal{X}}\overset{2}{T}_{00}\,, \quad {}^{\mathcal{Y}}\overset{2}{T}_{ij} = {}^{\mathcal{X}}\overset{2}{T}_{ij}\,, \quad {}^{\mathcal{Y}}\overset{3}{T}_{0i} = {}^{\mathcal{X}}\overset{3}{T}_{0i}\,,
$$

$$
{}^{\mathcal{Y}}\overset{4}{T}_{00} = {}^{\mathcal{X}}\overset{4}{T}_{00} + \overset{2}{\xi}_i\partial_i\,{}^{\mathcal{X}}\overset{2}{T}_{00}\,, \quad {}^{\mathcal{Y}}\overset{4}{T}_{ij} = {}^{\mathcal{X}}\overset{4}{T}_{ij}\,.
$$

- Energy-momentum tensor $\sim$ derivatives of PPN potentials.

Post-Newtonian field equations

- Expand energy-momentum tensor in velocity orders:

$$
\begin{aligned}
{}^{\mathcal{X}}T_{00} &= {}^{\mathcal{X}}\rho\left(1 - {}^{\mathcal{X}}\overset{2}{g}_{00} + ({}^{\mathcal{X}}v)^2 + {}^{\mathcal{X}}\Pi\right) + \mathcal{O}(6)\,,\\
{}^{\mathcal{X}}T_{0i} &= -{}^{\mathcal{X}}\rho\,{}^{\mathcal{X}}v_i + \mathcal{O}(5)\,,\\
{}^{\mathcal{X}}T_{ij} &= {}^{\mathcal{X}}\rho\,{}^{\mathcal{X}}v_i\,{}^{\mathcal{X}}v_j + {}^{\mathcal{X}}p\delta_{ij} + \mathcal{O}(6)
\end{aligned}
$$

- Behavior under gauge transformations:

$$
{}^{\mathcal{Y}}\overset{2}{T}_{00} = {}^{\mathcal{X}}\overset{2}{T}_{00}\,, \quad {}^{\mathcal{Y}}\overset{2}{T}_{ij} = {}^{\mathcal{X}}\overset{2}{T}_{ij}\,, \quad {}^{\mathcal{Y}}\overset{3}{T}_{0i} = {}^{\mathcal{X}}\overset{3}{T}_{0i}\,,
$$

$$
{}^{\mathcal{Y}}\overset{4}{T}_{00} = {}^{\mathcal{X}}\overset{4}{T}_{00} + \overset{2}{\xi}_i\partial_i\,{}^{\mathcal{X}}\overset{2}{T}_{00}\,, \quad {}^{\mathcal{Y}}\overset{4}{T}_{ij} = {}^{\mathcal{X}}\overset{4}{T}_{ij}\,.
$$

- Energy-momentum tensor ~ derivatives of PPN potentials.

⇒ Solve for PPN parameters by PPN expanding field equations.

Post-Newtonian field equations

- Expand energy-momentum tensor in velocity orders:

$$
\begin{aligned}
{}^{\mathcal{X}}T_{00} &= {}^{\mathcal{X}}\rho\left(1 - {}^{\mathcal{X}}\overset{2}{g}_{00} + ({}^{\mathcal{X}}v)^2 + {}^{\mathcal{X}}\Pi\right) + \mathcal{O}(6)\,,\\
{}^{\mathcal{X}}T_{0i} &= -{}^{\mathcal{X}}\rho\,{}^{\mathcal{X}}v_i + \mathcal{O}(5)\,,\\
{}^{\mathcal{X}}T_{ij} &= {}^{\mathcal{X}}\rho\,{}^{\mathcal{X}}v_i\,{}^{\mathcal{X}}v_j + {}^{\mathcal{X}}p\delta_{ij} + \mathcal{O}(6)
\end{aligned}
$$

- Behavior under gauge transformations:

$$
{}^{\mathcal{Y}}\overset{2}{T}_{00} = {}^{\mathcal{X}}\overset{2}{T}_{00}\,,\quad {}^{\mathcal{Y}}\overset{2}{T}_{ij} = {}^{\mathcal{X}}\overset{2}{T}_{ij}\,,\quad {}^{\mathcal{Y}}\overset{3}{T}_{0i} = {}^{\mathcal{X}}\overset{3}{T}_{0i}\,,
$$

$$
{}^{\mathcal{Y}}\overset{4}{T}_{00} = {}^{\mathcal{X}}\overset{4}{T}_{00} + \overset{2}{\xi}_i\partial_i\,{}^{\mathcal{X}}\overset{2}{T}_{00}\,,\quad {}^{\mathcal{Y}}\overset{4}{T}_{ij} = {}^{\mathcal{X}}\overset{4}{T}_{ij}\,.
$$

- Energy-momentum tensor ~ derivatives of PPN potentials.

⇒ Solve for PPN parameters by PPN expanding field equations.

↯ Equations may be gauge dependent & hard to solve.

Post-Newtonian field equations

- Expand energy-momentum tensor in velocity orders:

$$
\begin{aligned}
{}^{\mathcal{X}}T_{00} &= {}^{\mathcal{X}}\rho\left(1 - {}^{\mathcal{X}}\overset{2}{g}_{00} + ({}^{\mathcal{X}}v)^2 + {}^{\mathcal{X}}\Pi\right) + \mathcal{O}(6)\,,\\
{}^{\mathcal{X}}T_{0i} &= -{}^{\mathcal{X}}\rho\,{}^{\mathcal{X}}v_i + \mathcal{O}(5)\,,\\
{}^{\mathcal{X}}T_{ij} &= {}^{\mathcal{X}}\rho\,{}^{\mathcal{X}}v_i\,{}^{\mathcal{X}}v_j + {}^{\mathcal{X}}p\delta_{ij} + \mathcal{O}(6)
\end{aligned}
$$

- Behavior under gauge transformations:

$$
{}^{\mathcal{Y}}\overset{2}{T}_{00} = {}^{\mathcal{X}}\overset{2}{T}_{00}\,, \quad {}^{\mathcal{Y}}\overset{2}{T}_{ij} = {}^{\mathcal{X}}\overset{2}{T}_{ij}\,, \quad {}^{\mathcal{Y}}\overset{3}{T}_{0i} = {}^{\mathcal{X}}\overset{3}{T}_{0i}\,,
$$

$$
{}^{\mathcal{Y}}\overset{4}{T}_{00} = {}^{\mathcal{X}}\overset{4}{T}_{00} + \overset{2}{\xi}_i\partial_i\,{}^{\mathcal{X}}\overset{2}{T}_{00}\,, \quad {}^{\mathcal{Y}}\overset{4}{T}_{ij} = {}^{\mathcal{X}}\overset{4}{T}_{ij}\,.
$$

- Energy-momentum tensor ~ derivatives of PPN potentials.

⇒ Solve for PPN parameters by PPN expanding field equations.

↯ Equations may be gauge dependent & hard to solve.

⇝ Use gauge-invariant formalism to decouple equations.

Outline

Gauge transformation of the metric

- Allow only gauge transformations preserving PPN assumptions:

$$\overset{2}{\xi}_i\,, \quad \overset{3}{\xi}_0\,, \quad \overset{4}{\xi}_i\,.$$

Gauge transformation of the metric

- Allow only gauge transformations preserving PPN assumptions:

$$\overset{2}{\xi}_i\,, \quad \overset{3}{\xi}_0\,, \quad \overset{4}{\xi}_i\,.$$

- Relation between metrics in gauges $\mathcal{X}$ and $\mathcal{Y}$:

$$\begin{aligned}
{}^{\mathcal{Y}}\overset{2}{g}_{00} &= {}^{\mathcal{X}}\overset{2}{g}_{00}\,,\\
{}^{\mathcal{Y}}\overset{2}{g}_{ij} &= {}^{\mathcal{X}}\overset{2}{g}_{ij} + 2\partial_{(i}\overset{2}{\xi}_{j)}\,,\\
{}^{\mathcal{Y}}\overset{3}{g}_{0i} &= {}^{\mathcal{X}}\overset{3}{g}_{0i} + \partial_i\overset{3}{\xi}_0 + \partial_0\overset{2}{\xi}_i\,,\\
{}^{\mathcal{Y}}\overset{4}{g}_{00} &= {}^{\mathcal{X}}\overset{4}{g}_{00} + 2\partial_0\overset{3}{\xi}_0 + \overset{2}{\xi}_i\partial_i\,{}^{\mathcal{X}}\overset{2}{g}_{00}\,,\\
{}^{\mathcal{Y}}\overset{4}{g}_{ij} &= {}^{\mathcal{X}}\overset{4}{g}_{ij} + 2\partial_{(i}\overset{4}{\xi}_{j)} + 2\,{}^{\mathcal{X}}\overset{2}{g}_{k(i}\partial_{j)}\overset{2}{\xi}_k + \overset{2}{\xi}_k\partial_k\,{}^{\mathcal{X}}\overset{2}{g}_{ij} + \partial_{(i}\big(\overset{2}{\xi}_{|k}\partial_{k|}\overset{2}{\xi}_{j)}\big) + \partial_i\overset{2}{\xi}_k\partial_j\overset{2}{\xi}_k\,.
\end{aligned}$$

Gauge transformation of the metric

- Allow only gauge transformations preserving PPN assumptions:

$$\overset{2}{\xi}_i\,, \quad \overset{3}{\xi}_0\,, \quad \overset{4}{\xi}_i\,.$$

- Relation between metrics in gauges $\mathcal{X}$ and $\mathcal{Y}$:

$$\begin{aligned}
{}^{\mathcal{Y}}\overset{2}{g}_{00} &= {}^{\mathcal{X}}\overset{2}{g}_{00}\,,\\
{}^{\mathcal{Y}}\overset{2}{g}_{ij} &= {}^{\mathcal{X}}\overset{2}{g}_{ij} + 2\partial_{(i}\overset{2}{\xi}_{j)}\,,\\
{}^{\mathcal{Y}}\overset{3}{g}_{0i} &= {}^{\mathcal{X}}\overset{3}{g}_{0i} + \partial_i\overset{3}{\xi}_0 + \partial_0\overset{2}{\xi}_i\,,\\
{}^{\mathcal{Y}}\overset{4}{g}_{00} &= {}^{\mathcal{X}}\overset{4}{g}_{00} + 2\partial_0\overset{3}{\xi}_0 + \overset{2}{\xi}_i\partial_i\,{}^{\mathcal{X}}\overset{2}{g}_{00}\,,\\
{}^{\mathcal{Y}}\overset{4}{g}_{ij} &= {}^{\mathcal{X}}\overset{4}{g}_{ij} + 2\partial_{(i}\overset{4}{\xi}_{j)} + 2\,{}^{\mathcal{X}}\overset{2}{g}_{k(i}\partial_{j)}\overset{2}{\xi}_k + \overset{2}{\xi}_k\partial_k\,{}^{\mathcal{X}}\overset{2}{g}_{ij} + \partial_{(i}\big(\overset{2}{\xi}_{|k}\partial_{k|}\overset{2}{\xi}_{j)}\big) + \partial_i\overset{2}{\xi}_k\partial_j\overset{2}{\xi}_k\,.
\end{aligned}$$

- Use gauge transformation to eliminate metric components.

Gauge-invariant metric

- Definition of gauge-invariant metric components:

$$\mathbf{g}_{00} = \mathbf{g}^{\star}\,, \quad \mathbf{g}_{0i} = \mathbf{g}^{\diamond}_{i}\,, \quad \mathbf{g}_{ij} = \mathbf{g}^{\bullet}\delta_{ij} + \mathbf{g}^{\dagger}_{ij}\,.$$

Gauge-invariant metric

- Definition of gauge-invariant metric components:

$$\mathbf{g}_{00} = \mathbf{g}^{\star}\,, \quad \mathbf{g}_{0i} = \mathbf{g}^{\diamond}_{i}\,, \quad \mathbf{g}_{ij} = \mathbf{g}^{\bullet}\delta_{ij} + \mathbf{g}^{\dagger}_{ij}\,.$$

- Conditions imposed on components:

$$\partial^i \mathbf{g}^{\diamond}_{i} = 0\,, \quad \partial^i \mathbf{g}^{\dagger}_{ij} = 0\,, \quad \mathbf{g}^{\dagger}_{[ij]} = 0\,, \quad \mathbf{g}^{\dagger}_{ii} = 0\,.$$

Gauge-invariant metric

- Definition of gauge-invariant metric components:

$$\mathbf{g}_{00} = \mathbf{g}^{\star}\,, \quad \mathbf{g}_{0i} = \mathbf{g}^{\diamond}_{i}\,, \quad \mathbf{g}_{ij} = \mathbf{g}^{\bullet}\delta_{ij} + \mathbf{g}^{\dagger}_{ij}\,.$$

- Conditions imposed on components:

$$\partial^i\mathbf{g}^{\diamond}_{i} = 0\,, \quad \partial^i\mathbf{g}^{\dagger}_{ij} = 0\,, \quad \mathbf{g}^{\dagger}_{[ij]} = 0\,, \quad \mathbf{g}^{\dagger}_{ii} = 0\,.$$

- Relation to arbitrary gauge $\mathcal{X}$:

$$\begin{aligned}
{}^{\mathcal{X}}\overset{2}{g}_{00} &= \overset{2}{\mathbf{g}}{}^{\star}\,,\\
{}^{\mathcal{X}}\overset{2}{g}_{ij} &= \overset{2}{\mathbf{g}}{}^{\bullet}\delta_{ij} + \overset{2}{\mathbf{g}}{}^{\dagger}_{ij} + 2\partial_i\partial_j\overset{2}{X}{}^{\blacklozenge} + 2\partial_{(i}\overset{2}{X}{}^{\diamond}_{j)}\,,\\
{}^{\mathcal{X}}\overset{3}{g}_{0i} &= \overset{3}{\mathbf{g}}{}^{\diamond}_{i} + \partial_i\overset{3}{X}{}^{\star} + \partial_0\partial_i\overset{2}{X}{}^{\blacklozenge} + \partial_0\overset{2}{X}{}^{\diamond}_{i}\,,\\
{}^{\mathcal{X}}\overset{4}{g}_{00} &= \overset{4}{\mathbf{g}}{}^{\star} + 2\partial_0\overset{3}{X}{}^{\star} + (\partial_i\overset{2}{X}{}^{\blacklozenge} + \overset{2}{X}{}^{\diamond}_{i})\partial_i\overset{2}{\mathbf{g}}{}^{\star}\,,\\
{}^{\mathcal{X}}\overset{4}{g}_{ij} &= \overset{4}{\mathbf{g}}{}^{\bullet}\delta_{ij} + \overset{4}{\mathbf{g}}{}^{\dagger}_{ij} + 2\partial_i\partial_j\overset{4}{X}{}^{\blacklozenge} + \mathcal{O}(2)\cdot\mathcal{O}(2)\,.
\end{aligned}$$

Gauge-invariant metric

- Definition of gauge-invariant metric components:

$$\mathbf{g}_{00} = \mathbf{g}^{\star}\,, \quad \mathbf{g}_{0i} = \mathbf{g}^{\diamond}_{i}\,, \quad \mathbf{g}_{ij} = \mathbf{g}^{\bullet}\delta_{ij} + \mathbf{g}^{\dagger}_{ij}\,.$$

- Conditions imposed on components:

$$\partial^i\mathbf{g}^{\diamond}_{i} = 0\,, \quad \partial^i\mathbf{g}^{\dagger}_{ij} = 0\,, \quad \mathbf{g}^{\dagger}_{[ij]} = 0\,, \quad \mathbf{g}^{\dagger}_{ii} = 0\,.$$

- Relation to arbitrary gauge $\mathcal{X}$:

$$\begin{aligned}
{}^{\mathcal{X}}\overset{2}{g}_{00} &= \overset{2}{\mathbf{g}}{}^{\star}\,,\\
{}^{\mathcal{X}}\overset{2}{g}_{ij} &= \overset{2}{\mathbf{g}}{}^{\bullet}\delta_{ij} + \overset{2}{\mathbf{g}}{}^{\dagger}_{ij} + 2\partial_i\partial_j\overset{2}{X}{}^{\blacklozenge} + 2\partial_{(i}\overset{2}{X}{}^{\diamond}_{j)}\,,\\
{}^{\mathcal{X}}\overset{3}{g}_{0i} &= \overset{3}{\mathbf{g}}{}^{\diamond}_{i} + \partial_i\overset{3}{X}{}^{\star} + \partial_0\partial_i\overset{2}{X}{}^{\blacklozenge} + \partial_0\overset{2}{X}{}^{\diamond}_{i}\,,\\
{}^{\mathcal{X}}\overset{4}{g}_{00} &= \overset{4}{\mathbf{g}}{}^{\star} + 2\partial_0\overset{3}{X}{}^{\star} + (\partial_i\overset{2}{X}{}^{\blacklozenge} + \overset{2}{X}{}^{\diamond}_{i})\partial_i\overset{2}{\mathbf{g}}{}^{\star}\,,\\
{}^{\mathcal{X}}\overset{4}{g}_{ij} &= \overset{4}{\mathbf{g}}{}^{\bullet}\delta_{ij} + \overset{4}{\mathbf{g}}{}^{\dagger}_{ij} + 2\partial_i\partial_j\overset{4}{X}{}^{\blacklozenge} + \mathcal{O}(2)\cdot\mathcal{O}(2)\,.
\end{aligned}$$

- Gauge defining vector fields:

$$X_i = \partial_i X^{\blacklozenge} + X^{\diamond}_{i}\,, \quad X_0 = X^{\star}\,, \quad \partial^i X^{\diamond}_{i} = 0\,.$$

Decomposition of metric components

- Count number of independent components at each order:

total		invariant		pure gauge	
$\mathcal{X}\overset{2}{g}_{00}$	1	$\overset{2}{\mathbf{g}}^{\star}$	1	-	0
$\mathcal{X}\overset{2}{g}_{ij}$	6	$\overset{2}{\mathbf{g}}^{\star}, \overset{2}{\mathbf{g}}^{\dagger}_{ij}$	$1+2$	$\overset{2}{X}^{\blacklozenge}, \overset{2}{X}^{\diamond}_{i}$	$1+2$
$\mathcal{X}\overset{3}{g}_{0i}$	3	$\overset{3}{\mathbf{g}}^{\diamond}_{i}$	2	$\overset{3}{X}^{\star}$	1
$\mathcal{X}\overset{4}{g}_{00}$	1	$\overset{4}{\mathbf{g}}^{\star}$	1	-	0
$\mathcal{X}\overset{4}{g}_{ij}$	6	$\overset{4}{\mathbf{g}}^{\star}, \overset{4}{\mathbf{g}}^{\dagger}_{ij}$	$1+2$	$\overset{4}{X}^{\blacklozenge}, \overset{4}{X}^{\diamond}_{i}$	$1+2$

Decomposition of metric components

- Count number of independent components at each order:

total		invariant		pure gauge	
$\mathcal{X}\overset{2}{g}_{00}$	1	$\overset{2}{\mathbf{g}}^{\star}$	1	-	0
$\mathcal{X}\overset{2}{g}_{ij}$	6	$\overset{2}{\mathbf{g}}^{\star}, \overset{2}{\mathbf{g}}^{\dagger}_{ij}$	$1+2$	$\overset{2}{X}^{\blacklozenge}, \overset{2}{X}^{\diamond}_{i}$	$1+2$
$\mathcal{X}\overset{3}{g}_{0i}$	3	$\overset{3}{\mathbf{g}}^{\diamond}_{i}$	2	$\overset{3}{X}^{\star}$	1
$\mathcal{X}\overset{4}{g}_{00}$	1	$\overset{4}{\mathbf{g}}^{\star}$	1	-	0
$\mathcal{X}\overset{4}{g}_{ij}$	6	$\overset{4}{\mathbf{g}}^{\star}, \overset{4}{\mathbf{g}}^{\dagger}_{ij}$	$1+2$	$\overset{4}{X}^{\blacklozenge}, \overset{4}{X}^{\diamond}_{i}$	$1+2$

$\Rightarrow$ Components split into invariant and gauge parts.

Decomposition of metric components

- Count number of independent components at each order:

total		invariant		pure gauge	
$\mathcal{X}\overset{2}{g}_{00}$	1	$\overset{2}{\mathbf{g}}{}^{\star}$	1	-	0
$\mathcal{X}\overset{2}{g}_{ij}$	6	$\overset{2}{\mathbf{g}}{}^{\star}, \overset{2}{\mathbf{g}}{}^{\dagger}_{ij}$	$1+2$	$\overset{2}{X}{}^{\blacklozenge}, \overset{2}{X}{}^{\diamond}_{i}$	$1+2$
$\mathcal{X}\overset{3}{g}_{0i}$	3	$\overset{3}{\mathbf{g}}{}^{\diamond}_{i}$	2	$\overset{3}{X}{}^{\star}$	1
$\mathcal{X}\overset{4}{g}_{00}$	1	$\overset{4}{\mathbf{g}}{}^{\star}$	1	-	0
$\mathcal{X}\overset{4}{g}_{ij}$	6	$\overset{4}{\mathbf{g}}{}^{\star}, \overset{4}{\mathbf{g}}{}^{\dagger}_{ij}$	$1+2$	$\overset{4}{X}{}^{\blacklozenge}, \overset{4}{X}{}^{\diamond}_{i}$	$1+2$

$\Rightarrow$ Components split into invariant and gauge parts.

$\Rightarrow$ Possible to separate physical information from coordinate choice.

- Use relation between expansion coefficients:

$$\overset{\mathcal{P}}{\overset{k}{g}} = \sum_{0 \leq l_1 + 2l_2 + \ldots \leq k} \frac{1}{l_1!\, l_2! \ldots} \pounds^{l_1}_{\underset{P}{1}} \ldots \pounds^{l_k}_{\underset{P}{k}} \ldots \overset{k-l_1-2l_2-\ldots}{\mathbf{g}} .$$

Relation to standard PPN gauge

- Use relation between expansion coefficients:

$$ {}^{\mathcal{P}}\overset{k}{g} = \sum_{0 \leq l_1 + 2l_2 + \ldots \leq k} \frac{1}{l_1!\, l_2! \ldots} \pounds_{\underset{P}{1}}^{l_1} \ldots \pounds_{\underset{P}{k}}^{l_k} \ldots \overset{k-l_1-2l_2-\ldots}{\mathbf{g}} . $$

- Split components of ${}^{\mathcal{P}}g_{\mu\nu}$ into $\mathbf{g}_{\mu\nu}$ and P^{μ}.

Relation to standard PPN gauge

- Use relation between expansion coefficients:

$$\overset{\mathcal{P}\,k}{g} = \sum_{0 \leq l_1 + 2l_2 + \ldots \leq k} \frac{1}{l_1!\,l_2!\ldots} \pounds^{l_1}_{\overset{1}{P}} \cdots \pounds^{l_k}_{\overset{k}{P}} \cdots \overset{k-l_1-2l_2-\ldots}{\mathbf{g}}.$$

- Split components of ${}^{\mathcal{P}}g_{\mu\nu}$ into $\mathbf{g}_{\mu\nu}$ and P^{μ}.

⇒ Gauge defining vector fields:

$$\overset{2}{P}{}^{\blacklozenge} = 0\,, \quad \overset{2}{P}{}^{\diamond}_{i} = 0\,, \quad \overset{3}{P}{}^{\star} = -\frac{1}{4}(2 + 4\gamma + \alpha_1 - 2\alpha_2 + 2\zeta_1 - 4\xi)\chi_{,0}\,.$$

Relation to standard PPN gauge

- Use relation between expansion coefficients:

$$\overset{k}{{}^{\mathcal{P}}g} = \sum_{0 \leq l_1 + 2l_2 + \ldots \leq k} \frac{1}{l_1! l_2! \ldots} \mathfrak{L}^{l_1}_{\underset{P}{1}} \cdots \mathfrak{L}^{l_k}_{\underset{P}{k}} \cdots \overset{k-l_1-2l_2-\ldots}{\mathbf{g}} .$$

- Split components of ${}^{\mathcal{P}}g_{\mu\nu}$ into $\mathbf{g}_{\mu\nu}$ and P^{μ}.

⇒ Gauge defining vector fields:

$$\overset{2}{P}{}^{\blacklozenge} = 0 , \quad \overset{2}{P}{}^{\diamond}_{i} = 0 , \quad \overset{3}{P}{}^{\star} = -\frac{1}{4}(2 + 4\gamma + \alpha_1 - 2\alpha_2 + 2\zeta_1 - 4\xi)\chi_{,0} .$$

⇒ Gauge-invariant metric components:

$$\overset{2}{\mathbf{g}}{}^{\star} = 2\mathbf{U} , \quad \overset{2}{\mathbf{g}}{}^{\bullet} = 2\gamma\mathbf{U} , \quad \overset{2}{\mathbf{g}}{}^{\dagger}_{ij} = 0 , \quad \overset{3}{\mathbf{g}}{}^{\diamond}_{i} = -\left(1 + \gamma + \frac{\alpha_1}{4}\right)(\mathbf{V}_i + \mathbf{W}_i) ,$$

$$\overset{4}{\mathbf{g}}{}^{\star} = \frac{1}{2}(2-\alpha_1+2\alpha_2+2\alpha_3)\mathbf{\Phi}_1+2(1+3\gamma-2\beta+\zeta_2+\xi)\mathbf{\Phi}_2+2(1+\zeta_3)\mathbf{\Phi}_3+2(3\gamma+3\zeta_4-2\xi)\mathbf{\Phi}_4$$
$$-2\xi\mathbf{\Phi}_W - 2\beta\mathbf{U}^2 + \frac{1}{2}(2 + 4\gamma + \alpha_1 - 2\alpha_2)\mathfrak{A} + \frac{1}{2}(2 + 4\gamma + \alpha_1 - 2\alpha_2 + 2\zeta_1 - 4\xi)\mathfrak{B} .$$

- Perform similar decomposition of energy-momentum tensor:

$$\mathbf{T}^{\star} = \mathbf{T}_{00} = \boldsymbol{\rho}\left(1 - \overset{2}{\mathbf{g}}_{00} + \mathbf{v}^2 + \mathbf{\Pi}\right) + \mathcal{O}(6)\,,$$

$$\mathbf{T}^{\diamond}_{i} + \partial_i \mathbf{T}^{\blacklozenge} = \mathbf{T}_{0i} = -\boldsymbol{\rho}\mathbf{v}_i + \mathcal{O}(5)\,,$$

$$\mathbf{T}^{\bullet}\delta_{ij} + \triangle_{ij}\mathbf{T}^{\blacktriangle} + 2\partial_{(i}\mathbf{T}^{\triangle}_{j)} + \mathbf{T}^{\dagger}_{ij} = \mathbf{T}_{ij} = \boldsymbol{\rho}\mathbf{v}_i\mathbf{v}_j + \mathbf{p}\delta_{ij} + \mathcal{O}(6)\,.$$

Gauge-invariant field equations

- Perform similar decomposition of energy-momentum tensor:

$$\mathbf{T}^{\star} = \mathbf{T}_{00} = \boldsymbol{\rho}\left(1 - \overset{2}{\mathbf{g}}_{00} + \mathbf{v}^2 + \mathbf{\Pi}\right) + \mathcal{O}(6)\,,$$
$$\mathbf{T}^{\diamond}_{i} + \partial_i \mathbf{T}^{\blacklozenge} = \mathbf{T}_{0i} = -\boldsymbol{\rho}\mathbf{v}_i + \mathcal{O}(5)\,,$$
$$\mathbf{T}^{\bullet}\delta_{ij} + \triangle_{ij}\mathbf{T}^{\blacktriangle} + 2\partial_{(i}\mathbf{T}^{\triangle}_{j)} + \mathbf{T}^{\dagger}_{ij} = \mathbf{T}_{ij} = \boldsymbol{\rho}\mathbf{v}_i\mathbf{v}_j + \mathbf{p}\delta_{ij} + \mathcal{O}(6)\,.$$

- Express components in terms of PPN potentials:

$$\overset{2}{\mathbf{T}}{}^{\star} = \boldsymbol{\rho} = -\frac{1}{4\pi}\triangle\mathbf{U}\,, \quad \overset{3}{\mathbf{T}}{}^{\blacklozenge} = -\frac{1}{4\pi}\partial_0\mathbf{U}\,, \quad \overset{3}{\mathbf{T}}{}^{\diamond}_{i} = \frac{1}{8\pi}\triangle(\mathbf{V}_i + \mathbf{W}_i)\,,$$
$$\overset{4}{\mathbf{T}}{}^{\star} = \boldsymbol{\rho}\left(\mathbf{\Pi} + \mathbf{v}^2 - \overset{2}{\mathbf{g}}{}^{\star}\right) = -\frac{1}{4\pi}\triangle(\mathbf{\Phi}_3 + \mathbf{\Phi}_1 - 2\mathbf{\Phi}_2)\,,$$
$$\overset{4}{\mathbf{T}}{}^{\bullet} = \frac{1}{3}\boldsymbol{\rho}\mathbf{v}^2 + \mathbf{p} = -\frac{1}{12\pi}\triangle(\mathbf{\Phi}_1 + 3\mathbf{\Phi}_4)\,, \quad \overset{4}{\mathbf{T}}{}^{\blacktriangle} = \frac{1}{16\pi}(3\mathfrak{A} - \mathbf{\Phi}_1)\,.$$

Gauge-invariant field equations

- Perform similar decomposition of energy-momentum tensor:

$$\mathbf{T}^{\star} = \mathbf{T}_{00} = \rho\left(1 - \overset{2}{\mathbf{g}}_{00} + \mathbf{v}^2 + \mathbf{\Pi}\right) + \mathcal{O}(6)\,,$$

$$\mathbf{T}^{\diamond}_{i} + \partial_i \mathbf{T}^{\blacklozenge} = \mathbf{T}_{0i} = -\rho\mathbf{v}_i + \mathcal{O}(5)\,,$$

$$\mathbf{T}^{\bullet}\delta_{ij} + \triangle_{ij}\mathbf{T}^{\blacktriangle} + 2\partial_{(i}\mathbf{T}^{\triangle}_{j)} + \mathbf{T}^{\dagger}_{ij} = \mathbf{T}_{ij} = \rho\mathbf{v}_i\mathbf{v}_j + \mathbf{p}\delta_{ij} + \mathcal{O}(6)\,.$$

- Express components in terms of PPN potentials:

$$\overset{2}{\mathbf{T}}{}^{\star} = \rho = -\frac{1}{4\pi}\triangle\mathbf{U}\,, \quad \overset{3}{\mathbf{T}}{}^{\blacklozenge} = -\frac{1}{4\pi}\partial_0\mathbf{U}\,, \quad \overset{3}{\mathbf{T}}{}^{\diamond}_{i} = \frac{1}{8\pi}\triangle(\mathbf{V}_i + \mathbf{W}_i)\,,$$

$$\overset{4}{\mathbf{T}}{}^{\star} = \rho\left(\mathbf{\Pi} + \mathbf{v}^2 - \overset{2}{\mathbf{g}}{}^{\star}\right) = -\frac{1}{4\pi}\triangle(\mathbf{\Phi}_3 + \mathbf{\Phi}_1 - 2\mathbf{\Phi}_2)\,,$$

$$\overset{4}{\mathbf{T}}{}^{\bullet} = \frac{1}{3}\rho\mathbf{v}^2 + \mathbf{p} = -\frac{1}{12\pi}\triangle(\mathbf{\Phi}_1 + 3\mathbf{\Phi}_4)\,, \quad \overset{4}{\mathbf{T}}{}^{\blacktriangle} = \frac{1}{16\pi}(3\mathfrak{A} - \mathbf{\Phi}_1)\,.$$

- Decompose also gravity side $\simeq \triangle\mathbf{g}$.

Gauge-invariant field equations

- Perform similar decomposition of energy-momentum tensor:

$$\mathbf{T}^{\star} = \mathbf{T}_{00} = \rho\left(1 - \overset{2}{\mathbf{g}}_{00} + \mathbf{v}^2 + \mathbf{\Pi}\right) + \mathcal{O}(6)\,,$$

$$\mathbf{T}^{\diamond}_i + \partial_i \mathbf{T}^{\blacklozenge} = \mathbf{T}_{0i} = -\rho \mathbf{v}_i + \mathcal{O}(5)\,,$$

$$\mathbf{T}^{\bullet}\delta_{ij} + \triangle_{ij}\mathbf{T}^{\blacktriangle} + 2\partial_{(i}\mathbf{T}^{\triangle}_{j)} + \mathbf{T}^{\dagger}_{ij} = \mathbf{T}_{ij} = \rho \mathbf{v}_i \mathbf{v}_j + \mathbf{p}\delta_{ij} + \mathcal{O}(6)\,.$$

- Express components in terms of PPN potentials:

$$\overset{2}{\mathbf{T}}{}^{\star} = \rho = -\frac{1}{4\pi} \triangle \mathbf{U}\,, \quad \overset{3}{\mathbf{T}}{}^{\blacklozenge} = -\frac{1}{4\pi}\partial_0 \mathbf{U}\,, \quad \overset{3}{\mathbf{T}}{}^{\diamond}_i = \frac{1}{8\pi} \triangle (\mathbf{V}_i + \mathbf{W}_i)\,,$$

$$\overset{4}{\mathbf{T}}{}^{\star} = \rho\left(\mathbf{\Pi} + \mathbf{v}^2 - \overset{2}{\mathbf{g}}{}^{\star}\right) = -\frac{1}{4\pi} \triangle (\mathbf{\Phi}_3 + \mathbf{\Phi}_1 - 2\mathbf{\Phi}_2)\,,$$

$$\overset{4}{\mathbf{T}}{}^{\bullet} = \frac{1}{3}\rho \mathbf{v}^2 + \mathbf{p} = -\frac{1}{12\pi} \triangle (\mathbf{\Phi}_1 + 3\mathbf{\Phi}_4)\,, \quad \overset{4}{\mathbf{T}}{}^{\blacktriangle} = \frac{1}{16\pi}(3\mathfrak{A} - \mathbf{\Phi}_1)\,.$$

- Decompose also gravity side $\simeq \triangle \mathbf{g}$.

⇒ Find PPN parameters by comparing coefficients on both sides.

Outline

Teleparallel geometry

- Fundamental fields:
  - Coframe field $\theta^A = \theta^A{}_\mu \mathrm{d}x^\mu$.
  - Flat spin connection $\omega^A{}_B = \omega^A{}_{B\mu} \mathrm{d}x^\mu$.
  - Arbitrary matter fields χ.

Teleparallel geometry

- Fundamental fields:
  - Coframe field $\theta^A = \theta^A{}_\mu \mathrm{d}x^\mu$.
  - Flat spin connection $\omega^A{}_B = \omega^A{}_{B\mu} \mathrm{d}x^\mu$.
  - Arbitrary matter fields χ.
- Derived quantities:
  - Frame field $e_A = e_A{}^\mu \partial_\mu$ with $e_A{}^\mu \theta^B{}_\mu = \delta^B_A$ and $e_A{}^\mu \theta^A{}_\nu = \delta^\mu{}_\nu$.
  - Metric $g_{\mu\nu} = \eta_{AB} \theta^A{}_\mu \theta^B{}_\nu$.
  - Determinant $\theta = \det(\theta^A{}_\mu)$.
  - Teleparallel connection $\Gamma^\mu{}_{\nu\rho} = e_A{}^\mu (\partial_\rho \theta^A{}_\nu + \omega^A{}_{B\rho} \theta^B{}_\nu)$.
  - Levi-Civita connection $\overset{\circ}{\Gamma}{}^\mu{}_{\nu\rho} = \frac{1}{2} g^{\mu\sigma} (\partial_\nu g_{\sigma\rho} + \partial_\rho g_{\nu\sigma} - \partial_\sigma g_{\nu\rho})$.

Teleparallel geometry

- Fundamental fields:
  - Coframe field $\theta^A = \theta^A{}_\mu \mathrm{d}x^\mu$.
  - Flat spin connection $\omega^A{}_B = \omega^A{}_{B\mu} \mathrm{d}x^\mu$.
  - Arbitrary matter fields χ.
- Derived quantities:
  - Frame field $e_A = e_A{}^\mu \partial_\mu$ with $e_A{}^\mu \theta^B{}_\mu = \delta^B_A$ and $e_A{}^\mu \theta^A{}_\nu = \delta^\mu{}_\nu$.
  - Metric $g_{\mu\nu} = \eta_{AB} \theta^A{}_\mu \theta^B{}_\nu$.
  - Determinant $\theta = \det(\theta^A{}_\mu)$.
  - Teleparallel connection $\Gamma^\mu{}_{\nu\rho} = e_A{}^\mu (\partial_\rho \theta^A{}_\nu + \omega^A{}_{B\rho} \theta^B{}_\nu)$.
  - Levi-Civita connection $\overset{\circ}{\Gamma}{}^\mu{}_{\nu\rho} = \frac{1}{2} g^{\mu\sigma} (\partial_\nu g_{\sigma\rho} + \partial_\rho g_{\nu\sigma} - \partial_\sigma g_{\nu\rho})$.
- Properties of the teleparallel connection:
  - Vanishing curvature: $R^\mu{}_{\nu\rho\sigma} = \partial_\rho \Gamma^\mu{}_{\nu\sigma} - \partial_\sigma \Gamma^\mu{}_{\nu\rho} + \Gamma^\mu{}_{\tau\rho} \Gamma^\tau{}_{\nu\sigma} - \Gamma^\mu{}_{\tau\sigma} \Gamma^\tau{}_{\nu\rho} = 0$.
  - Vanishing nonmetricity: $Q_{\mu\nu\rho} = \nabla_\mu g_{\nu\rho} = 0$.
  - Nonvanishing torsion: $T^\mu{}_{\nu\rho} = \Gamma^\mu{}_{\rho\nu} - \Gamma^\mu{}_{\nu\rho} \neq 0$.

Teleparallel geometry

- Fundamental fields:
  - Coframe field $\theta^A = \theta^A{}_\mu \mathrm{d}x^\mu$.
  - Flat spin connection $\omega^A{}_B = \omega^A{}_{B\mu} \mathrm{d}x^\mu$.
  - Arbitrary matter fields χ.
- Derived quantities:
  - Frame field $e_A = e_A{}^\mu \partial_\mu$ with $e_A{}^\mu \theta^B{}_\mu = \delta^B_A$ and $e_A{}^\mu \theta^A{}_\nu = \delta^\mu{}_\nu$.
  - Metric $g_{\mu\nu} = \eta_{AB} \theta^A{}_\mu \theta^B{}_\nu$.
  - Determinant $\theta = \det(\theta^A{}_\mu)$.
  - Teleparallel connection $\Gamma^\mu{}_{\nu\rho} = e_A{}^\mu (\partial_\rho \theta^A{}_\nu + \omega^A{}_{B\rho} \theta^B{}_\nu)$.
  - Levi-Civita connection $\overset{\circ}{\Gamma}{}^\mu{}_{\nu\rho} = \frac{1}{2} g^{\mu\sigma} (\partial_\nu g_{\sigma\rho} + \partial_\rho g_{\nu\sigma} - \partial_\sigma g_{\nu\rho})$.
- Properties of the teleparallel connection:
  - Vanishing curvature: $R^\mu{}_{\nu\rho\sigma} = \partial_\rho \Gamma^\mu{}_{\nu\sigma} - \partial_\sigma \Gamma^\mu{}_{\nu\rho} + \Gamma^\mu{}_{\tau\rho} \Gamma^\tau{}_{\nu\sigma} - \Gamma^\mu{}_{\tau\sigma} \Gamma^\tau{}_{\nu\rho} = 0$.
  - Vanishing nonmetricity: $Q_{\mu\nu\rho} = \nabla_\mu g_{\nu\rho} = 0$.
  - Nonvanishing torsion: $T^\mu{}_{\nu\rho} = \Gamma^\mu{}_{\rho\nu} - \Gamma^\mu{}_{\nu\rho} \neq 0$.

$\Rightarrow$ Possible to use Weitzenböck gauge: $\omega^A{}_{B\mu} \equiv 0$.

Post-Newtonian tetrad

- Post-Newtonian tetrad expansion:

$$\mathcal{X}\theta^A{}_\mu = \mathcal{X}\overset{0}{\theta}{}^A{}_\mu + \mathcal{X}\overset{1}{\theta}{}^A{}_\mu + \mathcal{X}\overset{2}{\theta}{}^A{}_\mu + \mathcal{X}\overset{3}{\theta}{}^A{}_\mu + \mathcal{X}\overset{4}{\theta}{}^A{}_\mu + \mathcal{O}(5)\,.$$

Post-Newtonian tetrad

- Post-Newtonian tetrad expansion:

$$\mathcal{X}\theta^A{}_\mu = \mathcal{X}\overset{0}{\theta}{}^A{}_\mu + \mathcal{X}\overset{1}{\theta}{}^A{}_\mu + \mathcal{X}\overset{2}{\theta}{}^A{}_\mu + \mathcal{X}\overset{3}{\theta}{}^A{}_\mu + \mathcal{X}\overset{4}{\theta}{}^A{}_\mu + \mathcal{O}(5)\,.$$

- Background tetrad is diagonal: $\mathcal{X}\overset{0}{\theta}{}^A{}_\mu = \Delta^A{}_\mu = \text{diag}(1,1,1,1)$.

Post-Newtonian tetrad

- Post-Newtonian tetrad expansion:

$$\mathcal{X}\theta^A{}_\mu = \mathcal{X}\overset{0}{\theta}{}^A{}_\mu + \mathcal{X}\overset{1}{\theta}{}^A{}_\mu + \mathcal{X}\overset{2}{\theta}{}^A{}_\mu + \mathcal{X}\overset{3}{\theta}{}^A{}_\mu + \mathcal{X}\overset{4}{\theta}{}^A{}_\mu + \mathcal{O}(5)\,.$$

- Background tetrad is diagonal: $\mathcal{X}\overset{0}{\theta}{}^A{}_\mu = \Delta^A{}_\mu = \mathrm{diag}(1,1,1,1)$.
- Higher than fourth velocity order $\mathcal{O}(4)$ is not considered.

Post-Newtonian tetrad

- Post-Newtonian tetrad expansion:

$$\mathcal{X}\theta^A{}_\mu = \mathcal{X}\overset{0}{\theta}{}^A{}_\mu + \mathcal{X}\overset{1}{\theta}{}^A{}_\mu + \mathcal{X}\overset{2}{\theta}{}^A{}_\mu + \mathcal{X}\overset{3}{\theta}{}^A{}_\mu + \mathcal{X}\overset{4}{\theta}{}^A{}_\mu + \mathcal{O}(5)\,.$$

- Background tetrad is diagonal: $\mathcal{X}\overset{0}{\theta}{}^A{}_\mu = \Delta^A{}_\mu = \mathrm{diag}(1,1,1,1)$.
- Higher than fourth velocity order $\mathcal{O}(4)$ is not considered.
- Useful to define tetrad with lower spacetime indices:

$$\mathcal{X}\overset{k}{\theta}{}_{\mu\nu} = \Delta^A{}_\mu \eta_{AB} \mathcal{X}\overset{k}{\theta}{}^B{}_\nu\,.$$

Post-Newtonian tetrad

- Post-Newtonian tetrad expansion:

$$\mathcal{X}\theta^A{}_\mu = \mathcal{X}\overset{0}{\theta}{}^A{}_\mu + \mathcal{X}\overset{1}{\theta}{}^A{}_\mu + \mathcal{X}\overset{2}{\theta}{}^A{}_\mu + \mathcal{X}\overset{3}{\theta}{}^A{}_\mu + \mathcal{X}\overset{4}{\theta}{}^A{}_\mu + \mathcal{O}(5)\,.$$

- Background tetrad is diagonal: $\mathcal{X}\overset{0}{\theta}{}^A{}_\mu = \Delta^A{}_\mu = \mathrm{diag}(1,1,1,1)$.
- Higher than fourth velocity order $\mathcal{O}(4)$ is not considered.
- Useful to define tetrad with lower spacetime indices:

$$\mathcal{X}\overset{k}{\theta}{}_{\mu\nu} = \Delta^A{}_\mu \eta_{AB}\,\mathcal{X}\overset{k}{\theta}{}^B{}_\nu\,.$$

- Only certain components are relevant and non-vanishing:

$$\mathcal{X}\overset{2}{\theta}{}_{00}\,,\quad \mathcal{X}\overset{2}{\theta}{}_{ij}\,,\quad \mathcal{X}\overset{3}{\theta}{}_{0i}\,,\quad \mathcal{X}\overset{4}{\theta}{}_{00}\,,\quad \mathcal{X}\overset{4}{\theta}{}_{ij}\,.$$

Post-Newtonian tetrad

- Post-Newtonian tetrad expansion:

$$ {}^{\mathcal{X}}\theta^A{}_\mu = {}^{\mathcal{X}}\overset{0}{\theta}{}^A{}_\mu + {}^{\mathcal{X}}\overset{1}{\theta}{}^A{}_\mu + {}^{\mathcal{X}}\overset{2}{\theta}{}^A{}_\mu + {}^{\mathcal{X}}\overset{3}{\theta}{}^A{}_\mu + {}^{\mathcal{X}}\overset{4}{\theta}{}^A{}_\mu + \mathcal{O}(5)\,. $$

- Background tetrad is diagonal: ${}^{\mathcal{X}}\overset{0}{\theta}{}^A{}_\mu = \Delta^A{}_\mu = \mathrm{diag}(1, 1, 1, 1)$.
- Higher than fourth velocity order $\mathcal{O}(4)$ is not considered.
- Useful to define tetrad with lower spacetime indices:

$$ {}^{\mathcal{X}}\overset{k}{\theta}{}_{\mu\nu} = \Delta^A{}_\mu \eta_{AB}\, {}^{\mathcal{X}}\overset{k}{\theta}{}^B{}_\nu\,. $$

- Only certain components are relevant and non-vanishing:

$$ {}^{\mathcal{X}}\overset{2}{\theta}{}_{00}\,, \quad {}^{\mathcal{X}}\overset{2}{\theta}{}_{ij}\,, \quad {}^{\mathcal{X}}\overset{3}{\theta}{}_{0i}\,, \quad {}^{\mathcal{X}}\overset{4}{\theta}{}_{00}\,, \quad {}^{\mathcal{X}}\overset{4}{\theta}{}_{ij}\,. $$

- Relation to metric components:

$$ {}^{\mathcal{X}}\overset{2}{g}{}_{00} = 2\,{}^{\mathcal{X}}\overset{2}{\theta}{}_{00}\,, \quad {}^{\mathcal{X}}\overset{2}{g}{}_{ij} = 2\,{}^{\mathcal{X}}\overset{2}{\theta}{}_{(ij)}\,, \quad {}^{\mathcal{X}}\overset{3}{g}{}_{0i} = 2\,{}^{\mathcal{X}}\overset{3}{\theta}{}_{(i0)}\,, $$

$$ {}^{\mathcal{X}}\overset{4}{g}{}_{00} = -({}^{\mathcal{X}}\overset{2}{\theta}{}_{00})^2 + 2\,{}^{\mathcal{X}}\overset{4}{\theta}{}_{00}\,, \quad {}^{\mathcal{X}}\overset{4}{g}{}_{ij} = 2\,{}^{\mathcal{X}}\overset{4}{\theta}{}_{(ij)} + {}^{\mathcal{X}}\overset{2}{\theta}{}_{ki}\,{}^{\mathcal{X}}\overset{2}{\theta}{}_{kj}\,. $$

Gauge transformation of the tetrad

- Split tetrad perturbations in symmetric and antisymmetric parts:

$$\mathcal{X}\overset{k}{\theta}_{\mu\nu} = \mathcal{X}\overset{k}{s}_{\mu\nu} + \mathcal{X}\overset{k}{a}_{\mu\nu}\,, \quad \mathcal{X}\overset{k}{s}_{\mu\nu} = \mathcal{X}\overset{k}{\theta}_{(\mu\nu)}\,, \quad \mathcal{X}\overset{k}{a}_{\mu\nu} = \mathcal{X}\overset{k}{\theta}_{[\mu\nu]}\,.$$

Gauge transformation of the tetrad

- Split tetrad perturbations in symmetric and antisymmetric parts:

$$\mathcal{X}\overset{k}{\theta}_{\mu\nu} = \mathcal{X}\overset{k}{s}_{\mu\nu} + \mathcal{X}\overset{k}{a}_{\mu\nu}\,, \quad \mathcal{X}\overset{k}{s}_{\mu\nu} = \mathcal{X}\overset{k}{\theta}_{(\mu\nu)}\,, \quad \mathcal{X}\overset{k}{a}_{\mu\nu} = \mathcal{X}\overset{k}{\theta}_{[\mu\nu]}\,.$$

- Gauge transformation of the tetrad:

$$\begin{aligned}
\mathcal{Y}\overset{2}{\theta}_{00} &= \mathcal{X}\overset{2}{\theta}_{00}\,,\\
\mathcal{Y}\overset{2}{\theta}_{ij} &= \mathcal{X}\overset{2}{\theta}_{ij} + \partial_j\overset{2}{\xi}_i\,,\\
\mathcal{Y}\overset{3}{\theta}_{0i} &= \mathcal{X}\overset{3}{\theta}_{0i} + \partial_i\overset{3}{\xi}_0\,,\\
\mathcal{Y}\overset{3}{\theta}_{i0} &= \mathcal{X}\overset{3}{\theta}_{i0} + \partial_0\overset{2}{\xi}_i\,,\\
\mathcal{Y}\overset{4}{\theta}_{00} &= \mathcal{X}\overset{4}{\theta}_{00} + \partial_0\overset{3}{\xi}_0 + \overset{2}{\xi}_i\partial_i\,\mathcal{X}\overset{2}{\theta}_{00}\,,\\
\mathcal{Y}\overset{4}{\theta}_{ij} &= \mathcal{X}\overset{4}{\theta}_{ij} + \partial_j\overset{4}{\xi}_i + \partial_j\overset{2}{\xi}_k\,\mathcal{X}\overset{2}{\theta}_{ik} + \overset{2}{\xi}_k\partial_k\,\mathcal{X}\overset{2}{\theta}_{ij} + \frac{1}{2}\partial_j(\overset{2}{\xi}_k\partial_k\overset{2}{\xi}_i)\,.
\end{aligned}$$

Gauge transformation of the tetrad

- Split tetrad perturbations in symmetric and antisymmetric parts:

$$\mathcal{X}\overset{k}{\theta}_{\mu\nu} = \mathcal{X}\overset{k}{s}_{\mu\nu} + \mathcal{X}\overset{k}{a}_{\mu\nu}\,, \quad \mathcal{X}\overset{k}{s}_{\mu\nu} = \mathcal{X}\overset{k}{\theta}_{(\mu\nu)}\,, \quad \mathcal{X}\overset{k}{a}_{\mu\nu} = \mathcal{X}\overset{k}{\theta}_{[\mu\nu]}\,.$$

- Gauge transformation of the tetrad:

$$\begin{aligned}
\mathcal{Y}\overset{2}{\theta}_{00} &= \mathcal{X}\overset{2}{\theta}_{00}\,,\\
\mathcal{Y}\overset{2}{\theta}_{ij} &= \mathcal{X}\overset{2}{\theta}_{ij} + \partial_j\overset{2}{\xi}_i\,,\\
\mathcal{Y}\overset{3}{\theta}_{0i} &= \mathcal{X}\overset{3}{\theta}_{0i} + \partial_i\overset{3}{\xi}_0\,,\\
\mathcal{Y}\overset{3}{\theta}_{i0} &= \mathcal{X}\overset{3}{\theta}_{i0} + \partial_0\overset{2}{\xi}_i\,,\\
\mathcal{Y}\overset{4}{\theta}_{00} &= \mathcal{X}\overset{4}{\theta}_{00} + \partial_0\overset{3}{\xi}_0 + \overset{2}{\xi}_i\partial_i\,\mathcal{X}\overset{2}{\theta}_{00}\,,\\
\mathcal{Y}\overset{4}{\theta}_{ij} &= \mathcal{X}\overset{4}{\theta}_{ij} + \partial_j\overset{4}{\xi}_i + \partial_j\overset{2}{\xi}_k\,\mathcal{X}\overset{2}{\theta}_{ik} + \overset{2}{\xi}_k\partial_k\,\mathcal{X}\overset{2}{\theta}_{ij} + \frac{1}{2}\partial_j(\overset{2}{\xi}_k\partial_k\overset{2}{\xi}_i)\,.
\end{aligned}$$

- Use gauge transformation to eliminate certain tetrad components.

Gauge transformation of the tetrad

- Split tetrad perturbations in symmetric and antisymmetric parts:

$$\mathcal{X}\overset{k}{\theta}_{\mu\nu} = \mathcal{X}\overset{k}{s}_{\mu\nu} + \mathcal{X}\overset{k}{a}_{\mu\nu}\,, \quad \mathcal{X}\overset{k}{s}_{\mu\nu} = \mathcal{X}\overset{k}{\theta}_{(\mu\nu)}\,, \quad \mathcal{X}\overset{k}{a}_{\mu\nu} = \mathcal{X}\overset{k}{\theta}_{[\mu\nu]}\,.$$

- Gauge transformation of the tetrad:

$$\begin{aligned}
\mathcal{Y}\overset{2}{\theta}_{00} &= \mathcal{X}\overset{2}{\theta}_{00}\,,\\
\mathcal{Y}\overset{2}{\theta}_{ij} &= \mathcal{X}\overset{2}{\theta}_{ij} + \partial_j\overset{2}{\xi}_i\,,\\
\mathcal{Y}\overset{3}{\theta}_{0i} &= \mathcal{X}\overset{3}{\theta}_{0i} + \partial_i\overset{3}{\xi}_0\,,\\
\mathcal{Y}\overset{3}{\theta}_{i0} &= \mathcal{X}\overset{3}{\theta}_{i0} + \partial_0\overset{2}{\xi}_i\,,\\
\mathcal{Y}\overset{4}{\theta}_{00} &= \mathcal{X}\overset{4}{\theta}_{00} + \partial_0\overset{3}{\xi}_0 + \overset{2}{\xi}_i\partial_i\,\mathcal{X}\overset{2}{\theta}_{00}\,,\\
\mathcal{Y}\overset{4}{\theta}_{ij} &= \mathcal{X}\overset{4}{\theta}_{ij} + \partial_j\overset{4}{\xi}_i + \partial_j\overset{2}{\xi}_k\,\mathcal{X}\overset{2}{\theta}_{ik} + \overset{2}{\xi}_k\partial_k\,\mathcal{X}\overset{2}{\theta}_{ij} + \frac{1}{2}\partial_j(\overset{2}{\xi}_k\partial_k\overset{2}{\xi}_i)\,.
\end{aligned}$$

- Use gauge transformation to eliminate certain tetrad components.
- Gauge-invariant tetrad components:

$$\mathbf{s}_{00} = \boldsymbol{\theta}^{\star}\,, \quad \mathbf{s}_{0i} = \boldsymbol{\theta}^{\diamond}_i\,, \quad \mathbf{s}_{ij} = \boldsymbol{\theta}^{\bullet}\delta_{ij} + \boldsymbol{\theta}^{\dagger}_{ij}\,, \quad \mathbf{a}_{0i} = \partial_i\boldsymbol{\theta}^{\blacklozenge} + \boldsymbol{\theta}^{\circ}_i\,, \quad \mathbf{a}_{ij} = \epsilon_{ijk}(\partial_k\boldsymbol{\theta}^{\blacksquare} + \boldsymbol{\theta}^{\square}_k)\,.$$

Gauge-invariant tetrad in arbitrary gauge

- Gauge-invariant tetrad components:

$$\boldsymbol{\theta}_{00} = \boldsymbol{\theta}^{\star}\,, \quad \boldsymbol{\theta}_{0i} = \partial_i\boldsymbol{\theta}^{\blacklozenge} + \boldsymbol{\theta}^{\diamond}_i + \boldsymbol{\theta}^{\circ}_i\,, \quad \boldsymbol{\theta}_{i0} = -\partial_i\boldsymbol{\theta}^{\blacklozenge} + \boldsymbol{\theta}^{\diamond}_i - \boldsymbol{\theta}^{\circ}_i\,, \quad \boldsymbol{\theta}_{ij} = \boldsymbol{\theta}^{\bullet}\delta_{ij} + \boldsymbol{\theta}^{\dagger}_{ij} + \epsilon_{ijk}(\partial_k\boldsymbol{\theta}^{\blacksquare} + \boldsymbol{\theta}^{\square}_k)\,.$$

Gauge-invariant tetrad in arbitrary gauge

- Gauge-invariant tetrad components:

$$\theta_{00} = \theta^{\star}\,, \quad \theta_{0i} = \partial_i\theta^{\blacklozenge} + \theta^{\diamond}_i + \theta^{\circ}_i\,, \quad \theta_{i0} = -\partial_i\theta^{\blacklozenge} + \theta^{\diamond}_i - \theta^{\circ}_i\,, \quad \theta_{ij} = \theta^{\bullet}\delta_{ij} + \theta^{\dagger}_{ij} + \epsilon_{ijk}(\partial_k\theta^{\blacksquare} + \theta^{\square}_k)\,.$$

- Conditions imposed on components:

$$\partial^i\theta^{\diamond}_i = \partial^i\theta^{\circ}_i = \partial^i\theta^{\square}_i = 0\,, \quad \partial^i\theta^{\dagger}_{ij} = 0\,, \quad \theta^{\dagger}_{[ij]} = 0\,, \quad \theta^{\dagger}_{ii} = 0\,.$$

Gauge-invariant tetrad in arbitrary gauge

- Gauge-invariant tetrad components:

$$\boldsymbol{\theta}_{00} = \boldsymbol{\theta}^{\star}\,, \quad \boldsymbol{\theta}_{0i} = \partial_i\boldsymbol{\theta}^{\blacklozenge} + \boldsymbol{\theta}^{\diamond}_i + \boldsymbol{\theta}^{\circ}_i\,, \quad \boldsymbol{\theta}_{i0} = -\partial_i\boldsymbol{\theta}^{\blacklozenge} + \boldsymbol{\theta}^{\diamond}_i - \boldsymbol{\theta}^{\circ}_i\,, \quad \boldsymbol{\theta}_{ij} = \boldsymbol{\theta}^{\bullet}\delta_{ij} + \boldsymbol{\theta}^{\dagger}_{ij} + \epsilon_{ijk}(\partial_k\boldsymbol{\theta}^{\blacksquare} + \boldsymbol{\theta}^{\square}_k)\,.$$

- Conditions imposed on components:

$$\partial^i\boldsymbol{\theta}^{\diamond}_i = \partial^i\boldsymbol{\theta}^{\circ}_i = \partial^i\boldsymbol{\theta}^{\square}_i = 0\,, \quad \partial^i\boldsymbol{\theta}^{\dagger}_{ij} = 0\,, \quad \boldsymbol{\theta}^{\dagger}_{[ij]} = 0\,, \quad \boldsymbol{\theta}^{\dagger}_{ii} = 0\,.$$

- Transformation into arbitrary gauge $\mathcal{X}$ with defining vector fields $\overset{k}{X}$:

$$\overset{\mathcal{X}}{\overset{2}{\theta}}_{00} = \overset{2}{\boldsymbol{\theta}}{}^{\star}\,,$$

$$\overset{\mathcal{X}}{\overset{2}{\theta}}_{ij} = \overset{2}{\boldsymbol{\theta}}{}^{\bullet}\delta_{ij} + \overset{2}{\boldsymbol{\theta}}{}^{\dagger}_{ij} + \epsilon_{ijk}(\partial_k\overset{2}{\boldsymbol{\theta}}{}^{\blacksquare} + \overset{2}{\boldsymbol{\theta}}{}^{\square}_k) + \partial_i\partial_j\overset{2}{X}{}^{\blacklozenge} + \partial_j\overset{2}{X}{}^{\diamond}_i\,,$$

$$\overset{\mathcal{X}}{\overset{3}{\theta}}_{0i} = \partial_i\overset{3}{\boldsymbol{\theta}}{}^{\blacklozenge} + \overset{3}{\boldsymbol{\theta}}{}^{\diamond}_i + \overset{3}{\boldsymbol{\theta}}{}^{\circ}_i + \partial_i\overset{3}{X}{}^{\star}\,,$$

$$\overset{\mathcal{X}}{\overset{3}{\theta}}_{i0} = -\partial_i\overset{3}{\boldsymbol{\theta}}{}^{\blacklozenge} + \overset{3}{\boldsymbol{\theta}}{}^{\diamond}_i - \overset{3}{\boldsymbol{\theta}}{}^{\circ}_i + \partial_0\partial_i\overset{2}{X}{}^{\blacklozenge} + \partial_0\overset{2}{X}{}^{\diamond}_i\,,$$

$$\overset{\mathcal{X}}{\overset{4}{\theta}}_{00} = \overset{4}{\boldsymbol{\theta}}{}^{\star} + \partial_0\overset{3}{X}{}^{\star} + \partial_i\overset{2}{\boldsymbol{\theta}}{}^{\star}(\partial_i\overset{2}{X}{}^{\blacklozenge} + \overset{2}{X}{}^{\diamond}_i)\,,$$

$$\overset{\mathcal{X}}{\overset{4}{\theta}}_{ij} = \overset{4}{\boldsymbol{\theta}}{}^{\bullet}\delta_{ij} + \overset{4}{\boldsymbol{\theta}}{}^{\dagger}_{ij} + \epsilon_{ijk}(\partial_k\overset{4}{\boldsymbol{\theta}}{}^{\blacksquare} + \overset{4}{\boldsymbol{\theta}}{}^{\square}_k) + \partial_i\partial_j\overset{4}{X}{}^{\blacklozenge} + \partial_j\overset{4}{X}{}^{\diamond}_i + \mathcal{O}(2)\cdot\mathcal{O}(2)\,.$$

Standard PPN gauge and PPN parameters

- Compare gauge-invariant metric with metric in standard PPN gauge $\mathcal{P}$.

Standard PPN gauge and PPN parameters

- Compare gauge-invariant metric with metric in standard PPN gauge $\mathcal{P}$.

⇒ Gauge-invariant tetrad components in terms of PPN potentials and parameters:

$$\overset{2}{\boldsymbol{\theta}}{}^{\star} = \mathbf{U}\,, \quad \overset{2}{\boldsymbol{\theta}}{}^{\bullet} = \gamma\mathbf{U}\,, \quad \overset{2}{\boldsymbol{\theta}}{}^{\dagger}_{ij} = 0\,, \quad \overset{3}{\boldsymbol{\theta}}{}^{\diamond}_{i} = -\frac{1}{2}\left(1 + \gamma + \frac{\alpha_1}{4}\right)(\mathbf{V}_i + \mathbf{W}_i)\,,$$

$$\overset{4}{\boldsymbol{\theta}}{}^{\star} = \frac{1}{4}(2 - \alpha_1 + 2\alpha_2 + 2\alpha_3)\boldsymbol{\Phi}_1 + (1 + 3\gamma - 2\beta + \zeta_2 + \xi)\boldsymbol{\Phi}_2 + (1 + \zeta_3)\boldsymbol{\Phi}_3 + (3\gamma + 3\zeta_4 - 2\xi)\boldsymbol{\Phi}_4$$
$$-\xi\boldsymbol{\Phi}_W + \frac{1}{2}(1 - 2\beta)\mathbf{U}^2 + \frac{1}{4}(2 + 4\gamma + \alpha_1 - 2\alpha_2)\mathfrak{A} + \frac{1}{4}(2 + 4\gamma + \alpha_1 - 2\alpha_2 + 2\zeta_1 - 4\xi)\mathfrak{B}\,.$$

Standard PPN gauge and PPN parameters

- Compare gauge-invariant metric with metric in standard PPN gauge $\mathcal{P}$.

⇒ Gauge-invariant tetrad components in terms of PPN potentials and parameters:

$$\overset{2}{\boldsymbol{\theta}}{}^{\star} = \mathbf{U}\,, \quad \overset{2}{\boldsymbol{\theta}}{}^{\bullet} = \gamma\mathbf{U}\,, \quad \overset{2}{\boldsymbol{\theta}}{}^{\dagger}_{ij} = 0\,, \quad \overset{3}{\boldsymbol{\theta}}{}^{\diamond}_{i} = -\frac{1}{2}\left(1 + \gamma + \frac{\alpha_1}{4}\right)(\mathbf{V}_i + \mathbf{W}_i)\,,$$

$$\overset{4}{\boldsymbol{\theta}}{}^{\star} = \frac{1}{4}(2 - \alpha_1 + 2\alpha_2 + 2\alpha_3)\mathbf{\Phi}_1 + (1 + 3\gamma - 2\beta + \zeta_2 + \xi)\mathbf{\Phi}_2 + (1 + \zeta_3)\mathbf{\Phi}_3 + (3\gamma + 3\zeta_4 - 2\xi)\mathbf{\Phi}_4 - \xi\mathbf{\Phi}_W + \frac{1}{2}(1 - 2\beta)\mathbf{U}^2 + \frac{1}{4}(2 + 4\gamma + \alpha_1 - 2\alpha_2)\mathfrak{A} + \frac{1}{4}(2 + 4\gamma + \alpha_1 - 2\alpha_2 + 2\zeta_1 - 4\xi)\mathfrak{B}\,.$$

- PPN parameters can be obtained directly from solution for tetrad perturbations.

Outline

Action and field equations

- Action of scalar-tensor gravity with massless scalar field: [Nordtvedt '70]

$$S = \frac{1}{2\kappa^2}\int_M d^4x\sqrt{-g}\left(\psi R - \frac{\omega(\psi)}{\psi}\partial_\rho\psi\partial^\rho\psi\right) + S_m[g_{\mu\nu},\chi]\,.$$

- Action of scalar-tensor gravity with massless scalar field: [Nordtvedt '70]

$$S = \frac{1}{2\kappa^2}\int_M d^4x\sqrt{-g}\left(\psi R - \frac{\omega(\psi)}{\psi}\partial_\rho\psi\partial^\rho\psi\right) + S_m[g_{\mu\nu},\chi]\,.$$

- Free function $\omega(\psi)$ of the scalar field ψ.

Action and field equations

- Action of scalar-tensor gravity with massless scalar field: [Nordtvedt '70]

$$S = \frac{1}{2\kappa^2}\int_M d^4x\sqrt{-g}\left(\psi R - \frac{\omega(\psi)}{\psi}\partial_\rho\psi\partial^\rho\psi\right) + S_m[g_{\mu\nu},\chi]\,.$$

- Free function $\omega(\psi)$ of the scalar field ψ.
- Work in Jordan conformal frame: no direct coupling between matter and scalar field.

Action and field equations

- Action of scalar-tensor gravity with massless scalar field: [Nordtvedt '70]

$$S = \frac{1}{2\kappa^2}\int_M d^4x\sqrt{-g}\left(\psi R - \frac{\omega(\psi)}{\psi}\partial_\rho\psi\partial^\rho\psi\right) + S_m[g_{\mu\nu},\chi]\,.$$

- Free function $\omega(\psi)$ of the scalar field ψ.
- Work in Jordan conformal frame: no direct coupling between matter and scalar field.

⇒ Field equations:

$$\psi R_{\mu\nu} - \nabla_\mu\partial_\nu\psi - \frac{\omega}{\psi}\partial_\mu\psi\partial_\nu\psi + \frac{g_{\mu\nu}}{4\omega+6}\frac{d\omega}{d\psi}\partial_\rho\psi\partial^\rho\psi = \kappa^2\left(T_{\mu\nu} - \frac{\omega+1}{2\omega+3}g_{\mu\nu}T\right)\,,$$

$$(2\omega+3)\,\square\,\psi + \frac{d\omega}{d\psi}\partial_\rho\psi\partial^\rho\psi = \kappa^2 T\,.$$

Perturbative solution ansatz

- Consider arbitrary gauge $\mathcal{X}$ to pull equations and fields to reference spacetime M_0.

Perturbative solution ansatz

- Consider arbitrary gauge $\mathcal{X}$ to pull equations and fields to reference spacetime M_0.
- Relevant components of scalar field: ${}^{\mathcal{X}}\overset{0}{\psi} = \Psi, {}^{\mathcal{X}}\overset{2}{\psi}, {}^{\mathcal{X}}\overset{4}{\psi}$.

Perturbative solution ansatz

- Consider arbitrary gauge $\mathcal{X}$ to pull equations and fields to reference spacetime M_0.
- Relevant components of scalar field: ${}^{\mathcal{X}}\overset{0}{\psi} = \Psi, {}^{\mathcal{X}}\overset{2}{\psi}, {}^{\mathcal{X}}\overset{4}{\psi}$.
- Cosmological background value Ψ assumed to be constant.

Perturbative solution ansatz

- Consider arbitrary gauge $\mathcal{X}$ to pull equations and fields to reference spacetime M_0.
- Relevant components of scalar field: ${}^{\mathcal{X}}\overset{0}{\psi} = \Psi, {}^{\mathcal{X}}\overset{2}{\psi}, {}^{\mathcal{X}}\overset{4}{\psi}$.
- Cosmological background value Ψ assumed to be constant.
- Relation to gauge-invariant scalar field perturbations $\overset{2}{\boldsymbol{\psi}}$ and $\overset{2}{\boldsymbol{\psi}}$:

$$ {}^{\mathcal{X}}\overset{2}{\psi} = \overset{2}{\boldsymbol{\psi}} \,, \quad {}^{\mathcal{X}}\overset{4}{\psi} = \overset{4}{\boldsymbol{\psi}} + (\partial_i \overset{2}{X}^{\blacklozenge} + \overset{2}{X}{}_i^{\diamond})\overset{2}{\boldsymbol{\psi}}{}_{,i} \,. $$

Perturbative solution ansatz

- Consider arbitrary gauge $\mathcal{X}$ to pull equations and fields to reference spacetime M_0.
- Relevant components of scalar field: $\overset{0}{\mathcal{X}\psi} = \Psi, \overset{2}{\mathcal{X}\psi}, \overset{4}{\mathcal{X}\psi}$.
- Cosmological background value Ψ assumed to be constant.
- Relation to gauge-invariant scalar field perturbations $\overset{2}{\boldsymbol{\psi}}$ and $\overset{2}{\boldsymbol{\psi}}$:

$$\overset{2}{\mathcal{X}\psi} = \overset{2}{\boldsymbol{\psi}}, \quad \overset{4}{\mathcal{X}\psi} = \overset{4}{\boldsymbol{\psi}} + (\partial_i \overset{2}{X}{}^{\blacklozenge} + \overset{2}{X}{}^{\lozenge}_i)\overset{2}{\boldsymbol{\psi}}_{,i}.$$

- Taylor expansion of free function ω around cosmological background value:

$$\omega_0 = \omega(\Psi), \quad \omega_1 = \omega'(\Psi).$$

Perturbative solution ansatz

- Consider arbitrary gauge $\mathcal{X}$ to pull equations and fields to reference spacetime M_0.
- Relevant components of scalar field: $\mathcal{X}\overset{0}{\psi} = \Psi, \mathcal{X}\overset{2}{\psi}, \mathcal{X}\overset{4}{\psi}$.
- Cosmological background value Ψ assumed to be constant.
- Relation to gauge-invariant scalar field perturbations $\overset{2}{\boldsymbol{\psi}}$ and $\overset{2}{\boldsymbol{\psi}}$:

$$\mathcal{X}\overset{2}{\psi} = \overset{2}{\boldsymbol{\psi}}\,, \quad \mathcal{X}\overset{4}{\psi} = \overset{4}{\boldsymbol{\psi}} + (\partial_i \overset{2}{X}{}^{\blacklozenge} + \overset{2}{X}{}^{\diamond}_i)\overset{2}{\boldsymbol{\psi}}_{,i}\,.$$

- Taylor expansion of free function ω around cosmological background value:

$$\omega_0 = \omega(\Psi)\,, \quad \omega_1 = \omega'(\Psi)\,.$$

$\Rightarrow$ Zeroth order $\mathcal{X}\overset{0}{\psi} = \Psi, \mathcal{X}\overset{0}{g}_{\mu\nu} = \eta_{\mu\nu}$ solves (vacuum) field equations.

Newtonian limit

- Time component of second-order metric equation:

$$-\frac{1}{2}\Psi \triangle \mathcal{X}\overset{2}{g}_{00} = \kappa^2\left[\mathcal{X}\overset{2}{T}_{00} + \frac{\omega_0 + 1}{2\omega_0 + 3}(\mathcal{X}\overset{2}{T}_{ii} - \mathcal{X}\overset{2}{T}_{00})\right].$$

Newtonian limit

- Time component of second-order metric equation:

$$-\frac{1}{2}\Psi \triangle {}^{\mathcal{X}}\overset{2}{g}_{00} = \kappa^2\left[{}^{\mathcal{X}}\overset{2}{T}_{00} + \frac{\omega_0 + 1}{2\omega_0 + 3}\left({}^{\mathcal{X}}\overset{2}{T}_{ii} - {}^{\mathcal{X}}\overset{2}{T}_{00}\right)\right].$$

- Substitute metric by gauge-invariant component ${}^{\mathcal{X}}\overset{2}{g}_{00} = \overset{2}{\mathbf{g}}^{\star}$.

Newtonian limit

- Time component of second-order metric equation:

$$-\frac{1}{2}\Psi \triangle \,{}^{\mathcal{X}}\overset{2}{g}_{00} = \kappa^2\left[{}^{\mathcal{X}}\overset{2}{T}_{00} + \frac{\omega_0 + 1}{2\omega_0 + 3}\left({}^{\mathcal{X}}\overset{2}{T}_{ii} - {}^{\mathcal{X}}\overset{2}{T}_{00}\right)\right].$$

- Substitute metric by gauge-invariant component ${}^{\mathcal{X}}\overset{2}{g}_{00} = \overset{2}{\mathbf{g}}{}^{\star}$.
- Substitute energy-momentum tensor:

$${}^{\mathcal{X}}\overset{2}{T}_{00} = \overset{2}{\mathbf{T}}_{00} = \overset{2}{\mathbf{T}}{}^{\star} = \boldsymbol{\rho}\,, \quad {}^{\mathcal{X}}\overset{2}{T}_{ii} = \overset{2}{\mathbf{T}}_{ii} = 3\overset{2}{\mathbf{T}}{}^{\bullet} = 0\,.$$

Newtonian limit

- Time component of second-order metric equation:

$$-\frac{1}{2}\Psi \triangle {}^{\mathfrak{X}}\overset{2}{g}_{00} = \kappa^2\left[{}^{\mathfrak{X}}\overset{2}{T}_{00} + \frac{\omega_0 + 1}{2\omega_0 + 3}\left({}^{\mathfrak{X}}\overset{2}{T}_{ii} - {}^{\mathfrak{X}}\overset{2}{T}_{00}\right)\right].$$

- Substitute metric by gauge-invariant component ${}^{\mathfrak{X}}\overset{2}{g}_{00} = \overset{2}{\mathbf{g}}^{\star}$.
- Substitute energy-momentum tensor:

$${}^{\mathfrak{X}}\overset{2}{T}_{00} = \overset{2}{\mathbf{T}}_{00} = \overset{2}{\mathbf{T}}^{\star} = \boldsymbol{\rho}\,, \quad {}^{\mathfrak{X}}\overset{2}{T}_{ii} = \overset{2}{\mathbf{T}}_{ii} = 3\overset{2}{\mathbf{T}}^{\bullet} = 0\,.$$

⇒ Equation becomes fully gauge-invariant and can be solved:

$$-\frac{1}{2}\Psi \triangle \overset{2}{\mathbf{g}}^{\star} = \kappa^2\frac{\omega_0 + 2}{2\omega_0 + 3}\boldsymbol{\rho} \quad \Rightarrow \quad \overset{2}{\mathbf{g}}^{\star} = \frac{\kappa^2}{2\pi\Psi}\frac{\omega_0 + 2}{2\omega_0 + 3}\mathbf{U}\,.$$

Newtonian limit

- Time component of second-order metric equation:

$$-\frac{1}{2}\Psi \bigtriangleup {}^{\mathfrak{X}}\overset{2}{g}_{00} = \kappa^2\left[{}^{\mathfrak{X}}\overset{2}{T}_{00} + \frac{\omega_0 + 1}{2\omega_0 + 3}({}^{\mathfrak{X}}\overset{2}{T}_{ii} - {}^{\mathfrak{X}}\overset{2}{T}_{00})\right].$$

- Substitute metric by gauge-invariant component ${}^{\mathfrak{X}}\overset{2}{g}_{00} = \overset{2}{\mathbf{g}}^{\star}$.
- Substitute energy-momentum tensor:

$$ {}^{\mathfrak{X}}\overset{2}{T}_{00} = \overset{2}{\mathbf{T}}_{00} = \overset{2}{\mathbf{T}}^{\star} = \boldsymbol{\rho}\,, \quad {}^{\mathfrak{X}}\overset{2}{T}_{ii} = \overset{2}{\mathbf{T}}_{ii} = 3\overset{2}{\mathbf{T}}^{\bullet} = 0\,.$$

⇒ Equation becomes fully gauge-invariant and can be solved:

$$-\frac{1}{2}\Psi \bigtriangleup \overset{2}{\mathbf{g}}^{\star} = \kappa^2\frac{\omega_0 + 2}{2\omega_0 + 3}\boldsymbol{\rho} \quad \Rightarrow \quad \overset{2}{\mathbf{g}}^{\star} = \frac{\kappa^2}{2\pi\Psi}\frac{\omega_0 + 2}{2\omega_0 + 3}\mathbf{U}\,.$$

- Normalization $\overset{2}{\mathbf{g}}^{\star} = 2\mathbf{U}$ of the gravitational constant:

$$\kappa^2 = 4\pi\Psi\frac{2\omega_0 + 3}{\omega_0 + 2}\,.$$

Second-order scalar field

- Scalar field equation at second velocity order:

$$(2\omega_0 + 3) \triangle \, {}^{\mathcal{X}}\overset{2}{\psi} = \kappa^2 ({}^{\mathcal{X}}\overset{2}{T}_{ii} - {}^{\mathcal{X}}\overset{2}{T}_{00}) \,.$$

Second-order scalar field

- Scalar field equation at second velocity order:

$$(2\omega_0 + 3) \triangle {}^{\mathcal{X}}\overset{2}{\psi} = \kappa^2 ({}^{\mathcal{X}}\overset{2}{T}_{ii} - {}^{\mathcal{X}}\overset{2}{T}_{00}) \,.$$

- Perform substitutions to obtain gauge-invariant equation:

$$(2\omega_0 + 3) \triangle \overset{2}{\boldsymbol{\psi}} = -\kappa^2 \boldsymbol{\rho} \quad \Rightarrow \quad \overset{2}{\boldsymbol{\psi}} = \frac{\kappa^2}{4\pi(2\omega_0 + 3)} \mathbf{U} = \frac{\Psi}{\omega_0 + 2} \mathbf{U} \,.$$

Second-order scalar field

- Scalar field equation at second velocity order:

$$(2\omega_0 + 3) \triangle \, {}^{\mathcal{X}}\overset{2}{\psi} = \kappa^2 ({}^{\mathcal{X}}\overset{2}{T}_{ii} - {}^{\mathcal{X}}\overset{2}{T}_{00}) \,.$$

- Perform substitutions to obtain gauge-invariant equation:

$$(2\omega_0 + 3) \triangle \overset{2}{\boldsymbol{\psi}} = -\kappa^2 \boldsymbol{\rho} \quad \Rightarrow \quad \overset{2}{\boldsymbol{\psi}} = \frac{\kappa^2}{4\pi(2\omega_0 + 3)} \mathbf{U} = \frac{\Psi}{\omega_0 + 2} \mathbf{U} \,.$$

- Substitutions applied here:
  - Replaced scalar field ${}^{\mathcal{X}}\overset{2}{\psi} = \overset{2}{\boldsymbol{\psi}}$ with gauge-invariant term.

Second-order scalar field

- Scalar field equation at second velocity order:

$$(2\omega_0 + 3) \triangle {}^{\mathcal{X}}\overset{2}{\psi} = \kappa^2 ({}^{\mathcal{X}}\overset{2}{T}_{ii} - {}^{\mathcal{X}}\overset{2}{T}_{00}) \,.$$

- Perform substitutions to obtain gauge-invariant equation:

$$(2\omega_0 + 3) \triangle \overset{2}{\boldsymbol{\psi}} = -\kappa^2 \boldsymbol{\rho} \quad \Rightarrow \quad \overset{2}{\boldsymbol{\psi}} = \frac{\kappa^2}{4\pi(2\omega_0 + 3)} \mathbf{U} = \frac{\Psi}{\omega_0 + 2} \mathbf{U} \,.$$

- Substitutions applied here:
  - Replaced scalar field ${}^{\mathcal{X}}\overset{2}{\psi} = \overset{2}{\boldsymbol{\psi}}$ with gauge-invariant term.
  - Substituted energy-momentum tensor as for the Newtonian limit before.

Second-order scalar field

- Scalar field equation at second velocity order:

$$(2\omega_0 + 3)\,\triangle\, {}^{\mathcal{X}}\overset{2}{\psi} = \kappa^2 ({}^{\mathcal{X}}\overset{2}{T}_{ii} - {}^{\mathcal{X}}\overset{2}{T}_{00})\,.$$

- Perform substitutions to obtain gauge-invariant equation:

$$(2\omega_0 + 3)\,\triangle\, \overset{2}{\boldsymbol{\psi}} = -\kappa^2 \boldsymbol{\rho} \quad \Rightarrow \quad \overset{2}{\boldsymbol{\psi}} = \frac{\kappa^2}{4\pi(2\omega_0 + 3)}\mathbf{U} = \frac{\Psi}{\omega_0 + 2}\mathbf{U}\,.$$

- Substitutions applied here:
  - Replaced scalar field ${}^{\mathcal{X}}\overset{2}{\psi} = \overset{2}{\boldsymbol{\psi}}$ with gauge-invariant term.
  - Substituted energy-momentum tensor as for the Newtonian limit before.
  - Used normalization of the gravitational constant to substitute κ^2.

Second-order spatial equations

- Spatial components of second-order metric equations:

$$-\frac{1}{2}\Psi\left(\triangle\,{}^{\mathcal{X}}\overset{2}{g}_{ij} - {}^{\mathcal{X}}\overset{2}{g}_{00,ij} + {}^{\mathcal{X}}\overset{2}{g}_{kk,ij} - {}^{\mathcal{X}}\overset{2}{g}_{ik,jk} - {}^{\mathcal{X}}\overset{2}{g}_{jk,ik}\right) - {}^{\mathcal{X}}\overset{2}{\psi}_{,ij} = \kappa^2\left[{}^{\mathcal{X}}\overset{2}{T}_{ij} - \frac{\omega_0+1}{2\omega_0+3}\delta_{ij}({}^{\mathcal{X}}\overset{2}{T}_{ii} - {}^{\mathcal{X}}\overset{2}{T}_{00})\right].$$

Second-order spatial equations

- Spatial components of second-order metric equations:

$$-\frac{1}{2}\Psi\left(\triangle\,{}^{\mathcal{X}}\overset{2}{g}_{ij} - {}^{\mathcal{X}}\overset{2}{g}_{00,ij} + {}^{\mathcal{X}}\overset{2}{g}_{kk,ij} - {}^{\mathcal{X}}\overset{2}{g}_{ik,jk} - {}^{\mathcal{X}}\overset{2}{g}_{jk,ik}\right) - {}^{\mathcal{X}}\overset{2}{\psi}_{,ij} = \kappa^2\left[{}^{\mathcal{X}}\overset{2}{T}_{ij} - \frac{\omega_0 + 1}{2\omega_0 + 3}\delta_{ij}({}^{\mathcal{X}}\overset{2}{T}_{ii} - {}^{\mathcal{X}}\overset{2}{T}_{00})\right].$$

- Substitute gauge-invariant variables:

$$-\frac{1}{2}\Psi(\delta_{ij}\triangle\overset{2}{\mathbf{g}}^{\bullet} + \overset{2}{\mathbf{g}}^{\bullet}_{,ij} - \overset{2}{\mathbf{g}}^{\star}_{,ij} + \triangle\overset{2}{\mathbf{g}}^{\dagger}_{ij}) - \overset{2}{\boldsymbol{\psi}}_{,ij} = \kappa^2\frac{\omega_0 + 1}{2\omega_0 + 3}\delta_{ij}\boldsymbol{\rho}\,.$$

Second-order spatial equations

- Spatial components of second-order metric equations:

$$-\frac{1}{2}\Psi\left(\triangle \mathcal{X}\overset{2}{g}_{ij} - \mathcal{X}\overset{2}{g}_{00,ij} + \mathcal{X}\overset{2}{g}_{kk,ij} - \mathcal{X}\overset{2}{g}_{ik,jk} - \mathcal{X}\overset{2}{g}_{jk,ik}\right) - \mathcal{X}\overset{2}{\psi}_{,ij} = \kappa^2\left[\mathcal{X}\overset{2}{T}_{ij} - \frac{\omega_0+1}{2\omega_0+3}\delta_{ij}(\mathcal{X}\overset{2}{T}_{ii} - \mathcal{X}\overset{2}{T}_{00})\right].$$

- Substitute gauge-invariant variables:

$$-\frac{1}{2}\Psi(\delta_{ij}\triangle\overset{2}{\mathbf{g}}^{\bullet} + \overset{2}{\mathbf{g}}^{\bullet}_{,ij} - \overset{2}{\mathbf{g}}^{\star}_{,ij} + \triangle\overset{2}{\mathbf{g}}^{\dagger}_{ij}) - \overset{2}{\boldsymbol{\psi}}_{,ij} = \kappa^2\frac{\omega_0+1}{2\omega_0+3}\delta_{ij}\boldsymbol{\rho}\,.$$

- Canonical differential decomposition of gauge-invariant equations:
  - Trace part yields solution for $\overset{2}{\mathbf{g}}^{\bullet}$:

$$-\frac{1}{2}\Psi(4\triangle\overset{2}{\mathbf{g}}^{\bullet} - \triangle\overset{2}{\mathbf{g}}^{\star}) - \triangle\overset{2}{\boldsymbol{\psi}} = 3\kappa^2\frac{\omega_0+1}{2\omega_0+3}\boldsymbol{\rho} \quad\Rightarrow\quad \overset{2}{\mathbf{g}}^{\bullet} = \frac{\kappa^2}{2\pi\Psi}\frac{\omega_0+1}{2\omega_0+3}\mathbf{U} = 2\frac{\omega_0+1}{\omega_0+2}\mathbf{U}\,.$$

Second-order spatial equations

- Spatial components of second-order metric equations:

$$-\frac{1}{2}\Psi\left(\triangle \overset{\mathcal{X}}{\overset{2}{g}}_{ij} - \overset{\mathcal{X}}{\overset{2}{g}}_{00,ij} + \overset{\mathcal{X}}{\overset{2}{g}}_{kk,ij} - \overset{\mathcal{X}}{\overset{2}{g}}_{ik,jk} - \overset{\mathcal{X}}{\overset{2}{g}}_{jk,ik}\right) - \overset{\mathcal{X}}{\overset{2}{\psi}}_{,ij} = \kappa^2\left[\overset{\mathcal{X}}{\overset{2}{T}}_{ij} - \frac{\omega_0+1}{2\omega_0+3}\delta_{ij}(\overset{\mathcal{X}}{\overset{2}{T}}_{ii} - \overset{\mathcal{X}}{\overset{2}{T}}_{00})\right].$$

- Substitute gauge-invariant variables:

$$-\frac{1}{2}\Psi(\delta_{ij}\triangle \overset{2}{\mathbf{g}}^{\bullet} + \overset{2}{\mathbf{g}}^{\bullet}_{,ij} - \overset{2}{\mathbf{g}}^{\star}_{,ij} + \triangle \overset{2}{\mathbf{g}}^{\dagger}_{ij}) - \overset{2}{\boldsymbol{\psi}}_{,ij} = \kappa^2\frac{\omega_0+1}{2\omega_0+3}\delta_{ij}\boldsymbol{\rho}\,.$$

- Canonical differential decomposition of gauge-invariant equations:
  - Trace part yields solution for $\overset{2}{\mathbf{g}}^{\bullet}$:

$$-\frac{1}{2}\Psi(4\triangle \overset{2}{\mathbf{g}}^{\bullet} - \triangle \overset{2}{\mathbf{g}}^{\star}) - \triangle \overset{2}{\boldsymbol{\psi}} = 3\kappa^2\frac{\omega_0+1}{2\omega_0+3}\boldsymbol{\rho} \quad \Rightarrow \quad \overset{2}{\mathbf{g}}^{\bullet} = \frac{\kappa^2}{2\pi\Psi}\frac{\omega_0+1}{2\omega_0+3}\mathbf{U} = 2\frac{\omega_0+1}{\omega_0+2}\mathbf{U}\,.$$

  ⇒ Trace-free second derivative part is satisfied by preceding solutions:

$$-\triangle_{ij}\left[\frac{1}{2}\Psi(\overset{2}{\mathbf{g}}^{\bullet} - \overset{2}{\mathbf{g}}^{\star}) + \overset{2}{\boldsymbol{\psi}}\right] = 0\,.$$

Second-order spatial equations

- Spatial components of second-order metric equations:

$$-\frac{1}{2}\Psi\left(\triangle\,\mathcal{X}\overset{2}{g}_{ij} - \mathcal{X}\overset{2}{g}_{00,ij} + \mathcal{X}\overset{2}{g}_{kk,ij} - \mathcal{X}\overset{2}{g}_{ik,jk} - \mathcal{X}\overset{2}{g}_{jk,ik}\right) - \mathcal{X}\overset{2}{\psi}_{,ij} = \kappa^2\left[\mathcal{X}\overset{2}{T}_{ij} - \frac{\omega_0+1}{2\omega_0+3}\delta_{ij}(\mathcal{X}\overset{2}{T}_{ii} - \mathcal{X}\overset{2}{T}_{00})\right].$$

- Substitute gauge-invariant variables:

$$-\frac{1}{2}\Psi(\delta_{ij}\triangle\overset{2}{\mathbf{g}}^{\bullet} + \overset{2}{\mathbf{g}}^{\bullet}_{,ij} - \overset{2}{\mathbf{g}}^{\star}_{,ij} + \triangle\overset{2}{\mathbf{g}}^{\dagger}_{ij}) - \overset{2}{\psi}_{,ij} = \kappa^2\frac{\omega_0+1}{2\omega_0+3}\delta_{ij}\boldsymbol{\rho}\,.$$

- Canonical differential decomposition of gauge-invariant equations:
  - Trace part yields solution for $\overset{2}{\mathbf{g}}^{\bullet}$:

$$-\frac{1}{2}\Psi(4\triangle\overset{2}{\mathbf{g}}^{\bullet} - \triangle\overset{2}{\mathbf{g}}^{\star}) - \triangle\overset{2}{\psi} = 3\kappa^2\frac{\omega_0+1}{2\omega_0+3}\boldsymbol{\rho} \quad\Rightarrow\quad \overset{2}{\mathbf{g}}^{\bullet} = \frac{\kappa^2}{2\pi\Psi}\frac{\omega_0+1}{2\omega_0+3}\mathbf{U} = 2\frac{\omega_0+1}{\omega_0+2}\mathbf{U}\,.$$

  ⇒ Trace-free second derivative part is satisfied by preceding solutions:

$$-\triangle_{ij}\left[\frac{1}{2}\Psi(\overset{2}{\mathbf{g}}^{\bullet} - \overset{2}{\mathbf{g}}^{\star}) + \overset{2}{\psi}\right] = 0\,.$$

  - Trace-free, divergence-free part yields trivial solution: $\triangle\overset{2}{\mathbf{g}}^{\dagger}_{ij} = 0 \Rightarrow \overset{2}{\mathbf{g}}^{\dagger}_{ij} = 0$.

Second-order spatial equations

- Spatial components of second-order metric equations:

$$-\frac{1}{2}\Psi\left(\triangle \overset{2}{\mathcal{X}g}_{ij} - \overset{2}{\mathcal{X}g}_{00,ij} + \overset{2}{\mathcal{X}g}_{kk,ij} - \overset{2}{\mathcal{X}g}_{ik,jk} - \overset{2}{\mathcal{X}g}_{jk,ik}\right) - \overset{2}{\mathcal{X}\psi}_{,ij} = \kappa^2\left[\overset{2}{\mathcal{X}T}_{ij} - \frac{\omega_0+1}{2\omega_0+3}\delta_{ij}(\overset{2}{\mathcal{X}T}_{ii} - \overset{2}{\mathcal{X}T}_{00})\right].$$

- Substitute gauge-invariant variables:

$$-\frac{1}{2}\Psi(\delta_{ij}\triangle\overset{2}{\mathbf{g}}^{\bullet} + \overset{2}{\mathbf{g}}^{\bullet}_{,ij} - \overset{2}{\mathbf{g}}^{\star}_{,ij} + \triangle\overset{2}{\mathbf{g}}^{\dagger}_{ij}) - \overset{2}{\psi}_{,ij} = \kappa^2\frac{\omega_0+1}{2\omega_0+3}\delta_{ij}\boldsymbol{\rho}.$$

- Canonical differential decomposition of gauge-invariant equations:
  - Trace part yields solution for $\overset{2}{\mathbf{g}}^{\bullet}$:

$$-\frac{1}{2}\Psi(4\triangle\overset{2}{\mathbf{g}}^{\bullet} - \triangle\overset{2}{\mathbf{g}}^{\star}) - \triangle\overset{2}{\psi} = 3\kappa^2\frac{\omega_0+1}{2\omega_0+3}\boldsymbol{\rho} \quad\Rightarrow\quad \overset{2}{\mathbf{g}}^{\bullet} = \frac{\kappa^2}{2\pi\Psi}\frac{\omega_0+1}{2\omega_0+3}\mathbf{U} = 2\frac{\omega_0+1}{\omega_0+2}\mathbf{U}.$$

$\Rightarrow$ Trace-free second derivative part is satisfied by preceding solutions:

$$-\triangle_{ij}\left[\frac{1}{2}\Psi(\overset{2}{\mathbf{g}}^{\bullet} - \overset{2}{\mathbf{g}}^{\star}) + \overset{2}{\psi}\right] = 0.$$

  - Trace-free, divergence-free part yields trivial solution: $\triangle\overset{2}{\mathbf{g}}^{\dagger}_{ij} = 0 \Rightarrow \overset{2}{\mathbf{g}}^{\dagger}_{ij} = 0$.
  - Pure vector divergence part $\partial_{(i}\mathbf{E}_{j)}$ does not appear.

Third-order metric equations

- Metric equations at third velocity order:

$$-\frac{1}{2}\Psi\left(\triangle\,\mathcal{X}\overset{3}{g}_{0i} - \mathcal{X}\overset{3}{g}_{0j,ij} + \mathcal{X}\overset{2}{g}_{jj,0i} - \mathcal{X}\overset{2}{g}_{ij,0j}\right) - \mathcal{X}\overset{2}{\psi}_{,0i} = \kappa^2\,\mathcal{X}\overset{3}{T}_{0i}\,.$$

Third-order metric equations

- Metric equations at third velocity order:

$$-\frac{1}{2}\Psi(\triangle \mathcal{X}\overset{3}{g}_{0i} - \mathcal{X}\overset{3}{g}_{0j,ij} + \mathcal{X}\overset{2}{g}_{jj,0i} - \mathcal{X}\overset{2}{g}_{ij,0j}) - \mathcal{X}\overset{2}{\psi}_{,0i} = \kappa^2 \mathcal{X}\overset{3}{T}_{0i}\,.$$

- Substitute energy-momentum tensor:

$$\mathcal{X}\overset{3}{T}_{0i} = \overset{3}{\mathbf{T}}_{0i} = \overset{3}{\mathbf{T}}{}^{\diamond}_{i} + \partial_i\overset{3}{\mathbf{T}}{}^{\blacklozenge} = -\boldsymbol{\rho}\mathbf{v}_i\,.$$

Third-order metric equations

- Metric equations at third velocity order:

$$-\frac{1}{2}\Psi(\triangle\, {}^{\mathcal{X}}\overset{3}{g}_{0i} - {}^{\mathcal{X}}\overset{3}{g}_{0j,ij} + {}^{\mathcal{X}}\overset{2}{g}_{jj,0i} - {}^{\mathcal{X}}\overset{2}{g}_{ij,0j}) - {}^{\mathcal{X}}\overset{2}{\psi}_{,0i} = \kappa^2\, {}^{\mathcal{X}}\overset{3}{T}_{0i}\,.$$

- Substitute energy-momentum tensor:

$${}^{\mathcal{X}}\overset{3}{T}_{0i} = \overset{3}{\mathbf{T}}_{0i} = \overset{3}{\mathbf{T}}{}^{\diamond}_{i} + \partial_i \overset{3}{\mathbf{T}}{}^{\blacklozenge} = -\boldsymbol{\rho}\mathbf{v}_i\,.$$

- Gauge-invariant field equation:

$$-\frac{1}{2}\Psi(\triangle\overset{3}{\mathbf{g}}{}^{\diamond}_{i} + 2\overset{2}{\mathbf{g}}{}^{\bullet}_{,0i}) - \overset{2}{\psi}_{,0i} = \kappa^2(\overset{3}{\mathbf{T}}{}^{\diamond}_{i} + \partial_i\overset{3}{\mathbf{T}}{}^{\blacklozenge}) = -\kappa^2\boldsymbol{\rho}\mathbf{v}_i\,.$$

Third-order metric equations

- Metric equations at third velocity order:

$$-\frac{1}{2}\Psi\left(\triangle \mathcal{X}\overset{3}{g}_{0i} - \mathcal{X}\overset{3}{g}_{0j,ij} + \mathcal{X}\overset{2}{g}_{jj,0i} - \mathcal{X}\overset{2}{g}_{ij,0j}\right) - \mathcal{X}\overset{2}{\psi}_{,0i} = \kappa^2 \mathcal{X}\overset{3}{T}_{0i}\,.$$

- Substitute energy-momentum tensor:

$$\mathcal{X}\overset{3}{T}_{0i} = \overset{3}{\mathbf{T}}_{0i} = \overset{3}{\mathbf{T}}{}^{\diamond}_{i} + \partial_i \overset{3}{\mathbf{T}}{}^{\blacklozenge} = -\boldsymbol{\rho}\mathbf{v}_i\,.$$

- Gauge-invariant field equation:

$$-\frac{1}{2}\Psi\left(\triangle \overset{3}{\mathbf{g}}{}^{\diamond}_{i} + 2\overset{2}{\mathbf{g}}{}^{\bullet}_{,0i}\right) - \overset{2}{\psi}_{,0i} = \kappa^2\left(\overset{3}{\mathbf{T}}{}^{\diamond}_{i} + \partial_i \overset{3}{\mathbf{T}}{}^{\blacklozenge}\right) = -\kappa^2\boldsymbol{\rho}\mathbf{v}_i\,.$$

- Canonical differential decomposition:
  - ⇒ Pure divergence part is satisfied identically by previous solutions:

$$-\Psi\overset{2}{\mathbf{g}}{}^{\bullet}_{,0i} - \overset{2}{\psi}_{,0i} = \kappa^2\partial_i\overset{3}{\mathbf{T}}{}^{\blacklozenge} = -\frac{\kappa^2}{4\pi}\mathbf{U}_{,0i}\,.$$

Third-order metric equations

- Metric equations at third velocity order:

$$-\frac{1}{2}\Psi(\triangle \mathcal{X}\overset{3}{g}_{0i} - \mathcal{X}\overset{3}{g}_{0j,ij} + \mathcal{X}\overset{2}{g}_{jj,0i} - \mathcal{X}\overset{2}{g}_{ij,0j}) - \mathcal{X}\overset{2}{\psi}_{,0i} = \kappa^2 \mathcal{X}\overset{3}{T}_{0i}\,.$$

- Substitute energy-momentum tensor:

$$\mathcal{X}\overset{3}{T}_{0i} = \overset{3}{\mathbf{T}}_{0i} = \overset{3}{\mathbf{T}}{}^{\diamond}_{i} + \partial_i \overset{3}{\mathbf{T}}{}^{\blacklozenge} = -\boldsymbol{\rho}\mathbf{v}_i\,.$$

- Gauge-invariant field equation:

$$-\frac{1}{2}\Psi(\triangle\overset{3}{\mathbf{g}}{}^{\diamond}_{i} + 2\overset{2}{\mathbf{g}}{}^{\bullet}_{,0i}) - \overset{2}{\psi}_{,0i} = \kappa^2(\overset{3}{\mathbf{T}}{}^{\diamond}_{i} + \partial_i\overset{3}{\mathbf{T}}{}^{\blacklozenge}) = -\kappa^2\boldsymbol{\rho}\mathbf{v}_i\,.$$

- Canonical differential decomposition:
  - ⇒ Pure divergence part is satisfied identically by previous solutions:

$$-\Psi\overset{2}{\mathbf{g}}{}^{\bullet}_{,0i} - \overset{2}{\psi}_{,0i} = \kappa^2\partial_i\overset{3}{\mathbf{T}}{}^{\blacklozenge} = -\frac{\kappa^2}{4\pi}\mathbf{U}_{,0i}\,.$$

  - ∘ Divergence-free part yields solution for third-order metric component $\overset{3}{\mathbf{g}}{}^{\diamond}_{i}$:

$$-\frac{1}{2}\Psi\triangle\overset{3}{\mathbf{g}}{}^{\diamond}_{i} = \kappa^2\overset{3}{\mathbf{T}}{}^{\diamond}_{i} = \frac{\kappa^2}{8\pi}\triangle(\mathbf{V}_i + \mathbf{W}_i) \quad\Rightarrow\quad \overset{3}{\mathbf{g}}{}^{\diamond}_{i} = -\frac{\kappa^2}{4\pi\Psi}(\mathbf{V}_i + \mathbf{W}_i) = -\frac{2\omega_0 + 3}{\omega_0 + 2}(\mathbf{V}_i + \mathbf{W}_i)\,.$$

Fourth-order metric equation

- Metric equation at fourth velocity order:

$$-\overset{2}{\mathcal{X}\psi}_{,00}-\frac{1}{2}\Psi\left[\triangle\overset{4}{\mathcal{X}g}_{00}+\overset{2}{\mathcal{X}g}_{ii,00}-2\overset{3}{\mathcal{X}g}_{0i,0i}+\frac{1}{2}\overset{2}{\mathcal{X}g}_{00,i}\left(\overset{2}{\mathcal{X}g}_{00,i}-2\overset{2}{\mathcal{X}g}_{ij,j}+\overset{2}{\mathcal{X}g}_{jj,i}\right)-\overset{2}{\mathcal{X}g}_{ij}\overset{2}{\mathcal{X}g}_{00,ij}\right]$$
$$-\frac{1}{2}\overset{2}{\mathcal{X}\psi}\triangle\overset{2}{\mathcal{X}g}_{00}-\frac{1}{2}\overset{2}{\mathcal{X}g}_{00,i}\overset{2}{\mathcal{X}\psi}_{,i}-\frac{\omega_1}{4\omega_0+6}\overset{2}{\mathcal{X}\psi}_{,i}\overset{2}{\mathcal{X}\psi}_{,i}=\kappa^2\left[\overset{4}{\mathcal{X}T}_{00}-\frac{\omega_0+1}{2\omega_0+3}\overset{2}{\mathcal{X}g}_{00}\left(\overset{2}{\mathcal{X}T}_{ii}-\overset{2}{\mathcal{X}T}_{00}\right)\right.$$
$$\left.+\frac{\omega_1}{(2\omega_0+3)^2}\overset{2}{\mathcal{X}\psi}\left(\overset{2}{\mathcal{X}T}_{ii}-\overset{2}{\mathcal{X}T}_{00}\right)+\frac{\omega_0+1}{2\omega_0+3}\left(\overset{4}{\mathcal{X}T}_{ii}-\overset{4}{\mathcal{X}T}_{00}-\overset{2}{\mathcal{X}g}_{ij}\overset{2}{\mathcal{X}T}_{ij}-\overset{2}{\mathcal{X}g}_{00}\overset{2}{\mathcal{X}T}_{00}\right)\right].$$

Fourth-order metric equation

- Metric equation at fourth velocity order:

$$-\overset{2}{\psi}_{,00}-\frac{1}{2}\Psi\left[\triangle\overset{4}{g}_{00}+\overset{2}{g}_{ii,00}-2\overset{3}{g}_{0i,0i}+\frac{1}{2}\overset{2}{g}_{00,i}\left(\overset{2}{g}_{00,i}-2\overset{2}{g}_{ij,j}+\overset{2}{g}_{jj,i}\right)-\overset{2}{g}_{ij}\overset{2}{g}_{00,ij}\right]$$
$$-\frac{1}{2}\overset{2}{\psi}\triangle\overset{2}{g}_{00}-\frac{1}{2}\overset{2}{g}_{00,i}\overset{2}{\psi}_{,i}-\frac{\omega_1}{4\omega_0+6}\overset{2}{\psi}_{,i}\overset{2}{\psi}_{,i}=\kappa^2\left[\overset{4}{T}_{00}-\frac{\omega_0+1}{2\omega_0+3}\overset{2}{g}_{00}\left(\overset{2}{T}_{ii}-\overset{2}{T}_{00}\right)\right.$$
$$\left.+\frac{\omega_1}{(2\omega_0+3)^2}\overset{2}{\psi}\left(\overset{2}{T}_{ii}-\overset{2}{T}_{00}\right)+\frac{\omega_0+1}{2\omega_0+3}\left(\overset{4}{T}_{ii}-\overset{4}{T}_{00}-\overset{2}{g}_{ij}\overset{2}{T}_{ij}-\overset{2}{g}_{00}\overset{2}{T}_{00}\right)\right].$$

- Substitute gauge-invariant quantities:

$$-\frac{1}{2}\Psi\left[\triangle\overset{4}{\mathbf{g}}^{\star}+(\overset{2}{X}{}^{\blacklozenge}_{,i}+\overset{2}{X}{}^{\diamond}_{i})\triangle\overset{2}{\mathbf{g}}{}^{\star}_{,i}+3\overset{2}{\mathbf{g}}{}^{\bullet}_{,00}+\frac{1}{2}\overset{2}{\mathbf{g}}{}^{\star}_{,i}(\overset{2}{\mathbf{g}}{}^{\star}_{,i}+\overset{2}{\mathbf{g}}{}^{\bullet}_{,i})-\overset{2}{\mathbf{g}}{}^{\star}_{,ij}(\overset{2}{\mathbf{g}}{}^{\bullet}\delta_{ij}+\overset{2}{\mathbf{g}}{}^{\dagger}_{ij})\right]$$
$$-\frac{1}{2}\overset{2}{\psi}\triangle\overset{2}{\mathbf{g}}{}^{\star}-\overset{2}{\psi}_{,00}-\frac{1}{2}\overset{2}{\mathbf{g}}{}^{\star}_{,i}\overset{2}{\psi}_{,i}-\frac{\omega_1}{4\omega_0+6}\overset{2}{\psi}_{,i}\overset{2}{\psi}_{,i}$$
$$=\kappa^2\left\{\overset{4}{\mathbf{T}}{}^{\star}+(\overset{2}{X}{}^{\blacklozenge}_{,i}+\overset{2}{X}{}^{\diamond}_{i})\overset{2}{\mathbf{T}}{}^{\star}_{,i}+\frac{\omega_0+1}{2\omega_0+3}\left[3\overset{4}{\mathbf{T}}{}^{\bullet}-\overset{4}{\mathbf{T}}{}^{\star}-(\overset{2}{X}{}^{\blacklozenge}_{,i}+\overset{2}{X}{}^{\diamond}_{i})\overset{2}{\mathbf{T}}{}^{\star}_{,i}\right]-\frac{\omega_1}{(2\omega_0+3)^2}\overset{2}{\psi}\overset{2}{\mathbf{T}}{}^{\star}\right\}.$$

Fourth-order metric equation

- Metric equation at fourth velocity order:

$$-{}^{\mathcal{X}}\overset{2}{\psi}_{,00}-\frac{1}{2}\Psi\left[\triangle{}^{\mathcal{X}}\overset{4}{g}_{00}+{}^{\mathcal{X}}\overset{2}{g}_{ii,00}-2\,{}^{\mathcal{X}}\overset{3}{g}_{0i,0i}+\frac{1}{2}{}^{\mathcal{X}}\overset{2}{g}_{00,i}\left({}^{\mathcal{X}}\overset{2}{g}_{00,i}-2\,{}^{\mathcal{X}}\overset{2}{g}_{ij,j}+{}^{\mathcal{X}}\overset{2}{g}_{jj,i}\right)-{}^{\mathcal{X}}\overset{2}{g}_{ij}{}^{\mathcal{X}}\overset{2}{g}_{00,ij}\right]$$

$$-\frac{1}{2}{}^{\mathcal{X}}\overset{2}{\psi}\triangle{}^{\mathcal{X}}\overset{2}{g}_{00}-\frac{1}{2}{}^{\mathcal{X}}\overset{2}{g}_{00,i}{}^{\mathcal{X}}\overset{2}{\psi}_{,i}-\frac{\omega_1}{4\omega_0+6}{}^{\mathcal{X}}\overset{2}{\psi}_{,i}{}^{\mathcal{X}}\overset{2}{\psi}_{,i}=\kappa^2\left[{}^{\mathcal{X}}\overset{4}{T}_{00}-\frac{\omega_0+1}{2\omega_0+3}{}^{\mathcal{X}}\overset{2}{g}_{00}\left({}^{\mathcal{X}}\overset{2}{T}_{ii}-{}^{\mathcal{X}}\overset{2}{T}_{00}\right)\right.$$

$$\left.+\frac{\omega_1}{(2\omega_0+3)^2}{}^{\mathcal{X}}\overset{2}{\psi}\left({}^{\mathcal{X}}\overset{2}{T}_{ii}-{}^{\mathcal{X}}\overset{2}{T}_{00}\right)+\frac{\omega_0+1}{2\omega_0+3}\left({}^{\mathcal{X}}\overset{4}{T}_{ii}-{}^{\mathcal{X}}\overset{4}{T}_{00}-{}^{\mathcal{X}}\overset{2}{g}_{ij}{}^{\mathcal{X}}\overset{2}{T}_{ij}-{}^{\mathcal{X}}\overset{2}{g}_{00}{}^{\mathcal{X}}\overset{2}{T}_{00}\right)\right].$$

- Substitute gauge-invariant quantities:

$$-\frac{1}{2}\Psi\left[\triangle\overset{4}{\mathbf{g}}^{\star}+\left(\overset{2}{X}{}^{\blacklozenge}_{,i}+\overset{2}{X}{}^{\diamond}_{i}\right)\triangle\overset{2}{\mathbf{g}}^{\star}_{,i}+3\overset{2}{\mathbf{g}}^{\bullet}_{,00}+\frac{1}{2}\overset{2}{\mathbf{g}}^{\star}_{,i}\left(\overset{2}{\mathbf{g}}^{\star}_{,i}+\overset{2}{\mathbf{g}}^{\bullet}_{,i}\right)-\overset{2}{\mathbf{g}}^{\star}_{,ij}\left(\overset{2}{\mathbf{g}}^{\bullet}\delta_{ij}+\overset{2}{\mathbf{g}}^{\dagger}_{ij}\right)\right]$$

$$-\frac{1}{2}\overset{2}{\psi}\triangle\overset{2}{\mathbf{g}}^{\star}-\overset{2}{\psi}_{,00}-\frac{1}{2}\overset{2}{\mathbf{g}}^{\star}_{,i}\overset{2}{\psi}_{,i}-\frac{\omega_1}{4\omega_0+6}\overset{2}{\psi}_{,i}\overset{2}{\psi}_{,i}$$

$$=\kappa^2\left\{\overset{4}{\mathbf{T}}^{\star}+\left(\overset{2}{X}{}^{\blacklozenge}_{,i}+\overset{2}{X}{}^{\diamond}_{i}\right)\overset{2}{\mathbf{T}}^{\star}_{,i}+\frac{\omega_0+1}{2\omega_0+3}\left[3\overset{4}{\mathbf{T}}^{\bullet}-\overset{4}{\mathbf{T}}^{\star}-\left(\overset{2}{X}{}^{\blacklozenge}_{,i}+\overset{2}{X}{}^{\diamond}_{i}\right)\overset{2}{\mathbf{T}}^{\star}_{,i}\right]-\frac{\omega_1}{(2\omega_0+3)^2}\overset{2}{\psi}\overset{2}{\mathbf{T}}^{\star}\right\}.$$

↯ Gauge defining vector fields X appear on both sides of the equation…

Fourth-order metric equation

- Metric equation at fourth velocity order:

$$-{}^{\chi}\overset{2}{\psi}_{,00} - \frac{1}{2}\Psi\left[\triangle\, {}^{\chi}\overset{4}{g}_{00} + {}^{\chi}\overset{2}{g}_{ii,00} - 2\,{}^{\chi}\overset{3}{g}_{0i,0i} + \frac{1}{2}\, {}^{\chi}\overset{2}{g}_{00,i}\left({}^{\chi}\overset{2}{g}_{00,i} - 2\,{}^{\chi}\overset{2}{g}_{ij,j} + {}^{\chi}\overset{2}{g}_{jj,i}\right) - {}^{\chi}\overset{2}{g}_{ij}\, {}^{\chi}\overset{2}{g}_{00,ij}\right]$$

$$-\frac{1}{2}\, {}^{\chi}\overset{2}{\psi}\triangle {}^{\chi}\overset{2}{g}_{00} - \frac{1}{2}\, {}^{\chi}\overset{2}{g}_{00,i}\, {}^{\chi}\overset{2}{\psi}_{,i} - \frac{\omega_1}{4\omega_0+6}\, {}^{\chi}\overset{2}{\psi}_{,i}\, {}^{\chi}\overset{2}{\psi}_{,i} = \kappa^2\left[{}^{\chi}\overset{4}{T}_{00} - \frac{\omega_0+1}{2\omega_0+3}\, {}^{\chi}\overset{2}{g}_{00}\left({}^{\chi}\overset{2}{T}_{ii} - {}^{\chi}\overset{2}{T}_{00}\right)\right.$$

$$\left. + \frac{\omega_1}{(2\omega_0+3)^2}\, {}^{\chi}\overset{2}{\psi}\left({}^{\chi}\overset{2}{T}_{ii} - {}^{\chi}\overset{2}{T}_{00}\right) + \frac{\omega_0+1}{2\omega_0+3}\left({}^{\chi}\overset{4}{T}_{ii} - {}^{\chi}\overset{4}{T}_{00} - {}^{\chi}\overset{2}{g}_{ij}\, {}^{\chi}\overset{2}{T}_{ij} - {}^{\chi}\overset{2}{g}_{00}\, {}^{\chi}\overset{2}{T}_{00}\right)\right].$$

- Substitute gauge-invariant quantities:

$$-\frac{1}{2}\Psi\left[\triangle\overset{4}{\mathbf{g}}^{\star} + (\overset{2}{X}{}^{\blacklozenge}_{,i} + \overset{2}{X}{}^{\diamond}_{i})\triangle\overset{2}{\mathbf{g}}{}^{\star}_{,i} + 3\overset{2}{\mathbf{g}}{}^{\bullet}_{,00} + \frac{1}{2}\overset{2}{\mathbf{g}}{}^{\star}_{,i}(\overset{2}{\mathbf{g}}{}^{\star}_{,i} + \overset{2}{\mathbf{g}}{}^{\bullet}_{,i}) - \overset{2}{\mathbf{g}}{}^{\star}_{,ij}(\overset{2}{\mathbf{g}}{}^{\bullet}\delta_{ij} + \overset{2}{\mathbf{g}}{}^{\dagger}_{ij})\right]$$

$$-\frac{1}{2}\overset{2}{\psi}\triangle\overset{2}{\mathbf{g}}{}^{\star} - \overset{2}{\psi}_{,00} - \frac{1}{2}\overset{2}{\mathbf{g}}{}^{\star}_{,i}\overset{2}{\psi}_{,i} - \frac{\omega_1}{4\omega_0+6}\overset{2}{\psi}_{,i}\overset{2}{\psi}_{,i}$$

$$= \kappa^2\left\{\overset{4}{\mathbf{T}}{}^{\star} + (\overset{2}{X}{}^{\blacklozenge}_{,i} + \overset{2}{X}{}^{\diamond}_{i})\overset{2}{\mathbf{T}}{}^{\star}_{,i} + \frac{\omega_0+1}{2\omega_0+3}\left[3\overset{4}{\mathbf{T}}{}^{\bullet} - \overset{4}{\mathbf{T}}{}^{\star} - (\overset{2}{X}{}^{\blacklozenge}_{,i} + \overset{2}{X}{}^{\diamond}_{i})\overset{2}{\mathbf{T}}{}^{\star}_{,i}\right] - \frac{\omega_1}{(2\omega_0+3)^2}\overset{2}{\psi}\overset{2}{\mathbf{T}}{}^{\star}\right\}.$$

↯ Gauge defining vector fields X appear on both sides of the equation…

✓ …but cancel due to second order field equation.

Fourth-order solution and PPN parameters

- Gauge-invariant equation for metric component $\overset{4}{\mathbf{g}}{}^{\star}$:

$$\triangle \overset{4}{\mathbf{g}}{}^{\star} = 8\pi\left(\frac{3}{\omega_0+2} + \frac{\omega_1\Psi}{(2\omega_0+3)(\omega_0+2)^2}\right)\rho\mathbf{U} - 8\pi\frac{2\omega_0+3}{\omega_0+2}\rho\mathbf{v}^2 - 8\pi\rho\mathbf{\Pi} - 24\pi\frac{\omega_0+1}{\omega_0+2}\mathbf{p}$$
$$- 2\frac{3\omega_0+4}{\omega_0+2}\mathbf{U}_{,00} - \left(4 + \frac{\omega_1\Psi}{(2\omega_0+3)(\omega_0+2)^2}\right)\mathbf{U}_{,i}\mathbf{U}_{,i}\,.$$

Fourth-order solution and PPN parameters

- Gauge-invariant equation for metric component $\overset{4}{\mathbf{g}}^{\star}$:

$$\triangle\overset{4}{\mathbf{g}}^{\star} = 8\pi\left(\frac{3}{\omega_0+2} + \frac{\omega_1\Psi}{(2\omega_0+3)(\omega_0+2)^2}\right)\rho\mathbf{U} - 8\pi\frac{2\omega_0+3}{\omega_0+2}\rho\mathbf{v}^2 - 8\pi\rho\mathbf{\Pi} - 24\pi\frac{\omega_0+1}{\omega_0+2}\mathbf{p}$$
$$- 2\frac{3\omega_0+4}{\omega_0+2}\mathbf{U}_{,00} - \left(4 + \frac{\omega_1\Psi}{(2\omega_0+3)(\omega_0+2)^2}\right)\mathbf{U}_{,i}\mathbf{U}_{,i}\,.$$

⇒ Solution in terms of PPN potentials:

$$\overset{4}{\mathbf{g}}^{\star} = \frac{3\omega_0+4}{\omega_0+2}(\mathfrak{A}+\mathfrak{B}) + \mathbf{\Phi}_1 + \left(\frac{4\omega_0+2}{\omega_0+2} - \frac{\omega_1\Psi}{(2\omega_0+3)(\omega_0+2)^2}\right)\mathbf{\Phi}_2 + 3\mathbf{\Phi}_3 + 6\frac{\omega_0+1}{\omega_0+2}\mathbf{\Phi}_4$$
$$- 2\left(1 + \frac{\omega_1\Psi}{4(2\omega_0+3)(\omega_0+2)^2}\right)\mathbf{U}^2\,.$$

Fourth-order solution and PPN parameters

- Gauge-invariant equation for metric component $\overset{4}{\mathbf{g}}^\star$:

$$\triangle\overset{4}{\mathbf{g}}^\star = 8\pi\left(\frac{3}{\omega_0+2} + \frac{\omega_1\Psi}{(2\omega_0+3)(\omega_0+2)^2}\right)\rho\mathbf{U} - 8\pi\frac{2\omega_0+3}{\omega_0+2}\rho\mathbf{v}^2 - 8\pi\rho\mathbf{\Pi} - 24\pi\frac{\omega_0+1}{\omega_0+2}\mathbf{p}$$
$$- 2\frac{3\omega_0+4}{\omega_0+2}\mathbf{U}_{,00} - \left(4 + \frac{\omega_1\Psi}{(2\omega_0+3)(\omega_0+2)^2}\right)\mathbf{U}_{,i}\mathbf{U}_{,i}\,.$$

⇒ Solution in terms of PPN potentials:

$$\overset{4}{\mathbf{g}}^\star = \frac{3\omega_0+4}{\omega_0+2}(\mathfrak{A}+\mathfrak{B}) + \mathbf{\Phi}_1 + \left(\frac{4\omega_0+2}{\omega_0+2} - \frac{\omega_1\Psi}{(2\omega_0+3)(\omega_0+2)^2}\right)\mathbf{\Phi}_2 + 3\mathbf{\Phi}_3 + 6\frac{\omega_0+1}{\omega_0+2}\mathbf{\Phi}_4$$
$$- 2\left(1 + \frac{\omega_1\Psi}{4(2\omega_0+3)(\omega_0+2)^2}\right)\mathbf{U}^2\,.$$

⇒ PPN parameters reproduce well-known result: [Nordtvedt '70]

$$\gamma = \frac{\omega_0+1}{\omega_0+2}\,, \quad \beta = 1 + \frac{\omega_1\Psi}{4(2\omega_0+3)(\omega_0+2)^2}\,, \quad \alpha_1 = \alpha_2 = \alpha_3 = \zeta_1 = \zeta_2 = \zeta_3 = \zeta_4 = \xi = 0\,.$$

Solution in tetrad formulation

- Express metric in terms of tetrad and solve for tetrad components.

Solution in tetrad formulation

- Express metric in terms of tetrad and solve for tetrad components.

⇒ Solution for tetrad perturbations:

$$\overset{2}{\theta}{}^{\star} = \mathbf{U}\,, \quad \overset{2}{\theta}{}^{\bullet} = \frac{\omega_0 + 1}{\omega_0 + 2}\mathbf{U}\,, \quad \overset{2}{\theta}{}^{\dagger}_{ij} = 0\,, \quad \overset{3}{\theta}{}^{\diamond}_{i} = -\frac{2\omega_0 + 3}{2\omega_0 + 4}(\mathbf{V}_i + \mathbf{W}_i)\,,$$

$$\overset{4}{\theta}{}^{\star} = \frac{3\omega_0 + 4}{2\omega_0 + 4}(\mathfrak{A} + \mathfrak{B}) - \left(\frac{1}{2} + \frac{\omega_1\Psi}{4(2\omega_0 + 3)(\omega_0 + 2)^2}\right)\mathbf{U}^2$$

$$+\frac{1}{2}\mathbf{\Phi}_1 + \left(\frac{2\omega_0 + 1}{\omega_0 + 2} - \frac{\omega_1\Psi}{2(2\omega_0 + 3)(\omega_0 + 2)^2}\right)\mathbf{\Phi}_2 + \frac{3}{2}\mathbf{\Phi}_3 + 3\frac{\omega_0 + 1}{\omega_0 + 2}\mathbf{\Phi}_4\,.$$

Solution in tetrad formulation

- Express metric in terms of tetrad and solve for tetrad components.

⇒ Solution for tetrad perturbations:

$$\overset{2}{\theta}{}^{\star} = \mathbf{U}\,, \quad \overset{2}{\theta}{}^{\bullet} = \frac{\omega_0+1}{\omega_0+2}\mathbf{U}\,, \quad \overset{2}{\theta}{}^{\dagger}_{ij} = 0\,, \quad \overset{3}{\theta}{}^{\diamond}_{i} = -\frac{2\omega_0+3}{2\omega_0+4}(\mathbf{V}_i+\mathbf{W}_i)\,,$$

$$\overset{4}{\theta}{}^{\star} = \frac{3\omega_0+4}{2\omega_0+4}(\mathfrak{A}+\mathfrak{B}) - \left(\frac{1}{2} + \frac{\omega_1\Psi}{4(2\omega_0+3)(\omega_0+2)^2}\right)\mathbf{U}^2$$
$$+\frac{1}{2}\mathbf{\Phi}_1 + \left(\frac{2\omega_0+1}{\omega_0+2} - \frac{\omega_1\Psi}{2(2\omega_0+3)(\omega_0+2)^2}\right)\mathbf{\Phi}_2 + \frac{3}{2}\mathbf{\Phi}_3 + 3\frac{\omega_0+1}{\omega_0+2}\mathbf{\Phi}_4\,.$$

✓ Obtain same PPN parameters as in metric formulation.

Solution in tetrad formulation

- Express metric in terms of tetrad and solve for tetrad components.

⇒ Solution for tetrad perturbations:

$$\overset{2}{\theta}{}^{\star} = \mathbf{U}\,, \quad \overset{2}{\theta}{}^{\bullet} = \frac{\omega_0 + 1}{\omega_0 + 2}\mathbf{U}\,, \quad \overset{2}{\theta}{}^{\dagger}_{ij} = 0\,, \quad \overset{3}{\theta}{}^{\diamond}_{i} = -\frac{2\omega_0 + 3}{2\omega_0 + 4}(\mathbf{V}_i + \mathbf{W}_i)\,,$$

$$\overset{4}{\theta}{}^{\star} = \frac{3\omega_0 + 4}{2\omega_0 + 4}(\mathfrak{A} + \mathfrak{B}) - \left(\frac{1}{2} + \frac{\omega_1 \Psi}{4(2\omega_0 + 3)(\omega_0 + 2)^2}\right)\mathbf{U}^2$$

$$+\frac{1}{2}\mathbf{\Phi}_1 + \left(\frac{2\omega_0 + 1}{\omega_0 + 2} - \frac{\omega_1 \Psi}{2(2\omega_0 + 3)(\omega_0 + 2)^2}\right)\mathbf{\Phi}_2 + \frac{3}{2}\mathbf{\Phi}_3 + 3\frac{\omega_0 + 1}{\omega_0 + 2}\mathbf{\Phi}_4\,.$$

✓ Obtain same PPN parameters as in metric formulation.

- Tetrad formulation is more useful in teleparallel gravity etc.

Outline

Summary

- Gauge-invariant perturbation theory:
  - Distinguish between physical and background spacetime.
  - Gauge pulls physical metric to background spacetime.

  ⇒ Gauge dependent comparison between both metrics.
  - Decompose perturbations into physical data and gauge data.

Summary

- Gauge-invariant perturbation theory:
  - Distinguish between physical and background spacetime.
  - Gauge pulls physical metric to background spacetime.
  - $\Rightarrow$ Gauge dependent comparison between both metrics.
  - Decompose perturbations into physical data and gauge data.
- Parametrized post-Newtonian formalism:
  - Weak-field approximation of metric gravity theories.
  - Characterizes gravity theories by 10 (constant) parameters.
  - Parameters closely related to solar system observations.

Summary

- Gauge-invariant perturbation theory:
  - Distinguish between physical and background spacetime.
  - Gauge pulls physical metric to background spacetime.
  - ⇒ Gauge dependent comparison between both metrics.
  - Decompose perturbations into physical data and gauge data.
- Parametrized post-Newtonian formalism:
  - Weak-field approximation of metric gravity theories.
  - Characterizes gravity theories by 10 (constant) parameters.
  - Parameters closely related to solar system observations.
- Gauge-invariant PPN formalism:
  - Apply gauge-invariant perturbation theory to PPN formalism.
  - Decomposition of metric and field equations.
  - Avoids issues arising from necessity to choose a gauge.
  - ⇒ Simpler set of equations to determine PPN parameters.
  - Can also be formulated for tetrad instead of metric.

Summary

- Gauge-invariant perturbation theory:
  - Distinguish between physical and background spacetime.
  - Gauge pulls physical metric to background spacetime.
  - $\Rightarrow$ Gauge dependent comparison between both metrics.
  - Decompose perturbations into physical data and gauge data.
- Parametrized post-Newtonian formalism:
  - Weak-field approximation of metric gravity theories.
  - Characterizes gravity theories by 10 (constant) parameters.
  - Parameters closely related to solar system observations.
- Gauge-invariant PPN formalism:
  - Apply gauge-invariant perturbation theory to PPN formalism.
  - Decomposition of metric and field equations.
  - Avoids issues arising from necessity to choose a gauge.
  - $\Rightarrow$ Simpler set of equations to determine PPN parameters.
  - Can also be formulated for tetrad instead of metric.
- Post-Newtonian limit of scalar-tensor gravity:
  - Perturbative field equations simplify in gauge-invariant formulation.
  - Consistency check: obtain well-known PPN parameters.
  - Also possible to use tetrad formulation to calculate solution.

Outlook

- Extend formalism by including higher perturbation orders:
  - General covariant expansion instead of space-time split.
  - Allow also for fast-moving source masses.
  - Consider inspiral phase of black hole merger event.
  - Devise method for calculating gravitational waves.

Outlook

- Extend formalism by including higher perturbation orders:
  - General covariant expansion instead of space-time split.
  - Allow also for fast-moving source masses.
  - Consider inspiral phase of black hole merger event.
  - Devise method for calculating gravitational waves.
- Further possible extensions and modifications:
  - Apply to formalism based on modified density $\rho^* = \rho\sqrt{-g}u^0$. [Will '18]
  - Allow for multiple dynamical metrics / tetrads.
  - Take into account Vainshtein or other screening mechanisms.
  - Consider cosmological background and time variability of PPN parameters.
  - Include further PPN potentials appearing in higher derivative theories.

Outlook

- Extend formalism by including higher perturbation orders:
  - General covariant expansion instead of space-time split.
  - Allow also for fast-moving source masses.
  - Consider inspiral phase of black hole merger event.
  - Devise method for calculating gravitational waves.
- Further possible extensions and modifications:
  - Apply to formalism based on modified density $\rho^* = \rho\sqrt{-g}u^0$. [Will '18]
  - Allow for multiple dynamical metrics / tetrads.
  - Take into account Vainshtein or other screening mechanisms.
  - Consider cosmological background and time variability of PPN parameters.
  - Include further PPN potentials appearing in higher derivative theories.
- Apply formalism to complicated gravity theories:
  - Bimetric and multimetric gravity theories.
  - Multi-scalar Horndeski generalizations.
  - Theories involving generalized Proca fields.
  - Extensions based on metric-affine geometry.
  - Extensions of teleparallel and symmetric teleparallel gravity.

Literature

Further reading

MH,
"Gauge invariant approach to the parametrized post-Newtonian formalism",
arXiv:1910.09245 [gr-qc].

Literature

Further reading

MH,
"Gauge invariant approach to the parametrized post-Newtonian formalism",
arXiv:1910.09245 [gr-qc].

One-sentence summary

The gauge-invariant approach provides a significant simplification of the PPN formalism.