Gauge-invariant approach to the parameterized post-Newtonian formalism and the post-Newtonian limit of teleparallel gravity

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Outline

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- Open questions in gravity theory and cosmology:
  - Accelerating expansion of the universe.
  - Homogeneity of cosmic microwave background.
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  - Conservation of total energy-momentum.
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$\Rightarrow$ Metric theories of gravity.

Idea

- Parametrized post-Newtonian formalism:
  - Weak-field approximation of metric gravity theories.
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⇝ Improvements presented here:
  - Use gauge-invariant higher order perturbation theory.
  - Allow for tetrad instead of metric as fundamental field.

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- Reference spacetime:
  - Manifold M_0 with metric $g^{(0)}$ and coordinates (x^μ).
  - Usually some highly symmetric standard spacetime:
    - maximally symmetric spacetime: Minkowski, (anti-)de Sitter
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2. No possibility to compare g and $g^{(0)}$: different manifolds.

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2. Comparison between reference and physical metric:
   - Define pullback ${}^{\mathcal{X}}g = \mathcal{X}^*g$ of the metric g to M_0.
   - ${}^{\mathcal{X}}g$ and $g^{(0)}$ are tensors on the same manifold M_0.

Gauge and perturbations

- Parameter dependent physical metric:
  - Assume physical metric $g \equiv g_\epsilon$ depends on parameter $\epsilon \in \mathbb{R}$.
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$$ {}^{\mathcal{X}}g_\epsilon = \sum_{k=0}^{\infty} \frac{\epsilon^k}{k!} \left.\frac{\partial^k\, {}^{\mathcal{X}}g_\epsilon}{\partial\epsilon^k}\right|_{\epsilon=0} = \sum_{k=0}^{\infty} \frac{\epsilon^k}{k!}\, {}^{\mathcal{X}}g^{(k)}\,. $$

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  - Series coefficients ${}^{\mathcal{X}}g^{(k)}$ depend on gauge choice $\mathcal{X}$.

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⇒ Vector fields $\xi^{(k)}$ are “Taylor expansion” coefficients of Φ_ϵ.

Gauge transformation of tensor fields

- Metrics in different gauges are related:

$$^{\mathcal{Y}}g_{\epsilon} = \sum_{l_1=0}^{\infty} \cdots \sum_{l_k=0}^{\infty} \cdots \frac{\epsilon^{l_1+\cdots+kl_k+\cdots}}{(1!)^{l_1}\cdots(k!)^{l_k}\cdots l_1!\cdots l_k!\cdots} \mathfrak{L}_{\xi_{(1)}}^{l_1}\cdots\mathfrak{L}_{\xi_{(k)}}^{l_k}\cdots {}^{\mathcal{X}}g_{\epsilon}\,.$$

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⇒ Relation of Taylor coefficients:

$$^{\mathcal{Y}}g^{(k)} = \sum_{0\leq l_1+2l_2+\ldots\leq k} \frac{k!}{(k-l_1-2l_2-\ldots)!(1!)^{l_1}(2!)^{l_2}\cdots l_1!l_2!\cdots} \mathfrak{L}_{\xi_{(1)}}^{l_1} \cdots \mathfrak{L}_{\xi_{(k)}}^{l_k} \cdots {}^{\mathcal{X}}g^{(k-l_1-2l_2-\ldots)}\,.$$

Gauge invariant perturbations

- Choose a fixed “distinguished” gauge $\mathcal{S}_\epsilon : M_0 \to M_\epsilon$:
  - E.g., impose gauge conditions on the metric.
  - Examples: harmonic gauge used for GWs, standard PPN gauge.

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- Number # of independent components:

$$ \#({}^{\mathcal{X}}g_\epsilon) = \#(\mathbf{g}_\epsilon) + \#(X_{(k)}) \,. $$

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Post-Newtonian matter and velocity orders

- Energy-momentum tensor of a perfect fluid:

$$T^{\mu\nu} = (\rho + \rho\Pi + p)u^\mu u^\nu + pg^{\mu\nu}\,.$$

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  - Specific internal energy Π.
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- Universe rest frame and slow-moving source matter:
  - Consider some gauge $\mathcal{X} : M_0 \to M$ ("universe rest frame").
  - Pullback of metric and matter variables along $\mathcal{X}$.
  - Velocity of the source matter: ${}^{\mathcal{X}}v^i = {}^{\mathcal{X}}u^i / {}^{\mathcal{X}}u^0$.
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- Quasi-static: assign additional $\mathcal{O}(1)$ to time derivatives ∂_0.

Post-Newtonian metric

- Standard post-Newtonian metric expansion:

$$^{\mathcal{X}}g_{\mu\nu} = {}^{\mathcal{X}}\overset{0}{g}_{\mu\nu} + {}^{\mathcal{X}}\overset{1}{g}_{\mu\nu} + {}^{\mathcal{X}}\overset{2}{g}_{\mu\nu} + {}^{\mathcal{X}}\overset{3}{g}_{\mu\nu} + {}^{\mathcal{X}}\overset{4}{g}_{\mu\nu} + \mathcal{O}(5)\,.$$

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- Higher than fourth velocity order $\mathcal{O}(4)$ is not considered.

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$$^{\mathcal{X}}g_{\mu\nu} = {}^{\mathcal{X}}\overset{0}{g}_{\mu\nu} + {}^{\mathcal{X}}\overset{1}{g}_{\mu\nu} + {}^{\mathcal{X}}\overset{2}{g}_{\mu\nu} + {}^{\mathcal{X}}\overset{3}{g}_{\mu\nu} + {}^{\mathcal{X}}\overset{4}{g}_{\mu\nu} + \mathcal{O}(5)\,.$$

- Note difference in notation: ${}^{\mathcal{X}}\overset{k}{g} = {}^{\mathcal{X}}g^{(k)}\epsilon^k/k!$.
- Background metric given by Minkowski metric: ${}^{\mathcal{X}}\overset{0}{g}_{\mu\nu} = \eta_{\mu\nu}$.
- Higher than fourth velocity order $\mathcal{O}(4)$ is not considered.
- Only certain components are relevant and non-vanishing:

$${}^{\mathcal{X}}\overset{2}{g}_{00}\,, \quad {}^{\mathcal{X}}\overset{2}{g}_{ij}\,, \quad {}^{\mathcal{X}}\overset{3}{g}_{0i}\,, \quad {}^{\mathcal{X}}\overset{4}{g}_{00}\,, \quad {}^{\mathcal{X}}\overset{4}{g}_{ij}\,.$$

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- $\mathcal{X}\overset{4}{g}_{ij}$ not used in standard PPN formalism, but may couple.

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{}^{\mathcal{P}}\overset{4}{g}_{00} &= -2\beta\,{}^{\mathcal{P}}U^2 + (2 + 2\gamma + \alpha_3 + \zeta_1 - 2\xi)\,{}^{\mathcal{P}}\Phi_1 + 2(1 + 3\gamma - 2\beta + \zeta_2 + \xi)\,{}^{\mathcal{P}}\Phi_2 \\
&\quad + 2(1 + \zeta_3)\,{}^{\mathcal{P}}\Phi_3 + 2(3\gamma + 3\zeta_4 - 2\xi)\,{}^{\mathcal{P}}\Phi_4 - 2\xi\,{}^{\mathcal{P}}\Phi_W - (\zeta_1 - 2\xi)\,{}^{\mathcal{P}}\mathfrak{A}\,,
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  - Fourth-order temporal part ${}^{\mathcal{P}}\overset{4}{g}_{00}$ does not contain potential $\mathfrak{B}$.

PPN parameters

- PPN parameters are linked to physical properties:
  - γ: spatial curvature genrated by unit mass.
  - β: non-linearity in gravity superposition law.
  - $\alpha_1, \alpha_2, \alpha_3$: violation of local Lorentz invariance.
  - $\alpha_3, \zeta_1, \zeta_2, \zeta_3, \zeta_4$: violation of energy-momentum conservation.
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$\Rightarrow$ Fully conservative gravity theory:

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⇒ Other theories will receive bounds from experiments.

Experimental bounds

Par.	Bound	Effects	Experiment
$\gamma - 1$	$2.3 \cdot 10^{-5}$	Time delay, light deflection	Cassini tracking
$\beta - 1$	$8 \cdot 10^{-5}$	Perihelion shift	Perihelion shift
ξ	$4 \cdot 10^{-9}$	Spin precession	Millisecond pulsars
α_1	10^{-4}	Orbital polarization	Lunar laser ranging
α_1	$4 \cdot 10^{-5}$	Orbital polarization	PSR J1738+0333
α_2	$2 \cdot 10^{-9}$	Spin precession	Millisecond pulsars
α_3	$4 \cdot 10^{-20}$	Self-acceleration	Pulsar spin-down statistics
η_N[1]	$9 \cdot 10^{-4}$	Nordtvedt effect	Lunar Laser Ranging
ζ_1	0.02	Combined PPN bounds	—
ζ_2	$4 \cdot 10^{-5}$	Binary pulsar acceleration	PSR 1913+16
ζ_3	10^{-8}	Newton's 3rd law	Lunar acceleration
ζ_4	0.006	—	Kreuzer experiment

[1] $\eta_N = 4\beta - \gamma - 3 - \frac{10}{3}\xi - \alpha_1 + \frac{2}{3}\alpha_2 - \frac{2}{3}\zeta_1 - \frac{1}{3}\zeta_2$

PPN potentials

- Newtonian potential:

$$\overset{\mathcal{X}}{\chi} = -\int d^3x' \, \overset{\mathcal{X}}{\rho}'|\vec{x} - \vec{x}'| \,, \quad \overset{\mathcal{X}}{U} = \int d^3x' \frac{\overset{\mathcal{X}}{\rho}'}{|\vec{x} - \vec{x}'|} \,, \quad \overset{\mathcal{X}}{\rho}' \equiv \overset{\mathcal{X}}{\rho}(t, \vec{x}') \,.$$

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- Vector potentials:

$$\overset{\mathcal{X}}{V}_i = \int d^3x' \frac{\overset{\mathcal{X}}{\rho}' \overset{\mathcal{X}}{v}'_i}{|\vec{x} - \vec{x}'|} \, , \quad \overset{\mathcal{X}}{W}_i = \int d^3x' \frac{\overset{\mathcal{X}}{\rho}' \overset{\mathcal{X}}{v}'_j (x_i - x'_i)(x_j - x'_j)}{|\vec{x} - \vec{x}'|^3} \, .$$

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- Fourth-order scalar potentials:

$$\overset{\mathcal{X}}{\Phi}_1 = \int d^3x' \frac{\overset{\mathcal{X}}{\rho}' \overset{\mathcal{X}}{v}'^2}{|\vec{x} - \vec{x}'|} \,, \quad \overset{\mathcal{X}}{\Phi}_4 = \int d^3x' \frac{\overset{\mathcal{X}}{p}'}{|\vec{x} - \vec{x}'|} \,,$$

$$\overset{\mathcal{X}}{\Phi}_2 = \int d^3x' \frac{\overset{\mathcal{X}}{\rho}' \overset{\mathcal{X}}{U}'}{|\vec{x} - \vec{x}'|} \,, \quad \overset{\mathcal{X}}{\mathfrak{A}} = \int d^3x' \frac{\overset{\mathcal{X}}{\rho}' \left[\overset{\mathcal{X}}{v}'_i (x_i - x'_i)\right]^2}{|\vec{x} - \vec{x}'|^3} \,,$$

$$\overset{\mathcal{X}}{\Phi}_3 = \int d^3x' \frac{\overset{\mathcal{X}}{\rho}' \overset{\mathcal{X}}{\Pi}'}{|\vec{x} - \vec{x}'|} \,, \quad \overset{\mathcal{X}}{\mathfrak{B}} = \int d^3x' \frac{\overset{\mathcal{X}}{\rho}'}{|\vec{x} - \vec{x}'|} (x_i - x'_i) \frac{d \overset{\mathcal{X}}{v}'_i}{dt} \,,$$

$$\overset{\mathcal{X}}{\Phi}_W = \int d^3x' d^3x'' \, \overset{\mathcal{X}}{\rho}' \overset{\mathcal{X}}{\rho}'' \frac{x_i - x'_i}{|\vec{x} - \vec{x}'|^3} \left(\frac{x'_i - x''_i}{|\vec{x} - \vec{x}''|} - \frac{x_i - x''_i}{|\vec{x}' - \vec{x}''|} \right) .$$

Post-Newtonian field equations

- Expand energy-momentum tensor in velocity orders:

$$\begin{aligned}
{}^{\mathcal{X}}T_{00} &= {}^{\mathcal{X}}\rho\left(1 - {}^{\mathcal{X}}\overset{2}{g}_{00} + ({}^{\mathcal{X}}v)^2 + {}^{\mathcal{X}}\Pi\right) + \mathcal{O}(6)\,,\\
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- Behavior under gauge transformations:

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{}^{\mathcal{Y}}\overset{2}{T}_{00} = {}^{\mathcal{X}}\overset{2}{T}_{00}\,, \quad {}^{\mathcal{Y}}\overset{2}{T}_{ij} = {}^{\mathcal{X}}\overset{2}{T}_{ij}\,, \quad {}^{\mathcal{Y}}\overset{3}{T}_{0i} = {}^{\mathcal{X}}\overset{3}{T}_{0i}\,,\\
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⇝ Use gauge-invariant formalism to decouple equations.

Outline

Gauge transformation of the metric

- Allow only gauge transformations preserving PPN assumptions:

$$\overset{2}{\xi}_i\,,\quad \overset{3}{\xi}_0\,,\quad \overset{4}{\xi}_i\,.$$

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- Relation between metrics in gauges $\mathcal{X}$ and $\mathcal{Y}$:

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{}^{\mathcal{Y}}\overset{4}{g}_{ij} &= {}^{\mathcal{X}}\overset{4}{g}_{ij} + 2\partial_{(i}\overset{4}{\xi}_{j)} + 2\,{}^{\mathcal{X}}\overset{2}{g}_{k(i}\partial_{j)}\overset{2}{\xi}_k + \overset{2}{\xi}_k\partial_k\,{}^{\mathcal{X}}\overset{2}{g}_{ij} + \partial_{(i}\big(\overset{2}{\xi}_{|k}\partial_{k|}\overset{2}{\xi}_{j)}\big) + \partial_i\overset{2}{\xi}_k\partial_j\overset{2}{\xi}_k\,.
\end{aligned}$$

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\end{aligned}$$

- Use gauge transformation to eliminate metric components.

Gauge-invariant metric

- Definition of gauge-invariant metric components:

$$\mathbf{g}_{00} = \mathbf{g}^{\star}\,, \quad \mathbf{g}_{0i} = \mathbf{g}^{\diamond}_{i}\,, \quad \mathbf{g}_{ij} = \mathbf{g}^{\bullet}\delta_{ij} + \mathbf{g}^{\dagger}_{ij}\,.$$

Gauge-invariant metric

- Definition of gauge-invariant metric components:

$$\mathbf{g}_{00} = \mathbf{g}^{\star}\,, \quad \mathbf{g}_{0i} = \mathbf{g}^{\diamond}_{i}\,, \quad \mathbf{g}_{ij} = \mathbf{g}^{\bullet}\delta_{ij} + \mathbf{g}^{\dagger}_{ij}\,.$$

- Conditions imposed on components:

$$\partial^i\mathbf{g}^{\diamond}_{i} = 0\,, \quad \partial^i\mathbf{g}^{\dagger}_{ij} = 0\,, \quad \mathbf{g}^{\dagger}_{[ij]} = 0\,, \quad \mathbf{g}^{\dagger}_{ii} = 0\,.$$

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- Relation to arbitrary gauge $\mathcal{X}$:

$$\begin{aligned}
{}^{\mathcal{X}}\overset{2}{g}_{00} &= \overset{2}{\mathbf{g}}{}^{\star}\,,\\
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{}^{\mathcal{X}}\overset{3}{g}_{0i} &= \overset{3}{\mathbf{g}}{}^{\diamond}_{i} + \partial_i\overset{3}{X}{}^{\star} + \partial_0\partial_i\overset{2}{X}{}^{\blacklozenge} + \partial_0\overset{2}{X}{}^{\diamond}_{i}\,,\\
{}^{\mathcal{X}}\overset{4}{g}_{00} &= \overset{4}{\mathbf{g}}{}^{\star} + 2\partial_0\overset{3}{X}{}^{\star} + (\partial_i\overset{2}{X}{}^{\blacklozenge} + \overset{2}{X}{}^{\diamond}_{i})\partial_i\overset{2}{\mathbf{g}}{}^{\star}\,,\\
{}^{\mathcal{X}}\overset{4}{g}_{ij} &= \overset{4}{\mathbf{g}}{}^{\bullet}\delta_{ij} + \overset{4}{\mathbf{g}}{}^{\dagger}_{ij} + 2\partial_i\partial_j\overset{4}{X}{}^{\blacklozenge} + \mathcal{O}(2)\cdot\mathcal{O}(2)\,.
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- Gauge defining vector fields:

$$X_i = \partial_i X^{\blacklozenge} + X^{\diamond}_{i}\,, \quad X_0 = X^{\star}\,, \quad \partial^i X^{\diamond}_{i} = 0\,.$$

Decomposition of metric components

- Count number of independent components at each order:

total		invariant		pure gauge	
$\mathcal{X}\overset{2}{g}_{00}$	1	$\overset{2}{\mathbf{g}}{}^{\star}$	1	-	0
$\mathcal{X}\overset{2}{g}_{ij}$	6	$\overset{2}{\mathbf{g}}{}^{\star}, \overset{2}{\mathbf{g}}{}^{\dagger}_{ij}$	$1+2$	$\overset{2}{X}{}^{\blacklozenge}, \overset{2}{X}{}^{\diamond}_{i}$	$1+2$
$\mathcal{X}\overset{3}{g}_{0i}$	3	$\overset{3}{\mathbf{g}}{}^{\diamond}_{i}$	2	$\overset{3}{X}{}^{\star}$	1
$\mathcal{X}\overset{4}{g}_{00}$	1	$\overset{4}{\mathbf{g}}{}^{\star}$	1	-	0
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$\Rightarrow$ Components split into invariant and gauge parts.

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⇒ Components split into invariant and gauge parts.

⇒ Possible to separate physical information from coordinate choice.

Relation to standard PPN gauge

- Use relation between expansion coefficients:

$$ {}^{\mathcal{P}}\overset{k}{g} = \sum_{0 \leq l_1 + 2l_2 + \ldots \leq k} \frac{1}{l_1!\,l_2!\ldots} \pounds^{l_1}_{\overset{1}{P}} \cdots \pounds^{l_k}_{\overset{k}{P}} \cdots \overset{k-l_1-2l_2-\ldots}{\mathbf{g}}\,. $$

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$$\overset{2}{P}{}^{\blacklozenge} = 0 , \quad \overset{2}{P}{}^{\diamond}_{i} = 0 , \quad \overset{3}{P}{}^{\star} = -\frac{1}{4}(2 + 4\gamma + \alpha_1 - 2\alpha_2 + 2\zeta_1 - 4\xi)\chi_{,0} .$$

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$$\overset{2}{\mathbf{g}}{}^{\star} = 2\mathbf{U}\,, \quad \overset{2}{\mathbf{g}}{}^{\bullet} = 2\gamma\mathbf{U}\,, \quad \overset{2}{\mathbf{g}}{}^{\dagger}_{ij} = 0\,, \quad \overset{3}{\mathbf{g}}{}^{\diamond}_{i} = -\left(1 + \gamma + \frac{\alpha_1}{4}\right)(\mathbf{V}_i + \mathbf{W}_i)\,,$$

$$\begin{aligned}\overset{4}{\mathbf{g}}{}^{\star} = {} & \frac{1}{2}(2-\alpha_1+2\alpha_2+2\alpha_3)\mathbf{\Phi}_1+2(1+3\gamma-2\beta+\zeta_2+\xi)\mathbf{\Phi}_2+2(1+\zeta_3)\mathbf{\Phi}_3+2(3\gamma+3\zeta_4-2\xi)\mathbf{\Phi}_4 \\ & - 2\xi\mathbf{\Phi}_W - 2\beta\mathbf{U}^2 + \frac{1}{2}(2 + 4\gamma + \alpha_1 - 2\alpha_2)\mathfrak{A} + \frac{1}{2}(2 + 4\gamma + \alpha_1 - 2\alpha_2 + 2\zeta_1 - 4\xi)\mathfrak{B}\,.\end{aligned}$$

Gauge-invariant field equations

- Perform similar decomposition of energy-momentum tensor:

$$\begin{aligned}
\mathbf{T}^{\star} &= \mathbf{T}_{00} = \boldsymbol{\rho}\left(1 - \overset{2}{\mathbf{g}}_{00} + \mathbf{v}^2 + \mathbf{\Pi}\right) + \mathcal{O}(6)\,, \\
\mathbf{T}^{\diamond}_{i} + \partial_i \mathbf{T}^{\blacklozenge} &= \mathbf{T}_{0i} = -\boldsymbol{\rho}\mathbf{v}_i + \mathcal{O}(5)\,, \\
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- Express components in terms of PPN potentials:

$$\overset{2}{\mathbf{T}}{}^{\star} = \boldsymbol{\rho} = -\frac{1}{4\pi}\triangle\mathbf{U}\,, \quad \overset{3}{\mathbf{T}}{}^{\blacklozenge} = -\frac{1}{4\pi}\partial_0\mathbf{U}\,, \quad \overset{3}{\mathbf{T}}{}^{\diamond}_{i} = \frac{1}{8\pi}\triangle(\mathbf{V}_i + \mathbf{W}_i)\,,$$

$$\overset{4}{\mathbf{T}}{}^{\star} = \boldsymbol{\rho}\left(\mathbf{\Pi} + \mathbf{v}^2 - \overset{2}{\mathbf{g}}{}^{\star}\right) = -\frac{1}{4\pi}\triangle(\mathbf{\Phi}_3 + \mathbf{\Phi}_1 - 2\mathbf{\Phi}_2)\,,$$

$$\overset{4}{\mathbf{T}}{}^{\bullet} = \frac{1}{3}\boldsymbol{\rho}\mathbf{v}^2 + \mathbf{p} = -\frac{1}{12\pi}\triangle(\mathbf{\Phi}_1 + 3\mathbf{\Phi}_4)\,, \quad \overset{4}{\mathbf{T}}{}^{\blacktriangle} = \frac{1}{16\pi}(3\mathfrak{A} - \mathbf{\Phi}_1)\,.$$

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$\Rightarrow$ Find PPN parameters by comparing coefficients on both sides.

Outline

Teleparallel geometry

- Fundamental fields:
  - Coframe field $\theta^A = \theta^A{}_\mu \mathrm{d}x^\mu$.
  - Flat spin connection $\omega^A{}_B = \omega^A{}_{B\mu} \mathrm{d}x^\mu$.
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- Derived quantities:
  - Frame field $e_A = e_A{}^\mu \partial_\mu$ with $e_A{}^\mu \theta^B{}_\mu = \delta^B_A$ and $e_A{}^\mu \theta^A{}_\nu = \delta^\mu{}_\nu$.
  - Metric $g_{\mu\nu} = \eta_{AB} \theta^A{}_\mu \theta^B{}_\nu$.
  - Determinant $\theta = \det(\theta^A{}_\mu)$.
  - Teleparallel connection $\Gamma^\mu{}_{\nu\rho} = e_A{}^\mu (\partial_\rho \theta^A{}_\nu + \omega^A{}_{B\rho} \theta^B{}_\nu)$.
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- Properties of the teleparallel connection:
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  - Vanishing nonmetricity: $Q_{\mu\nu\rho} = \nabla_\mu g_{\nu\rho} = 0$.
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$\Rightarrow$ Possible to use Weitzenböck gauge: $\omega^A{}_{B\mu} \equiv 0$.

Post-Newtonian tetrad

- Post-Newtonian tetrad expansion:

$$\mathcal{X}\theta^A{}_\mu = \mathcal{X}\overset{0}{\theta}{}^A{}_\mu + \mathcal{X}\overset{1}{\theta}{}^A{}_\mu + \mathcal{X}\overset{2}{\theta}{}^A{}_\mu + \mathcal{X}\overset{3}{\theta}{}^A{}_\mu + \mathcal{X}\overset{4}{\theta}{}^A{}_\mu + \mathcal{O}(5)\,.$$

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- Only certain components are relevant and non-vanishing:

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- Background tetrad is diagonal: $\chi\overset{0}{\theta}{}^A{}_\mu = \Delta^A{}_\mu = \mathrm{diag}(1,1,1,1)$.
- Higher than fourth velocity order $\mathcal{O}(4)$ is not considered.
- Useful to define tetrad with lower spacetime indices:

$$\chi\overset{k}{\theta}{}_{\mu\nu} = \Delta^A{}_\mu \eta_{AB}\,\chi\overset{k}{\theta}{}^B{}_\nu\,.$$

- Only certain components are relevant and non-vanishing:

$$\chi\overset{2}{\theta}{}_{00}\,,\quad \chi\overset{2}{\theta}{}_{ij}\,,\quad \chi\overset{3}{\theta}{}_{0i}\,,\quad \chi\overset{4}{\theta}{}_{00}\,,\quad \chi\overset{4}{\theta}{}_{ij}\,.$$

- Relation to metric components:

$$\chi\overset{2}{g}{}_{00} = 2\,\chi\overset{2}{\theta}{}_{00}\,,\quad \chi\overset{2}{g}{}_{ij} = 2\,\chi\overset{2}{\theta}{}_{(ij)}\,,\quad \chi\overset{3}{g}{}_{0i} = 2\,\chi\overset{3}{\theta}{}_{(i0)}\,,$$

$$\chi\overset{4}{g}{}_{00} = -(\chi\overset{2}{\theta}{}_{00})^2 + 2\,\chi\overset{4}{\theta}{}_{00}\,,\quad \chi\overset{4}{g}{}_{ij} = 2\,\chi\overset{4}{\theta}{}_{(ij)} + \chi\overset{2}{\theta}{}_{ki}\,\chi\overset{2}{\theta}{}_{kj}\,.$$

Sketch of gauge-invariant tetrad PPN

1. Split tetrad perturbations in symmetric and antisymmetric:

$$\mathcal{X}\overset{k}{\theta}_{\mu\nu} = \mathcal{X}\overset{k}{s}_{\mu\nu} + \mathcal{X}\overset{k}{a}_{\mu\nu}\,, \quad \mathcal{X}\overset{k}{s}_{\mu\nu} = \mathcal{X}\overset{k}{\theta}_{(\mu\nu)}\,, \quad \mathcal{X}\overset{k}{a}_{\mu\nu} = \mathcal{X}\overset{k}{\theta}_{[\mu\nu]}\,.$$

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6. Find PPN parameters by matching terms on both sides.

Outline

$\mathcal{F}(\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3)$ - action and field equations

- Gravitational part of the action:

$$S_g[\theta, \omega] = \frac{1}{2\kappa^2} \int_M \mathcal{F}(\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3)\, \theta\, \mathrm{d}^4 x\,.$$

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$$\mathcal{T}_1 = T^{\mu\nu\rho} T_{\mu\nu\rho}\,, \quad \mathcal{T}_2 = T^{\mu\nu\rho} T_{\rho\nu\mu}\,, \quad \mathcal{T}_3 = T^{\mu}{}_{\mu\rho} T_{\nu}{}^{\nu\rho}\,.$$

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$$\kappa^2 \Theta_{\mu\nu} = \frac{1}{2}\mathcal{F} g_{\mu\nu} + 2\overset{\circ}{\nabla}{}^{\rho}\left(\mathcal{F}_{,1} T_{\nu\mu\rho} + \mathcal{F}_{,2} T_{[\rho\mu]\nu} + \mathcal{F}_{,3} T^{\sigma}{}_{\sigma[\rho} g_{\mu]\nu}\right) + \mathcal{F}_{,1} T^{\rho\sigma}{}_{\mu}\left(T_{\nu\rho\sigma} - 2T_{[\rho\sigma]\nu}\right)$$
$$+ \frac{1}{2}\mathcal{F}_{,2}\left[T_{\mu}{}^{\rho\sigma}\left(2T_{\rho\sigma\nu} - T_{\nu\rho\sigma}\right) + T^{\rho\sigma}{}_{\mu}\left(2T_{[\rho\sigma]\nu} - T_{\nu\rho\sigma}\right)\right] - \frac{1}{2}\mathcal{F}_{,3} T^{\sigma}{}_{\sigma\rho}\left(T^{\rho}{}_{\mu\nu} + 2T_{(\mu\nu)}{}^{\rho}\right),$$

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- Use $\Theta_{\mu\nu}$ instead of $T_{\mu\nu}$ to avoid confusion with torsion.

$\mathcal{F}(\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3)$ - PPN parameters

- PPN parameters:

$$\beta - 1 = -\frac{\varepsilon}{2}\,, \quad \gamma - 1 = -2\varepsilon\,, \quad \varepsilon = \frac{2F_{,1} + F_{,2} + F_{,3}}{2(2F_{,1} + F_{,2} + 2F_{,3})}\,,$$

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- Bounds on theory parameters from Cassini tracking:

$$\gamma - 1 = -2\varepsilon = (2.1 \pm 2.3) \cdot 10^{-5}\,.$$

$\mathcal{F}(\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3)$ - axial-vector-tensor decomposition

- Irreducible decomposition of torsion components:

$$T_{\text{ax}} = \frac{1}{18}(\mathcal{T}_1 - 2\mathcal{T}_2)\,, \quad T_{\text{ten}} = \frac{1}{2}(\mathcal{T}_1 + \mathcal{T}_2 - \mathcal{T}_3)\,, \quad T_{\text{vec}} = \mathcal{T}_3\,.$$

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$$\varepsilon = \frac{G_{,v} + G_{,t}}{4G_{,v} + G_{,t}}\,.$$

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$\Rightarrow$ Purely axial modifications do not affect PPN parameters.

$\mathcal{F}(\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3)$ - particular theories

- New general relativity [Hayashi, Shirafuji '79]:
  - Most general action linear in torsion scalars:

$$\mathcal{F}(\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3) = t_1\mathcal{T}_1 + t_2\mathcal{T}_2 + t_3\mathcal{T}_3 \,.$$

⇒ Taylor coefficients given by $F_{,i} = t_i$, $i = 1, \ldots, 3$.

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$$\varepsilon = \frac{2t_1 + t_2 + t_3}{2(2t_1 + t_2 + 2t_3)} \,.$$

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- $f(T)$ gravity theories [Bengochea, Ferraro '08; Linder '10]:
  - Lagrangian defined as function of linear combination:

$$\mathcal{F}(\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3) = f(T) \,, \quad T = \frac{1}{4}\mathcal{T}_1 + \frac{1}{2}\mathcal{T}_2 - \mathcal{T}_3 \,.$$

⇒ Taylor coefficients given by:

$$F_{,1} = \frac{1}{4}f'(0) \,, \quad F_{,2} = \frac{1}{2}f'(0) \,, \quad F_{,3} = -f'(0)$$

⇒ Indistinguishable from GR, since $\varepsilon \equiv 0$.

Scalar-torsion - action

- Gravitational part of the action:

$$S_g[\theta,\omega,\phi] = \frac{1}{2\kappa^2}\int_M \left[-\mathcal{A}(\phi)T + 2\mathcal{B}(\phi)X + 2\mathcal{C}(\phi)Y - 2\kappa^2\mathcal{V}(\phi)\right]\theta\,\mathrm{d}^4x\,.$$

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- Scalar quantities appearing in the action:

$$T = \frac{1}{2}T^{\rho}{}_{\mu\nu}S_{\rho}{}^{\mu\nu}\,,\quad X = -\frac{1}{2}g^{\mu\nu}\phi_{,\mu}\phi_{,\nu}\,,\quad Y = g^{\mu\nu}T^{\rho}{}_{\rho\mu}\phi_{,\nu}\,.$$

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- Superpotential:

$$S_{\rho\mu\nu} = \frac{1}{2} \left(T_{\nu\mu\rho} + T_{\rho\mu\nu} - T_{\mu\nu\rho} \right) - g_{\rho\mu} T^{\sigma}{}_{\sigma\nu} + g_{\rho\nu} T^{\sigma}{}_{\sigma\mu} \,.$$

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- Taylor expansion of functions around background $\overset{0}{\phi} = \Phi$:

$$A = \mathcal{A}(\Phi) \,, \quad A' = \mathcal{A}'(\Phi) \,, \quad A'' = \mathcal{A}''(\Phi) \,, \quad A''' = \mathcal{A}'''(\Phi) \,, \quad \ldots$$

Scalar-torsion - field equations

- Field equations:
  - Tetrad field equations - symmetric part:

$$\kappa^2\Theta_{\mu\nu} = (\mathcal{A}' + \mathcal{C})\, S_{(\mu\nu)}{}^{\rho}\phi_{,\rho} + \mathcal{A}\left(\overset{\circ}{R}_{\mu\nu} - \frac{1}{2}\overset{\circ}{R}g_{\mu\nu}\right) - (\mathcal{B} - \mathcal{C}')\phi_{,\mu}\phi_{,\nu} + \left(\frac{1}{2}\mathcal{B} - \mathcal{C}'\right)\phi_{,\rho}\phi_{,\sigma}g^{\rho\sigma}g_{\mu\nu} + \mathcal{C}\left(\overset{\circ}{\nabla}_{\mu}\overset{\circ}{\nabla}_{\nu}\phi - \overset{\circ}{\square}\phi g_{\mu\nu}\right) + \kappa^2\mathcal{V}g_{\mu\nu}\,.$$

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$$+ \left(\frac{1}{2}\mathcal{B} - \mathcal{C}'\right)\phi_{,\rho}\phi_{,\sigma}g^{\rho\sigma}g_{\mu\nu} + \mathcal{C}\left(\overset{\circ}{\nabla}_{\mu}\overset{\circ}{\nabla}_{\nu}\phi - \overset{\circ}{\square}\phi g_{\mu\nu}\right) + \kappa^2\mathcal{V}g_{\mu\nu}\,.$$

  - Tetrad field equations - antisymmetric part:

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$$0 = (\mathcal{A}' + \mathcal{C})\, T^{\rho}{}_{[\mu\nu}\phi_{,\rho]}\,.$$

  - Scalar field equation:

$$0 = \frac{1}{2}\mathcal{A}'T - \mathcal{B}\overset{\circ}{\Box}\phi - \frac{1}{2}\mathcal{B}'g^{\mu\nu}\phi_{,\mu}\phi_{,\nu} + \mathcal{C}\overset{\circ}{\nabla}_{\mu}T_{\nu}{}^{\nu\mu} + \kappa^2\mathcal{V}'\,.$$

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- Field equations:
  - Tetrad field equations - symmetric part:

$$\kappa^2\Theta_{\mu\nu} = (\mathcal{A}' + \mathcal{C})\, S_{(\mu\nu)}{}^{\rho}\phi_{,\rho} + \mathcal{A}\left(\overset{\circ}{R}_{\mu\nu} - \frac{1}{2}\overset{\circ}{R}g_{\mu\nu}\right) - (\mathcal{B} - \mathcal{C}')\phi_{,\mu}\phi_{,\nu} + \left(\frac{1}{2}\mathcal{B} - \mathcal{C}'\right)\phi_{,\rho}\phi_{,\sigma}g^{\rho\sigma}g_{\mu\nu} + \mathcal{C}\left(\overset{\circ}{\nabla}_{\mu}\overset{\circ}{\nabla}_{\nu}\phi - \overset{\circ}{\square}\phi g_{\mu\nu}\right) + \kappa^2\mathcal{V}g_{\mu\nu}\,.$$

  - Tetrad field equations - antisymmetric part:

$$0 = (\mathcal{A}' + \mathcal{C})\, T^{\rho}{}_{[\mu\nu}\phi_{,\rho]}\,.$$

  - Scalar field equation:

$$0 = \frac{1}{2}\mathcal{A}'T - \mathcal{B}\,\overset{\circ}{\square}\,\phi - \frac{1}{2}\mathcal{B}'g^{\mu\nu}\phi_{,\mu}\phi_{,\nu} + \mathcal{C}\overset{\circ}{\nabla}_{\mu}T_{\nu}{}^{\nu\mu} + \kappa^2\mathcal{V}'\,.$$

- Helpful steps to decouple / simplify:
  - Trace reversal: $\overset{\circ}{R}_{\mu\nu} - \frac{1}{2}\overset{\circ}{R}g_{\mu\nu} \to \overset{\circ}{R}_{\mu\nu}$.

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  - Eliminate second-order tetrad derivatives using tetrad equations.

Scalar-torsion - simplified field equations

- Trace-reversed symmetric tetrad field equations:

$$\bar{\Theta}_{\mu\nu} = (\mathcal{A}' + \mathcal{C})\left(S_{(\mu\nu)}{}^{\rho} + g_{\mu\nu}T_{\chi}{}^{\chi\rho}\right)\phi_{,\rho} + \mathcal{A}\overset{\circ}{R}_{\mu\nu} + \frac{1}{2}\mathcal{C}'g_{\mu\nu}\phi_{,\rho}\phi_{,\sigma}g^{\rho\sigma} - (\mathcal{B} - \mathcal{C}')\phi_{,\mu}\phi_{,\nu} + \mathcal{C}\overset{\circ}{\nabla}_{\mu}\overset{\circ}{\nabla}_{\nu}\phi + \frac{1}{2}\mathcal{C}\overset{\circ}{\square}\phi g_{\mu\nu} - \kappa^2\mathcal{V}g_{\mu\nu}$$

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- Effective gravitational constant:

$$G_{\text{eff}} = \frac{\kappa^2}{8\pi A}\left(1 + \frac{C^2 e^{-m_\phi r}}{2AB + 3C^2}\right)$$

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⇒ Corrections vanish for infinite mass $m_\phi \to \infty$.

Scalar-torsion - PPN parameters (massless case)

- PPN parameters:

$$\gamma = 1 - \frac{C^2}{AB + 2C^2}\,,$$

$$\beta = 1 - \frac{C\{6C^4(C + A') + ABC^2(7C + 8A') + A^2[2B^2(C + A') + B'C^2 - 2BCC']\}}{4(AB + 2C^2)^2(2AB + 3C^2)}\,,$$

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⇒ Identical to GR $\gamma = \beta = 1$ for $C = 0$.

- Choose non-minimal coupling function $\mathcal{C} = -\mathcal{A}'$.

Scalar-torsion - equivalent of scalar-curvature gravity

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$\Rightarrow$ Action functional after integration by parts [MH '18]:

$$S_g = \frac{1}{2\kappa^2} \int_M \left[\mathcal{A}(\phi) \overset{\circ}{R} + 2\mathcal{B}(\phi) X - 2\kappa^2 \mathcal{V}(\phi) \right] \theta \mathrm{d}^4 x \,.$$

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- Re-obtain well-known PPN parameters as consistency check.

Scalar-torsion - minimally coupled theories

- Numerous minimally coupled ($\mathcal{C} = 0$) theories:
  - Teleparallel dark energy [Geng, Lee, Saridakis, Wu '11]:

$$\mathcal{A} = 1 + 2\kappa^2\xi\phi^2\,, \quad \mathcal{B} = -\kappa^2\,.$$

  - Interacting dark energy [Otalora '13]:

$$\mathcal{A} = 1 + 2\kappa^2\xi F(\phi)\,, \quad \mathcal{B} = -\kappa^2\,.$$

  - Brans-Dicke type with a general coupling to torsion [Izumi, Gu, Ong '13]:

$$\mathcal{A} = F(\phi)\,, \quad \mathcal{B} = 2\kappa^2\omega\,.$$

  - Brans-Dicke type with a dynamical kinetic term [Chen, Wu, Wei '14]:

$$\mathcal{A} = \phi\,, \quad \mathcal{B} = 2\kappa^2\omega(\phi)/\phi\,.$$

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⇒ Theories are indistinguishable from GR by PPN parameters.

Scalar-torsion - non-minimally coupled boundary term

- Action functional [Bahamonde, Wright '15]:

$$S_g = \int_M \left[-\frac{T}{2\kappa^2} + \frac{1}{2}\left(g^{\mu\nu}\phi_{,\mu}\phi_{,\nu} - \xi\phi^2 T - \chi\phi^2 B\right) - \mathcal{V}(\phi) \right] \theta \mathrm{d}^4 x \, .$$

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⇒ Depends on background value Φ (determined from potential $\mathcal{V}$).

Outline

Summary

- Gauge-invariant perturbation theory:
  - Distinguish between physical and background spacetime.
  - Gauge pulls physical metric to background spacetime.
  - $\Rightarrow$ Gauge dependent comparison between both metrics.
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- Post-Newtonian limit of teleparallel gravity theories:
  - Obtained PPN parameters for different teleparallel theories.
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  - Large, widely used subclasses have same PPN limit as GR.

Outlook

- Extend formalism by including higher perturbation orders:
  - General covariant expansion instead of space-time split.
  - Allow also for fast-moving source masses.
  - Consider inspiral phase of black hole merger event.
  - Devise method for calculating gravitational waves.

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- Consider more general teleparallel gravity theories:
  - Theories with modified constitutive laws [MH, Järv, Krššák, Pfeifer '17].
  - Lagrangian as free function $L(T, X, Y, \phi)$[2] [MH '18].
  - Teleparallel extension to Horndeski gravity [Bahamonde, Dialektopoulos, Said '19].
  - Theories obtained from disformal transformations [MH '19].
  - Coupling of scalar fields to $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3$.
  - Theories with multiple tetrads.

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Literature

- MH,
  "Gauge invariant approach to the parametrized post-Newtonian formalism",
  to appear.
- U. Ualikhanova and MH,
  "Parameterized post-Newtonian limit of general teleparallel gravity theories",
  arXiv:1907.08178 [gr-qc].
- E. D. Emtsova and MH,
  "Post-Newtonian limit of scalar-torsion theories of gravity as analogue to scalar-curvature theories",
  arXiv:1909.09355 [gr-qc].