# Perturbative methods in gravity theory

## A computer algebra approach

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# Outline

# Motivation

- Perturbative approach to the study of gravity theories:
  - Consider approximation of gravitational field around well-known, simple, exact solution.
  - Background solution assumed to be symmetric (Minkowski, cosmological, spherical. . . )
  - Characterizes gravity theories by dynamics of the perturbations.
  - Dynamics of perturbations related to observations (CMB, gravitational waves. . . ).

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  - ✓ Solve field equations with increasing perturbation order, improve on each step.
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  - ↯ Equations may become coupled and difficult to disentangle at higher orders.
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- Approach in this talk: implementation as packages using *xAct* for Mathematica:
  - Mathematica offers powerful routines for symbolic calculations.
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  - Mathematica offers powerful routines for symbolic calculations.
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  - *xAct* can easily be extended with new functionality.
- This talk will focus on metric-affine and teleparallel gravity theories.

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  - Defines length of and angle between tangent vectors.
  - Defines length of curves and proper time.
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- Three characteristic quantities:
  - Curvature:

$$R^{\mu}{}_{\nu\rho\sigma} = \partial_{\rho}\Gamma^{\mu}{}_{\nu\sigma} - \partial_{\sigma}\Gamma^{\mu}{}_{\nu\rho} + \Gamma^{\mu}{}_{\tau\rho}\Gamma^{\tau}{}_{\nu\sigma} - \Gamma^{\mu}{}_{\tau\sigma}\Gamma^{\tau}{}_{\nu\rho} \,. \tag{1}$$

  - Torsion:

$$T^{\mu}{}_{\nu\rho} = \Gamma^{\mu}{}_{\rho\nu} - \Gamma^{\mu}{}_{\nu\rho} \,. \tag{2}$$

  - Nonmetricity:

$$Q_{\mu\nu\rho} = \nabla_{\mu}g_{\nu\rho} = \partial_{\mu}g_{\nu\rho} - \Gamma^{\sigma}{}_{\nu\mu}g_{\sigma\rho} - \Gamma^{\sigma}{}_{\rho\mu}g_{\nu\sigma} \,. \tag{3}$$

# Teleparallel geometries

- Fundamental fields in the Palatini / metric-affine formulation:
  - Metric tensor $g_{\mu\nu}$.
  - Flat affine connection $\Gamma^{\mu}{}_{\nu\rho} = 0$: vanishing curvature

$$R^{\rho}{}_{\sigma\mu\nu} = \partial_{\mu}\Gamma^{\rho}{}_{\sigma\nu} - \partial_{\nu}\Gamma^{\rho}{}_{\sigma\mu} + \Gamma^{\rho}{}_{\lambda\mu}\Gamma^{\lambda}{}_{\sigma\nu} - \Gamma^{\rho}{}_{\lambda\nu}\Gamma^{\lambda}{}_{\sigma\mu} = 0\,. \tag{4}$$

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- The flavors of teleparallel geometries: vanishing curvature
  - Metric teleparallel geometry: vanishing nonmetricity

$$Q_{\rho\mu\nu} = \nabla_{\rho}g_{\mu\nu} = 0\,. \tag{5}$$

  - Symmetric teleparallel geometry: vanishing torsion

$$T^{\rho}{}_{\mu\nu} = \Gamma^{\rho}{}_{\nu\mu} - \Gamma^{\rho}{}_{\mu\nu} = 0\,. \tag{6}$$

  - General teleparallel geometry: allow both torsion $T^{\rho}{}_{\mu\nu}$ and nonmetricity $Q_{\rho\mu\nu}$.

# Metric teleparallel geometry: tetrad and spin connection

- Metric teleparallelism conventionally formulated using:
  - Tetrad / coframe: $\theta^A = \theta^A{}_\mu \mathrm{d}x^\mu$ with inverse $e_A = e_A{}^\mu \partial_\mu$.
  - Spin connection: $\omega^A{}_B = \omega^A{}_{B\mu} \mathrm{d}x^\mu$.

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- Induced metric-affine geometry:
  - Metric:

$$g_{\mu\nu} = \eta_{AB} \theta^A{}_\mu \theta^B{}_\nu \,. \tag{7}$$

  - Affine connection:

$$\Gamma^\mu{}_{\nu\rho} = e_A{}^\mu \left( \partial_\rho \theta^A{}_\nu + \omega^A{}_{B\rho} \theta^B{}_\nu \right) \,. \tag{8}$$

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$$\Gamma^\mu{}_{\nu\rho} = e_A{}^\mu \left(\partial_\rho \theta^A{}_\nu + \omega^A{}_{B\rho}\theta^B{}_\nu\right) \,. \tag{8}$$
- Conditions on the spin connection:
  - Flatness $R = 0$:
$$\partial_\mu \omega^A{}_{B\nu} - \partial_\nu \omega^A{}_{B\mu} + \omega^A{}_{C\mu}\omega^C{}_{B\nu} - \omega^A{}_{C\nu}\omega^C{}_{B\mu} = 0 \,. \tag{9}$$
  - Metric compatibility $Q = 0$:
$$\eta_{AC}\omega^C{}_{B\mu} + \eta_{BC}\omega^C{}_{A\mu} = 0 \,. \tag{10}$$

# Local Lorentz invariance

- Local Lorentz transformation of the tetrad only:

$$\theta^A{}_\mu \mapsto \theta'^A{}_\mu = \Lambda^A{}_B \theta^B{}_\mu \,. \tag{11}$$

✓ Metric is invariant: $g'_{\mu\nu} = g_{\mu\nu}$.

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$$\omega^A{}_{B\mu} \mapsto \omega'^A{}_{B\mu} = \Lambda^A{}_C (\Lambda^{-1})^D{}_B \omega^C{}_{D\mu} + \Lambda^A{}_C \partial_\mu (\Lambda^{-1})^C{}_B \,. \tag{12}$$

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⇒ Metric-affine geometry equivalently described by:
  - ○ Metric $g_{\mu\nu}$ and affine connection $\Gamma^\mu{}_{\nu\rho}$.
  - ○ Equivalence class of tetrad $\theta^A{}_\mu$ and spin connection $\omega^A{}_{B\mu}$.
  - ○ Equivalence defined with respect to local Lorentz transformations.

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- Teleparallel geometry admits Weitzenböck gauge: $\omega^A{}_{B\mu} \equiv 0$.

# Outline

# Linear perturbations of affine connections

- General affine connection perturbation: $\Gamma^{\mu}{}_{\nu\rho} = \bar{\Gamma}^{\mu}{}_{\nu\rho} + \delta\Gamma^{\mu}{}_{\nu\rho}$.
  - $\Rightarrow$ Curvature perturbation:

$$\delta R^{\rho}{}_{\sigma\mu\nu} = \bar{\nabla}_{\mu}\delta\Gamma^{\rho}{}_{\sigma\nu} - \bar{\nabla}_{\nu}\delta\Gamma^{\rho}{}_{\sigma\mu} + \bar{T}^{\omega}{}_{\mu\nu}\delta\Gamma^{\rho}{}_{\sigma\omega}\,. \tag{13}$$

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- Restriction to particular geometries:
  - Vanishing torsion $T^{\mu}{}_{\nu\rho} \equiv 0$:
$$0 = \delta T^{\mu}{}_{\nu\rho} \quad \Leftrightarrow \quad \delta\Gamma^{\mu}{}_{\nu\rho} = \delta\Gamma^{\mu}{}_{\rho\nu}\,. \tag{15}$$
  - Vanishing curvature $R^{\rho}{}_{\sigma\mu\nu} \equiv 0$:
$$0 = \delta R^{\rho}{}_{\sigma\mu\nu} \quad \Leftrightarrow \quad \delta\Gamma^{\mu}{}_{\nu\rho} = \bar{\nabla}_{\rho}\tau^{\mu}{}_{\nu}\,. \tag{16}$$
  - Vanishing torsion $T^{\mu}{}_{\nu\rho} \equiv 0$ and curvature $R^{\rho}{}_{\sigma\mu\nu} \equiv 0$:
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# Linear perturbations of affine connections

- General affine connection perturbation: $\Gamma^{\mu}{}_{\nu\rho} = \bar{\Gamma}^{\mu}{}_{\nu\rho} + \delta\Gamma^{\mu}{}_{\nu\rho}$. 64 components
  - ⇒ Curvature perturbation:

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- Restriction to particular geometries:
  - Vanishing torsion $T^{\mu}{}_{\nu\rho} \equiv 0$: 40 components

$$0 = \delta T^{\mu}{}_{\nu\rho} \quad \Leftrightarrow \quad \delta\Gamma^{\mu}{}_{\nu\rho} = \delta\Gamma^{\mu}{}_{\rho\nu}\,. \tag{15}$$

  - Vanishing curvature $R^{\rho}{}_{\sigma\mu\nu} \equiv 0$: 16 components

$$0 = \delta R^{\rho}{}_{\sigma\mu\nu} \quad \Leftrightarrow \quad \delta\Gamma^{\mu}{}_{\nu\rho} = \bar{\nabla}_{\rho}\tau^{\mu}{}_{\nu}\,. \tag{16}$$

  - Vanishing torsion $T^{\mu}{}_{\nu\rho} \equiv 0$ and curvature $R^{\rho}{}_{\sigma\mu\nu} \equiv 0$: 4 components

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- In the following, focus on teleparallel case.

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- Restriction to particular geometries:
  - Riemann-Cartan geometry $Q_{\rho\mu\nu} \equiv 0$:

$$0 = \delta Q_{\rho\mu\nu} \quad \Leftrightarrow \quad \bar{g}_{\sigma\nu}\delta\Gamma^\sigma{}_{\mu\rho} + \bar{g}_{\mu\sigma}\delta\Gamma^\sigma{}_{\nu\rho} = \bar{\nabla}_\rho \delta g_{\mu\nu} \,. \tag{19}$$

  - Riemannian geometry $Q_{\rho\mu\nu} \equiv 0$ and $T^\mu{}_{\nu\rho} \equiv 0$:

$$0 = \delta T^\mu{}_{\nu\rho} \wedge 0 = \delta Q_{\rho\mu\nu} \quad \Leftrightarrow \quad \delta\Gamma^\rho{}_{\mu\nu} = \frac{1}{2}\bar{g}^{\rho\sigma}\left(\bar{\nabla}_\mu \delta g_{\sigma\nu} + \bar{\nabla}_\nu \delta g_{\mu\sigma} - \bar{\nabla}_\sigma \delta g_{\mu\nu}\right) \,. \tag{20}$$

  - Metric teleparallel geometry $Q_{\rho\mu\nu} \equiv 0$ and $R^\rho{}_{\sigma\mu\nu} \equiv 0$:

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- In the following, focus on metric teleparallel and Riemannian cases.

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# Post-Newtonian matter and velocity orders

- Energy-momentum tensor of a perfect fluid:

$$\Theta^{\mu\nu} = (\rho + \rho\Pi + p)u^{\mu}u^{\nu} + pg^{\mu\nu} .$$

  - Rest mass density $\rho$.
  - Specific internal energy $\Pi$.
  - Pressure $p$.
  - Four-velocity $u^{\mu}$.

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  - Four-velocity $u^\mu$.
- Universe rest frame and slow-moving source matter:
  - Velocity of the source matter: $v^i = u^i/u^0$.
  - Assume that source matter is slow-moving: $|\vec{v}| \ll 1$.
  - Use $\epsilon = |\vec{v}|$ as perturbation parameter.

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- Assign velocity orders $\mathcal{O}(n) \sim \epsilon^n$ to all quantities based on solar system.

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  - Use $\epsilon = |\vec{v}|$ as perturbation parameter.
- Assign velocity orders $\mathcal{O}(n) \sim \epsilon^n$ to all quantities based on solar system.
- Quasi-static: assign additional $\mathcal{O}(1)$ to time derivatives $\partial_0$.

# Post-Newtonian expansion of gravitational field

- Standard post-Newtonian metric expansion:
  - Expand metric up to fourth order of velocity of the source matter:

$$g_{\mu\nu} = \overset{0}{g}_{\mu\nu} + \overset{1}{g}_{\mu\nu} + \overset{2}{g}_{\mu\nu} + \overset{3}{g}_{\mu\nu} + \overset{4}{g}_{\mu\nu} + \mathcal{O}(5)\,. \tag{22}$$

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# PPN potentials

- Newtonian potential:

$$\chi = -\int d^3x' \rho' |\vec{x} - \vec{x}'| \,, \quad U = \int d^3x' \frac{\rho'}{|\vec{x} - \vec{x}'|} \,, \quad \rho' \equiv \rho(t, \vec{x}') \,.$$

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$$\Phi_1 = \int d^3x' \frac{\rho' v'^2}{|\vec{x} - \vec{x}'|} \,, \quad \Phi_4 = \int d^3x' \frac{p'}{|\vec{x} - \vec{x}'|} \,,$$

$$\Phi_2 = \int d^3x' \frac{\rho' U'}{|\vec{x} - \vec{x}'|} \,, \quad \mathcal{A} = \int d^3x' \frac{\rho' \left[v_i'(x_i - x_i')\right]^2}{|\vec{x} - \vec{x}'|^3} \,,$$

$$\Phi_3 = \int d^3x' \frac{\rho' \Pi'}{|\vec{x} - \vec{x}'|} \,, \quad \mathcal{B} = \int d^3x' \frac{\rho'}{|\vec{x} - \vec{x}'|} (x_i - x_i') \frac{dv_i'}{dt} \,,$$

$$\Phi_W = \int d^3x' d^3x'' \rho' \rho'' \frac{x_i - x_i'}{|\vec{x} - \vec{x}'|^3} \left( \frac{x_i' - x_i''}{|\vec{x} - \vec{x}''|} - \frac{x_i - x_i''}{|\vec{x}' - \vec{x}''|} \right) \,.$$

# Post-Newtonian field equations

- Expand energy-momentum tensor in velocity orders:

$$\Theta_{00} = \rho\left(1 - \overset{2}{g}_{00} + v^2 + \Pi\right) + \mathcal{O}(6)\,, \tag{26a}$$

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↯ Difficulties and demands on a computer algebra approach to PPN formalism:

1. Symbolic tensor algebra in order to manipulate and solve gravity field equations.
2. Perturbation of fields and equations to higher than linear order.
3. Proper split of spacetime indices into space and time components.
4. Assignment of different perturbation order to time and space derivatives.
5. Application of known rules for post-Newtonian matter source and potentials.

# Outline

# Key features

1. Pre-defined geometric objects:
   - Metric and tetrad based geometries.
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3. Algorithms typically used in PPN formalism:
    - $3 + 1$ decomposition of tensors and connection coefficients into time and space.
    - Perturbative expansion and decomposition into velocity orders.
    - Correct assignment of velocity order $+1$ to time derivative.
    - Both built-in rules and user-defined rules for perturbative expansion.
    - Known transformation rules for transforming between PPN potentials.
    - Transformation of derivatives on PPN potentials to matter source terms.
    - Application of Euler equations of motion to fluid variables.

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4. Open some example from `Examples` folder and run all code:
   - `GeneralRelativity.wl` - General Relativity (GR).
   - `BransDicke.wl` - Brans-Dicke type scalar-tensor gravity with dynamical coupling.
   - `NewGeneralRelativity.wl` - New GR class of teleparallel gravity.
   - `ScalarTorsion.wl` - General scalar-torsion class of teleparallel gravity.
   - `NewerGeneralRelativity.wl` - Newer GR class of symmetric teleparallel gravity.

   NB! For some examples, calculations are time consuming!

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```

Out[]= $\overset{3}{g}_{0a}$

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3. Extract second velocity order $\overset{2}{R}_{00}$:

```
In[]:= VelocityOrder[%[[1, 1]], 2]
```
Out[]= $-\frac{1}{2}\partial_a\partial^a\overset{2}{g}_{00}$

# Example: PPN expansion of the Ricci tensor

1. Ricci tensor of Levi-Civita connection is pre-defined:

```
In[]:= RicciCD[-T4α, -T4β]
```

Out[]= $R_{\alpha\beta}$

2. Perform $3+1$ decomposition into all possible space and time components:

```
In[]:= SpaceTimeSplits[%, {-T4α → -T3a, -T4β → -T3b}]
```

Out[]= $\{\{R_{00}, R_{0b}\}, \{R_{a0}, R_{ab}\}\}$

3. Extract second velocity order $\overset{2}{R}_{00}$:

```
In[]:= VelocityOrder[%[[1, 1]], 2]
```

Out[]= $-\frac{1}{2}\partial_a\partial^a\overset{2}{g}_{00}$

4. Extract third velocity order $\overset{3}{R}_{a0}$:

```
In[]:= Factor[SortPDs[ToCanonical[VelocityOrder[%%[[2, 1]], 3]]]]
```

Out[]= $\frac{1}{2}\left(-\partial_0\partial_a\overset{2}{g}{}^b{}_b + \partial_0\partial_b\overset{2}{g}{}_a{}^b + \partial_b\partial_a\overset{3}{g}{}_0{}^b - \partial_b\partial^b\overset{3}{g}{}_{0a}\right)$

# Example: energy-momentum conservation and Euler equations

1. Consider energy-momentum conservation equation:

```
In[]:= InvMet[T4β, T4γ] CD[-T4γ][EnergyMomentum[-T4β, -T4α]]
```

Out[]= $g^{\beta\gamma}\mathring{\nabla}_{\gamma}\Theta_{\beta\alpha}$

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```

Out[]= $g^{\beta\gamma}\mathring{\nabla}_{\gamma}\Theta_{\beta\alpha}$

2. Extract third order time component:

```
In[]:= ChangeCovD[%, CD, PD];
In[]:= SpaceTimeSplit[%, {-T4α → -LI[0]}];
In[]:= VelocityOrder[%, 3];
In[]:= ContractMetric[%];
In[]:= ToCanonical[%]
```

Out[]= $-\partial_0\rho - v^a\partial_a\rho - \rho\partial_a v^a$

# Example: energy-momentum conservation and Euler equations

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In[]:= ContractMetric[%];
In[]:= ToCanonical[%]
```

Out[]= $-\partial_0\rho - v^a\partial_a\rho - \rho\partial_a v^a$

3. Apply Euler equation of perfect fluid:

```
In[]:= TimeRhoToEuler[%]
```

Out[]= $0$

# Example: third order metric and vector PPN potentials

1. Standard PPN expansion of third-order metric perturbation:

```
In[]:= MetricToStandard[PPN[Met, 3][-LI[0], -T3a]];
In[]:= Collect[%, {PotentialV[-T3a], PotentialW[-T3a]}, Factor]
```

Out[]= $\frac{1}{2}(-3 - \alpha_1 + \alpha_2 - 4\gamma + 2\xi - \zeta_1)V_a + \frac{1}{2}(-1 - \alpha_2 - 2\xi + \zeta_1)W_a$

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2. Well-known relations satisfied by vector potentials:
   - Sum of vector potentials is divergence-free vector:

```
In[]:= PD[-T3a][PotentialV[T3a] + PotentialW[T3a]]
```

Out[]= $\partial_a V^a + \partial_a W^a$

```
In[]:= PotentialVToW[%]
```

Out[]= $0$

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```
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```
In[]:= PD[-T3a][PotentialV[T3a] + PotentialW[T3a]]
```

Out[]= $\partial_a V^a + \partial_a W^a$

```
In[]:= PotentialVToW[%]
```

Out[]= $0$

   - Difference of vector potentials is pure divergence:

```
In[]:= PotentialV[-T3a] - PotentialW[-T3a]
```

Out[]= $V_a - W_a$

```
In[]:= PotentialVToChiW[%]
```

Out[]= $\partial_0\partial_a\chi$

# Example: defining a new scalar field and its expansion

1. Define scalar field $\psi$ and its constant background value $\Psi$:

```
In[]:= DefTensor[psi[], {MfSpacetime}, PrintAs → "ψ"]
In[]:= DefConstantSymbol[psi0, PrintAs → "Ψ"]
```

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```

2. Define rules $\overset{0}{\psi} = \Psi$, $\overset{1}{\psi} = \overset{3}{\psi} = 0$ for PPN expansion:

```
In[]:= OrderSet[PPN[psi, 0][], psi0];
In[]:= OrderSet[PPN[psi, 1][], 0];
In[]:= OrderSet[PPN[psi, 3][], 0];
```

# Example: defining a new scalar field and its expansion

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In[]:= OrderSet[PPN[psi, 0][], psi0];
In[]:= OrderSet[PPN[psi, 1][], 0];
In[]:= OrderSet[PPN[psi, 3][], 0];
```

3. Rules are now used automatically, e.g., second-order space component of $\partial^\beta(\psi g_{\beta\alpha})$:

```
In[]:= PD[T4β][Met[-T4β, -T4α] psi[]]
```

Out[]= $\psi\partial^\beta g_{\beta\alpha} + g_{\beta\alpha}\partial^\beta\psi$

```
In[]:= SpaceTimeSplit[%, {-T4α → -T3a}];
In[]:= VelocityOrder[%, 2];
In[]:= ToCanonical[ContractMetric[%]]
```

Out[]= $\partial_a\overset{2}{\psi} + \Psi\partial_b\overset{2}{g}_a{}^b$

# PPN metric and parameters

## PPN metric

To read off the PPN parameters, we use the following metric components.

In[∘]:=
```
metcomp = {PPN[Met,2][-LI[0],-LI[0]], PPN[Met,2][-T3a,-T3b], PPN[Met,3][-LI[0],-T3a], PPN[Met,4][-LI[0],-LI[0]]}
```

Out[∘]= $\{\overset{2}{g}_{00}, \overset{2}{g}_{ab}, \overset{3}{g}_{0a}, \overset{4}{g}_{00}\}$

Insert the solution we obtained into the metric components.

In[∘]:=
```
metcomp /. sol2ru /. sol3ru /. sol4ru;
ToCanonical[%];
Expand[%];
ppnmet = Simplify[%];
metdef = MapThread[Equal, {metcomp, %}, 1]
```

Out[∘]= $\left\{\overset{2}{g}_{00} == \frac{\kappa^2\,U}{4\,\pi}, \overset{2}{g}_{ab} == \frac{\kappa^2\,\delta_{ab}\,U}{4\,\pi}, \overset{3}{g}_{0a} == -\frac{\kappa^2\,(7\,V_a + W_a)}{16\,\pi}, \overset{4}{g}_{00} == \frac{8\,\kappa^2\,\pi\,(2\,\Phi_1 + \Phi_3 + 3\,\Phi_4) + \kappa^4\,(2\,\Phi_2 - U^2)}{32\,\pi^2}\right\}$

## PPN parameters

Finally, solve the equations and determine the PPN parameters.

In[∘]:=
```
parsol = FullSimplify[Solve[# == 0& /@ eqns, pars][[1]]]
```

Out[∘]= $\{\beta \to 1, \gamma \to 1, \xi \to 0, \alpha_1 \to 0, \alpha_2 \to 0, \alpha_3 \to 0, \zeta_1 \to 0, \zeta_2 \to 0, \zeta_3 \to 0, \zeta_4 \to 0\}$

# Closer look at one example: general relativity

1. Starting point is trace-reversed Einstein equation:

$$\mathcal{E}_{\alpha\beta} == -\kappa^2\left(\Theta_{\alpha\beta} - \frac{1}{2}\Theta_{\gamma\delta}\ \mathtt{i}\mathtt{g}^{\gamma\delta}\ \mathtt{g}_{\alpha\beta}\right) + \mathrm{R}[\overset{\circ}{\nabla}]_{\alpha\beta}$$

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2. Example: second velocity order.
   (1) Extract second-order field equations:

$$\left\{\frac{1}{2}\left(-\kappa^2\rho - \partial_a\partial^a \overset{2}{g}_{00}\right) == 0,\ \frac{1}{2}\left(-\kappa^2\ \delta_{ba}\ \rho + \partial_b\partial_a \overset{2}{g}_{00} - \partial_b\partial_a \overset{2}{g}{}^c{}_c + \partial_c\partial_a \overset{2}{g}_b{}^c + \partial_c\partial_b \overset{2}{g}_a{}^c - \partial_c\partial^c \overset{2}{g}_{ab}\right) == 0\right\}$$

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   (2) Make ansatz for second-order metric components:

$$\left\{ \overset{2}{g}_{00} == a_1 U, \; \overset{2}{g}_{ab} == a_2 \delta_{ab} U + a_3 U_{ab} \right\}$$

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   (3) Solve for constant coefficients to obtain solution:

$$\left\{ \overset{2}{g}_{00} == \frac{\kappa^2 U}{4\pi}, \; \overset{2}{g}_{ab} == \frac{\kappa^2 \delta_{ab} U}{4\pi} \right\}$$

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   (3) Solve for constant coefficients to obtain solution:

$$\left\{\overset{2}{\mathrm{g}}_{00} == \frac{\kappa^2\,\mathrm{U}}{4\pi},\ \overset{2}{\mathrm{g}}_{ab} == \frac{\kappa^2\ \delta_{ab}\ \mathrm{U}}{4\pi}\right\}$$

3. Perform same steps to obtain all necessary metric components:

$$\left\{\overset{2}{\mathrm{g}}_{00} == \frac{\kappa^2\,\mathrm{U}}{4\pi},\ \overset{2}{\mathrm{g}}_{ab} == \frac{\kappa^2\ \delta_{ab}\ \mathrm{U}}{4\pi},\ \overset{3}{\mathrm{g}}_{0a} == -\frac{\kappa^2\left(7\ \mathrm{V}_a + \mathrm{W}_a\right)}{16\pi},\ \overset{4}{\mathrm{g}}_{00} == \frac{8\kappa^2\pi\,(2\Phi_1 + \Phi_3 + 3\Phi_4) + \kappa^4\,(2\Phi_2 - \mathrm{U}^2)}{32\pi^2}\right\}$$

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4. Obtain PPN parameters by comparing with standard PPN metric:

$$\{\beta == 1,\ \gamma == 1,\ \xi == 0,\ \alpha_1 == 0,\ \alpha_2 == 0,\ \alpha_3 == 0,\ \zeta_1 == 0,\ \zeta_2 == 0,\ \zeta_3 == 0,\ \zeta_4 == 0\}$$

# A look under the hood: expanded Einstein equations

$$\Big\{ \overset{0}{\mathcal{E}}_{00} == 0,\ \overset{1}{\mathcal{E}}_{00} == 0,\ \overset{2}{\mathcal{E}}_{00} == \frac{1}{2}\left(-\kappa^2 \rho - \partial_a\partial^a \overset{2}{g}_{00}\right),\ \overset{3}{\mathcal{E}}_{00} == 0,$$

$$\overset{4}{\mathcal{E}}_{00} == \frac{1}{4}\Big(-2\kappa^2\rho\Pi - 6\kappa^2 p - 4\kappa^2\rho\, v_a v^a + 4\,\partial_0\partial_a \overset{3}{g}_0{}^a - 2\,\partial_0\partial_0 \overset{2}{g}{}^a{}_a - 2\,\partial_a\partial^a \overset{4}{g}_{00} - \partial_a \overset{2}{g}_{00}\, \partial^a \overset{2}{g}_{00} -$$

$$\partial_a \overset{2}{g}{}^b{}_b\, \partial^a \overset{2}{g}_{00} + 2\,\partial^a \overset{2}{g}_{00}\, \partial_b \overset{2}{g}_a{}^b + 2\kappa^2\rho\, \overset{2}{g}_{00} + 2\,\partial^b\partial^a \overset{2}{g}_{00}\, \overset{2}{g}_{ab}\Big),\ \overset{0}{\mathcal{E}}_{0a} == 0,\ \overset{0}{\mathcal{E}}_{0b} == 0,\ \overset{1}{\mathcal{E}}_{0a} == 0,$$

$$\overset{1}{\mathcal{E}}_{0b} == 0,\ \overset{2}{\mathcal{E}}_{0a} == 0,\ \overset{2}{\mathcal{E}}_{0b} == 0,\ \overset{3}{\mathcal{E}}_{0a} == \frac{1}{2}\left(2\kappa^2\rho\, v_a - \partial_0\partial_a \overset{2}{g}{}^b{}_b + \partial_0\partial_b \overset{2}{g}_a{}^b + \partial_b\partial_a \overset{3}{g}_0{}^b - \partial_b\partial^b \overset{3}{g}_{0a}\right),$$

$$\overset{3}{\mathcal{E}}_{0b} == \frac{1}{2}\left(2\kappa^2\rho\, v_b + \partial_0\partial_a \overset{2}{g}_b{}^a - \partial_0\partial_b \overset{2}{g}{}^a{}_a - \partial_a\partial^a \overset{3}{g}_{0b} + \partial_a\partial_b \overset{3}{g}_0{}^a\right),\ \overset{4}{\mathcal{E}}_{0a} == 0,\ \overset{4}{\mathcal{E}}_{0b} == 0,\ \overset{0}{\mathcal{E}}_{ab} == 0,$$

$$\overset{1}{\mathcal{E}}_{ab} == 0,\ \overset{2}{\mathcal{E}}_{ab} == \frac{1}{2}\left(-\kappa^2\delta_{ab}\,\rho + \partial_b\partial_a \overset{2}{g}_{00} - \partial_b\partial_a \overset{2}{g}{}^c{}_c + \partial_c\partial_a \overset{2}{g}_b{}^c + \partial_c\partial_b \overset{2}{g}_a{}^c - \partial_c\partial^c \overset{2}{g}_{ab}\right),\ \overset{3}{\mathcal{E}}_{ab} == 0,$$

$$\overset{4}{\mathcal{E}}_{ab} == \frac{1}{4}\Big(2\kappa^2\delta_{ab}(-\rho\Pi + p) - 4\kappa^2\rho\, v_a v_b - 2\,\partial_0\partial_a \overset{3}{g}_{0b} - 2\,\partial_0\partial_b \overset{3}{g}_{0a} + 2\,\partial_0\partial_0 \overset{2}{g}_{ab} + 2\,\partial_b\partial_a \overset{4}{g}_{00} -$$

$$2\,\partial_b\partial_a \overset{4}{g}{}^c{}_c + \partial_a \overset{2}{g}_{00}\, \partial_b \overset{2}{g}_{00} + \partial_a \overset{2}{g}{}^{cd}\, \partial_b \overset{2}{g}_{cd} + 2\,\partial_c\partial_a \overset{4}{g}_b{}^c + 2\,\partial_c\partial_b \overset{4}{g}_a{}^c - 2\,\partial_c\partial^c \overset{4}{g}_{ab} + \partial_a \overset{2}{g}_b{}^c\, \partial_c \overset{2}{g}{}^d{}_d +$$

$$\partial_b \overset{2}{g}_a{}^c\, \partial_c \overset{2}{g}{}^d{}_d - \partial_a \overset{2}{g}_{bc}\, \partial^c \overset{2}{g}_{00} - \partial_b \overset{2}{g}_{ac}\, \partial^c \overset{2}{g}_{00} + \partial_c \overset{2}{g}_{ab}\, \partial^c \overset{2}{g}_{00} - \partial_c \overset{2}{g}{}^d{}_d\, \partial^c \overset{2}{g}_{ab} - 2\,\partial_a \overset{2}{g}_b{}^c\, \partial_d \overset{2}{g}_c{}^d -$$

$$2\,\partial_b \overset{2}{g}_a{}^c\, \partial_d \overset{2}{g}_c{}^d + 2\,\partial^c \overset{2}{g}_{ab}\, \partial_d \overset{2}{g}_c{}^d - 2\,\partial_c \overset{2}{g}_{bd}\, \partial^d \overset{2}{g}_a{}^c + 2\,\partial_d \overset{2}{g}_{bc}\, \partial^d \overset{2}{g}_a{}^c + 2\,\partial_b\partial_a \overset{2}{g}_{00}\, \overset{2}{g}_{00} -$$

$$2\kappa^2\rho\, \overset{2}{g}_{ab} + 2\,\partial_b\partial_a \overset{2}{g}_{cd}\, \overset{2}{g}{}^{cd} - 2\,\partial_d\partial_a \overset{2}{g}_{bc}\, \overset{2}{g}{}^{cd} - 2\,\partial_d\partial_b \overset{2}{g}_{ac}\, \overset{2}{g}{}^{cd} + 2\,\partial_d\partial_c \overset{2}{g}_{ab}\, \overset{2}{g}{}^{cd}\Big)\Big\}$$

# Outline

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# Cosmological metric teleparallel background geometry

- Friedmann-Lemaître-Robertson-Walker metric:

$$\bar{g}_{\mu\nu}\mathrm{d}x^{\mu} \otimes \mathrm{d}x^{\nu} = -n_{\mu}n_{\nu} + h_{\mu\nu} = -N^2\mathrm{d}t \otimes \mathrm{d}t + A^2\gamma_{ab}\mathrm{d}x^a \otimes \mathrm{d}x^b\,. \tag{27}$$

⇒ Scale factor $A(t)$ and lapse function $N(t)$ depend on time $t$, metric $\gamma_{ab}$ does not.

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⇒ Scale factor $A(t)$ and lapse function $N(t)$ depend on time $t$, metric $\gamma_{ab}$ does not.

- Cosmologically symmetric torsion and contortion tensors:

$$\bar{T}_{\mu\nu\rho} = \frac{2\mathscr{V}h_{\mu[\nu}n_{\rho]} + 2\mathscr{A}\varepsilon_{\mu\nu\rho}}{A}\,, \quad \bar{K}_{\mu\nu\rho} = \frac{2\mathscr{V}h_{\rho[\mu}n_{\nu]} - \mathscr{A}\varepsilon_{\mu\nu\rho}}{A}\,. \tag{28}$$

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- Friedmann-Lemaître-Robertson-Walker metric:

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- Two branches of cosmologically symmetric teleparallel geometries: [MH '20]
  1. "Vector" branch:

$$\mathscr{V} = \mathcal{H} \pm iu\,, \quad \mathscr{A} = 0\,, \tag{29}$$

  2. "Axial" branch:

$$\mathscr{V} = \mathcal{H}\,, \quad \mathscr{A} = \pm u\,. \tag{30}$$

⇒ Torsion depends on constant $k = u^2$ and conformal Hubble parameter $\mathcal{H} = N^{-1}\partial_t A$.

# Spatial geometry and $3 + 1$ decomposition

- Decomposition of Friedmann-Lemaître-Robertson-Walker metric:

$$\bar{g}_{\mu\nu} = -n_\mu n_\nu + h_{\mu\nu} \,. \tag{31}$$

# Spatial geometry and $3 + 1$ decomposition

- Decomposition of Friedmann-Lemaître-Robertson-Walker metric:

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# Projectors and tensor decomposition

- Introduce covariant and contravariant spatial projectors:

$$\Pi^a_\mu \partial_a \otimes \mathrm{d}x^\mu = A\delta^a_b\,\partial_a \otimes \mathrm{d}x^b\,, \quad \Pi^\mu_a \partial_\mu \otimes \mathrm{d}x^a = A^{-1}\delta^b_a\,\partial_b \otimes \mathrm{d}x^a\,. \tag{35}$$

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⇒ Indices of decomposed components are raised and lowered with Minkowski metric:

$$X^\mu = g^{\mu\nu}X_\nu \quad \Leftrightarrow \quad \hat{X}^0 = -\hat{X}_0\,, \quad \hat{X}^a = \gamma^{ab}\hat{X}_b\,. \tag{39}$$

# Derivative decomposition

- Space-time split of Levi-Civita covariant derivative:

$$\overset{\circ}{\nabla}_\alpha X^\beta =$$

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(40)

# Derivative decomposition

- Space-time split of Levi-Civita covariant derivative:

$$\mathring{\nabla}_\alpha X^\beta = (h^\gamma_\alpha - n_\alpha n^\gamma)(h^\beta_\delta - n^\beta n_\delta)\mathring{\nabla}_\gamma(n^\delta \hat{X}^0 + \Pi^\delta_a \hat{X}^a)$$
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- Introduce projectors for space-time split.

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- Introduce projectors for space-time split.
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  1. Derivative in time direction yields time derivatives.

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- Hubble parameter enters through derivative of projectors:
  - Eulerian observers move on geodesics ⇒ acceleration vanishes:

$$a_\mu = n^\nu \mathring{\nabla}_\nu n_\mu = 0\,. \tag{41}$$

  - Spatial geometry is maximally symmetric ⇒ extrinsic curvature:

$$K_{\mu\nu} = \mathring{\nabla}_\mu n_\nu + n_\mu a_\nu = H h_{\mu\nu}\,. \tag{42}$$

# Time coordinate and derivatives

- Lapse function $N$ can be fixed by choice of time coordinate:
  - Cosmological time $t \equiv \hat{t}$: lapse function $N \equiv 1$.
  - Conformal time $t \equiv \mathfrak{t}$: lapse function $N \equiv A$.

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- Common notation for derivatives of scalar function $f = f(t)$:
  - Cosmological time derivative:

$$\dot{f} = \frac{\mathrm{d}f}{\mathrm{d}\hat{t}} = \frac{1}{N}\partial_t f = \mathcal{L}_n f\,. \tag{44}$$

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- Example: cosmological and conformal Hubble parameters $H, \mathcal{H}$:

$$\mathcal{H} = \frac{A'}{A} = \dot{A} = AH\,. \tag{46}$$

# Outline

# Perturbation decomposition

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3. Irreducible decomposition of tetrad components:

$$\hat{\tau}_{00} = \hat{\phi}\,, \tag{47a}$$

$$\hat{\tau}_{0b} = \mathrm{d}_b\hat{j} + \hat{b}_b\,, \tag{47b}$$

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$$\mathrm{d}_a\hat{b}^a = \mathrm{d}_a\hat{v}^a = \mathrm{d}_a\hat{c}^a = \mathrm{d}_a\hat{w}^a = 0\,, \quad \mathrm{d}_a\hat{q}^{ab} = 0\,, \quad \hat{q}_{[ab]} = 0\,, \quad \hat{q}_a{}^a = 0\,. \tag{48}$$

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5. Note that the term $\mathrm{d}_b\hat{c}_a$ is not symmetrized: [Golovnev, Koivisto '18]
   - Antisymmetric part $\mathrm{d}_{[a}\hat{c}_{b]} = \frac{1}{2}\upsilon_{abc}\upsilon^{dec}\mathrm{d}_d\hat{c}_e$ can be absorbed into $\hat{w}^a$.
   - Vanishing divergence follows from Bianchi identity

$$\mathrm{d}_c(\upsilon^{dec}\mathrm{d}_d\hat{c}_e) = \upsilon^{dec}\mathrm{d}_{[c}\mathrm{d}_{d]}\hat{c}_e = \frac{1}{2}\upsilon^{dec}R^f{}_{ecd}\hat{c}_f = 0\,. \tag{49}$$

# Gauge-invariant perturbations

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⇒ Gauge-invariant cosmological tetrad perturbations remain invariant: [MH '20]

1. Scalar perturbations - 3 scalars + 1 pseudo-scalar:

$$\hat{\boldsymbol{\xi}} = \hat{\xi} + \mathscr{A}\hat{\sigma}\,, \tag{50a}$$
$$\hat{\mathbf{y}} = \hat{y} - \hat{\sigma}' - (\mathcal{H} - \mathscr{V})\hat{\sigma}\,, \tag{50b}$$
$$\hat{\boldsymbol{\psi}} = \hat{\psi} + \mathcal{H}[\hat{j} + (\mathcal{H} - \mathscr{V})\hat{\sigma}]\,, \tag{50c}$$
$$\hat{\boldsymbol{\phi}} = \hat{\phi} - \mathcal{H}[\hat{j} + (\mathcal{H} - \mathscr{V})\hat{\sigma}] + [\hat{j} + (\mathcal{H} - \mathscr{V})\hat{\sigma}]'\,. \tag{50d}$$

2. Vector perturbations - 2 divergence-free vectors + 1 pseudo-vector:

$$\hat{\mathbf{v}}_a = \hat{v}_a + (\mathscr{V} - \mathcal{H})\hat{c}_a - \hat{c}'_a\,, \tag{50e}$$
$$\hat{\mathbf{b}}_a = \hat{b}_a + (\mathcal{H} - \mathscr{V})\hat{c}_a\,, \tag{50f}$$
$$\hat{\mathbf{w}}_a = \hat{w}_a + \mathscr{A}\hat{c}_a\,, \tag{50g}$$

3. Tensor perturbation - 1 symmetric, trace-free, divergence-free tensor:

$$\hat{\mathbf{q}}_{ab} = \hat{q}_{ab}\,. \tag{50h}$$

# Perturbed gravitational field equations

- Perturbative expansion of gravitational field equations:

$$\bar{E}_A{}^{\mu} + \mathfrak{E}_A{}^{\mu} = E_A{}^{\mu} = \Theta_A{}^{\mu} = \bar{\Theta}_A{}^{\mu} + \mathfrak{T}_A{}^{\mu}\,, \tag{51}$$

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$$\mathfrak{N} n_{\mu} n_{\nu} + \mathfrak{H} h_{\mu\nu} = \bar{E}_{\mu\nu} = \bar{\theta}^A{}_{\mu} \bar{g}_{\nu\rho} \bar{E}_A{}^{\rho} = \bar{\theta}^A{}_{\mu} \bar{g}_{\nu\rho} \bar{T}_A{}^{\rho} = \bar{\Theta}_{\mu\nu} = \bar{\rho} n_{\mu} n_{\nu} + \bar{p} h_{\mu\nu}\,. \tag{52}$$

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- Quantities $\mathfrak{N}$, $\mathfrak{H}$ and $\mathfrak{E}_{\mu\nu}$ determined from gravity theory.

# Irreducible decomposition of perturbed equations

- Decomposition of perturbed gravitational field tensor similar to tetrad:

$$\hat{\mathfrak{E}}_{00} = \hat{\Phi}\,, \tag{55a}$$

$$\hat{\mathfrak{E}}_{0b} = \mathrm{d}_b\hat{J} + \hat{B}_b\,, \tag{55b}$$

$$\hat{\mathfrak{E}}_{a0} = \mathrm{d}_a\hat{Y} + \hat{V}_a\,, \tag{55c}$$

$$\hat{\mathfrak{E}}_{ab} = \hat{\Psi}\gamma_{ab} + \mathrm{d}_a\mathrm{d}_b\hat{\Sigma} + \mathrm{d}_a\hat{C}_b + \upsilon_{abc}(\mathrm{d}^c\hat{\Xi} + \hat{W}^c) + \frac{1}{2}\hat{Q}_{ab}\,. \tag{55d}$$

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$$\hat{\mathfrak{E}}_{ab} = \hat{\Psi}\gamma_{ab} + d_a d_b\hat{\Sigma} + d_a\hat{C}_b + \upsilon_{abc}(d^c\hat{\Xi} + \hat{W}^c) + \frac{1}{2}\hat{Q}_{ab}\,. \tag{55d}$$

- Decomposition of perturbed energy-momentum around perfect fluid:

$$\hat{\mathfrak{T}}_{00} = \delta\hat{\rho} + \bar{\rho}\hat{\phi}\,, \tag{56a}$$
$$\hat{\mathfrak{T}}_{0b} = -\left[(\bar{\rho} + \bar{p})\delta\hat{u}_b + \bar{p}(\hat{v}_b + d_b\hat{y})\right]\,, \tag{56b}$$
$$\hat{\mathfrak{T}}_{a0} = -\left[(\bar{\rho} + \bar{p})(\delta\hat{u}_a + \hat{v}_a + d_a\hat{y}) + \bar{p}(\hat{b}_a + d_a\hat{j})\right]\,, \tag{56c}$$
$$\hat{\mathfrak{T}}_{ab} = \delta\hat{p}\gamma_{ab} + \hat{\pi}_{ab} - \bar{p}\left[\hat{\psi}\gamma_{ab} + d_b d_a\hat{\sigma} + d_a\hat{c}_b - \upsilon_{abc}(d^c\hat{\xi} + \hat{w}^c) + \frac{1}{2}\hat{q}_{ab}\right]\,. \tag{56d}$$

# Gauge-invariant components of gravitational side

- Scalar components:

$$\hat{\boldsymbol{\Psi}} = \hat{\Psi} - (\mathfrak{H}\mathcal{H} - \mathfrak{H}')[\hat{j} + (\mathcal{H} - \mathscr{V})\hat{\sigma}]\,, \tag{57a}$$

$$\hat{\boldsymbol{\Sigma}} = \hat{\Sigma} + \mathfrak{H}\hat{\sigma}\,, \tag{57b}$$

$$\hat{\boldsymbol{\Xi}} = \hat{\Xi} + \mathscr{A}\mathfrak{H}\hat{\sigma}\,, \tag{57c}$$

$$\hat{\mathbf{J}} = \hat{J} - (\mathscr{V} - \mathcal{H})\mathfrak{H}\hat{\sigma} - \mathfrak{N}\hat{\sigma}'\,, \tag{57d}$$

$$\hat{\mathbf{Y}} = \hat{Y} + (\mathcal{H} - \mathscr{V})(\mathfrak{N} + \mathfrak{H})\hat{\sigma} + \mathfrak{H}\hat{j}\,, \tag{57e}$$

$$\hat{\boldsymbol{\Phi}} = \hat{\Phi} - (\mathfrak{N}\mathcal{H} - \mathfrak{N}')[\hat{j} + (\mathcal{H} - \mathscr{V})\hat{\sigma}] - \mathfrak{N}[\hat{j} + (\mathcal{H} - \mathscr{V})\hat{\sigma}]'\,. \tag{57f}$$

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- Vector components:

$$\hat{\mathbf{V}}_a = \hat{V}_a + (\mathcal{H} - \mathscr{V})\mathfrak{N}\hat{c}_a\,, \qquad \hat{\mathbf{W}}_a = \hat{W}_a + \mathscr{A}\mathfrak{H}\hat{c}_a\,, \tag{58a}$$

$$\hat{\mathbf{B}}_a = \hat{B}_a - (\mathscr{V} - \mathcal{H})\mathfrak{H}\hat{c}_a - \mathfrak{N}\hat{c}_a'\,, \qquad \hat{\mathbf{C}}_a = \hat{C}_a + \mathfrak{H}\hat{c}_a\,. \tag{58b}$$

# Gauge-invariant components of gravitational side

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$$\hat{\mathbf{J}} = \hat{J} - (\mathscr{V} - \mathcal{H})\mathfrak{H}\hat{\sigma} - \mathfrak{N}\hat{\sigma}'\,, \tag{57d}$$

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- Vector components:

$$\hat{\mathbf{V}}_a = \hat{V}_a + (\mathcal{H} - \mathscr{V})\mathfrak{N}\hat{c}_a\,, \qquad \hat{\mathbf{W}}_a = \hat{W}_a + \mathscr{A}\mathfrak{H}\hat{c}_a\,, \tag{58a}$$

$$\hat{\mathbf{B}}_a = \hat{B}_a - (\mathscr{V} - \mathcal{H})\mathfrak{H}\hat{c}_a - \mathfrak{N}\hat{c}'_a\,, \qquad \hat{\mathbf{C}}_a = \hat{C}_a + \mathfrak{H}\hat{c}_a\,. \tag{58b}$$

- Tensor component:

$$\hat{\mathbf{Q}}_{ab} = \hat{Q}_{ab}\,, \tag{59}$$

# Gauge-invariant matter variables

- Gauge-invariant density perturbation:

$$\hat{\mathcal{E}} = \delta\hat{\rho} + \bar{\rho}'[\hat{j} + (\mathcal{H} - \mathscr{V})\hat{\sigma}]\,. \tag{60}$$

# Gauge-invariant matter variables

- Gauge-invariant density perturbation:

$$\hat{\mathcal{E}} = \delta\hat{\rho} + \bar{\rho}'[\hat{j} + (\mathcal{H} - \mathscr{V})\hat{\sigma}]\,. \tag{60}$$

- Gauge-invariant pressure perturbation:

$$\hat{\mathcal{P}} = \delta\hat{p} + \bar{p}'[\hat{j} + (\mathcal{H} - \mathscr{V})\hat{\sigma}]\,. \tag{61}$$

# Gauge-invariant matter variables

- Gauge-invariant density perturbation:

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- Gauge-invariant pressure perturbation:

$$\hat{\mathcal{P}} = \delta\hat{p} + \bar{p}'[\hat{j} + (\mathcal{H} - \mathscr{V})\hat{\sigma}]\,. \tag{61}$$

- Decompose velocity perturbation into transverse and longitudinal part:

$$\hat{\mathcal{X}}_a + \mathrm{d}_a\hat{\mathcal{L}} = \delta\hat{u}_a + (\hat{c}_a + \mathrm{d}_a\hat{\sigma})'\,. \tag{62}$$

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- Anisotropic stress is gauge-invariant; decompose into scalar, vector, tensor:

$$\mathrm{d}_a\mathrm{d}_b\hat{\mathcal{S}} - \frac{1}{3}\triangle\hat{\mathcal{S}}\gamma_{ab} + \mathrm{d}_{(a}\hat{\mathcal{V}}_{b)} + \hat{\mathcal{T}}_{ab} = \hat{\pi}_{ab}\,. \tag{63}$$

# Gauge-invariant field equations

- Decompose perturbed field equations into irreducible components:
  - Scalar components:

$$\hat{\mathbf{J}} = -(\bar{\rho} + \bar{p})\hat{\mathcal{L}} - \bar{p}\hat{\mathbf{y}}\,, \qquad \hat{\mathbf{Y}} = -(\bar{\rho} + \bar{p})(\hat{\mathcal{L}} + \hat{\mathbf{y}})\,, \tag{64a}$$

$$\hat{\boldsymbol{\Sigma}} = \hat{\mathcal{S}}\,, \qquad \hat{\boldsymbol{\Xi}} = \bar{p}\hat{\xi}\,, \tag{64b}$$

$$\hat{\boldsymbol{\Psi}} = \hat{\mathcal{P}} - \frac{1}{3}\triangle\hat{\mathcal{S}} - \bar{p}\hat{\psi}\,, \qquad \hat{\boldsymbol{\Phi}} = \hat{\mathcal{E}} + \bar{\rho}\hat{\phi}\,. \tag{64c}$$

  - Vector components:

$$\hat{\mathbf{V}}_a = -(\bar{\rho} + \bar{p})(\hat{\mathcal{X}}_a + \hat{\mathbf{v}}_a) - \bar{p}\hat{\mathbf{b}}_a\,, \qquad \hat{\mathbf{W}}_a = \bar{p}\hat{\mathbf{w}}_a - \frac{1}{2}\upsilon_{abc}\mathrm{d}^b\hat{\mathcal{V}}^c\,, \tag{65a}$$

$$\hat{\mathbf{B}}_a = -(\bar{\rho} + \bar{p})\hat{\mathcal{X}}_b - \bar{p}\hat{\mathbf{v}}_b\,, \qquad \hat{\mathbf{C}}_a = \hat{\mathcal{V}}_a\,. \tag{65b}$$

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$$\hat{\mathbf{Q}}_{ab} = 2\hat{\mathcal{T}}_{ab} - \bar{p}\hat{\mathbf{q}}_{ab}\,. \tag{66}$$

✓ Equations are fully gauge-invariant.

# Gauge-invariant field equations

- Decompose perturbed field equations into irreducible components:
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$$\hat{\mathbf{J}} = -(\bar{\rho} + \bar{p})\hat{\mathcal{L}} - \bar{p}\hat{\mathbf{y}}\,, \qquad \hat{\mathbf{Y}} = -(\bar{\rho} + \bar{p})(\hat{\mathcal{L}} + \hat{\mathbf{y}})\,, \tag{64a}$$

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  - Tensor component:

$$\hat{\mathbf{Q}}_{ab} = 2\hat{\mathcal{T}}_{ab} - \bar{p}\hat{\mathbf{q}}_{ab}\,. \tag{66}$$

✓ Equations are fully gauge-invariant.

- Remaining task: determine components of gravity side from gravity theory.

# Outline

1. Introduction
2. Perturbations of metric-affine and teleparallel geometries
   - Classes of metric-affine geometries
   - Perturbations of fundamental fields
3. Parametrized post-Newtonian formalism
   - Overview of the PPN formalism
   - *xPPN*: implementation of the PPN formalism using *xAct*
4. Cosmological perturbations
   - Cosmological background geometry and $3 + 1$ split
   - Gauge-invariant cosmological perturbations in teleparallel gravity
   - Computer algebra approach
5. Conclusion

# Key features needed from implementation (WIP)

1. Pre-defined geometric objects:
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   - Different connections: Levi-Civita and metric teleparallel.
   - Tensors related to curvature and torsion.

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   - Energy-momentum variables: density, pressure, velocity, anisotropic stress.
   - Spatial geometry with metric $\gamma_{ab}$ and Levi-Civita derivative $\mathrm{d}_a$.
   - Projectors $\Pi_a^\mu$ and $\Pi_\mu^a$ to facilitate $3+1$ split.
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   - Irreducible components of tetrad perturbation and perturbed field equations.
3. Algorithms typically used in cosmological perturbations:
   - Linear perturbation of all quantities with respect to tetrad perturbation.
   - $3+1$ decomposition of tensors and connection coefficients into time and space.
   - Substitution of background values for cosmologically symmetric tensors.
   - Irreducible decomposition of perturbations.
   - Transformation from and to gauge-invariant variables.
   - Transformation between different choice of time coordinate.

# Work in progress: some known quantities

1. Scalar functions of time:

```
In[]:= {LapseF[], ScaleF[], Hubble[],
   CHubble[], VecTor[], AxiTor[]}
```

Out[]= $\{N, A, H, \mathcal{H}, \mathscr{V}, \mathscr{A}\}$

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2. Background metric and its decomposition:

```
In[]:= SMet[-T4α, -T4β] - Orth[-T4α] * Orth[-T4β]
```

Out[]= $-n_\alpha n_\beta + h_{\alpha\beta}$

```
In[]:= ProjectorToMetric[%]
```

Out[]= $g_{\alpha\beta}$

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3. Projector fields:

```
In[]:= {ProjCon[-T4α, T3a], ProjCov[T4α, -T3a]}
```

Out[]= $\{\Pi^a_\alpha, \Pi^\alpha_a\}$

# Work in progress: $3+1$ decomposition of tensors

1. Usual $3+1$ decomposition $g_{\mu\nu} \rightsquigarrow g_{00}, g_{a0}, g_{0b}, g_{ab}$ uses lapse and scale factor:

```
In[]:= SpaceTimeSplits[Met[-T4α, -T4β],
    {-T4α → -T3a, -T4β → -T3b}]
```

Out[]= $\{\{N^2\hat{g}_{00}, NA\hat{g}_{0b}\}, \{NA\hat{g}_{a0}, A^2\hat{g}_{ab}\}\}$

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2. Alternative approach using projectors and without explicit factors:

```
In[]:= SpaceTimeExpand[Met[-T4α, -T4β]]
```

Out[]= $n_\alpha n_\beta \hat{g}_{00} - n_\beta \Pi^a_\alpha \hat{g}_{0a} - n_\alpha \Pi^a_\beta \hat{g}_{0a} + \Pi^a_\alpha \Pi^b_\beta \hat{g}_{ab}$

```
In[]:= SpaceTimeSplits[%, {-T4α → -T3a, -T4β → -T3b}]
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```
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```

Out[]= $\{\{N^2\hat{g}_{00}, NA\hat{g}_{0b}\}, \{NA\hat{g}_{a0}, A^2\hat{g}_{ab}\}\}$

3. Use automatic background substitution $\hat{g}_{00} = -1, \hat{g}_{0a} = 0, \hat{g}_{ab} = \gamma_{ab}$:

```
In[]:= SpaceTimeSplits[Met[-T4α, -T4β],
    {-T4α → -T3a, -T4β → -T3b}, UseCosmoRules → True]
```

Out[]= $\{\{N^2, 0\}, \{0, A^2\gamma_{ab}\}\}$

```
In[]:= SpaceTimeExpand[Met[-T4α, -T4β], UseCosmoRules → True]
```

Out[]= $-n_\alpha n_\beta + \Pi^a_\alpha \Pi^b_\beta \gamma_{ab}$

# Work in progress: $3 + 1$ decomposition of derivatives

1. Partial derivative of scalar:

```
In[]:= DefTensor[S[], {MfSpacetime}]
In[]:= SpaceTimeSplits[PD[-T4α][S[]], {-T4α → -T3a}]
Out[]= {∂₀Ŝ, ∂ₐŜ}
```

# Work in progress: $3+1$ decomposition of derivatives

1. Partial derivative of scalar:

```
In[]:= DefTensor[S[], {MfSpacetime}]
In[]:= SpaceTimeSplits[PD[-T4α][S[]], {-T4α → -T3a}]
```

Out[]= $\{\partial_0\hat{S}, \partial_a\hat{S}\}$

2. Levi-Civita covariant derivative of vector field:

```
In[]:= DefTensor[X[T4α], {MfSpacetime}]
In[]:= SpaceTimeSplits[CD[-T4α][X[T4β]],
    {-T4α → -T3a, T4β→ T3b}]
```

Out[]= $\left\{\left\{\frac{\partial_0\hat{X}^0}{N}, \frac{\partial_0\hat{X}^b}{A}\right\}, \left\{\frac{\mathrm{d}_a\hat{X}^0}{N} + \frac{\gamma_{ab}HA\hat{X}^b}{N}, \frac{\mathrm{d}_a\hat{X}^b}{A} + \delta^b_a H\hat{X}^0\right\}\right\}$

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3. Purely spatial part:

```
In[]:= SpaceTimeSplits[SD[-T4α][ProjectorSMet[X[T4β]]],
    {-T4α → -T3a, T4β→ T3b}]
```

Out[]= $\left\{\{0,0\}, \left\{0, \frac{\mathrm{d}_a\hat{X}^b}{A}\right\}\right\}$

# Work in progress: calculating perturbations

1. Tetrad perturbation is expanded into $\tau_{\alpha\beta}$:

```
In[]:= Perturbation[Tet[L4Γ, -T4α]]
Out[]= τ^β_α θ^Γ_β
In[]:= Perturbation[InvTet[-L4Γ, T4α]]
Out[]= -e_Γ^β τ^α_β
```

# Work in progress: calculating perturbations

1. Tetrad perturbation is expanded into $\tau_{\alpha\beta}$:

```
In[]:= Perturbation[Tet[L4Γ, -T4α]]
```

Out[]= $\tau^{\beta}{}_{\alpha}\theta^{\Gamma}{}_{\beta}$

```
In[]:= Perturbation[InvTet[-L4Γ, T4α]]
```

Out[]= $-e_{\Gamma}{}^{\beta}\tau^{\alpha}{}_{\beta}$

2. Perturbations of common tensors:

```
In[]:= Perturbation[Met[-T4α, -T4β]]
```

Out[]= $\tau_{\alpha\beta} + \tau_{\beta\alpha}$

```
In[]:= Perturbation[TorsionFD[T4α, -T4β, -T4γ]]
```

Out[]= $\dot{\nabla}_{\beta}\tau^{\alpha}{}_{\gamma} - \dot{\nabla}_{\gamma}\tau^{\alpha}{}_{\beta}$

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1. Tetrad perturbation is expanded into $\tau_{\alpha\beta}$:

```
In[]:= Perturbation[Tet[L4Γ, -T4α]]
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Out[]= $\tau^{\beta}{}_{\alpha}\theta^{\Gamma}{}_{\beta}$
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2. Perturbations of common tensors:

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In[]:= Perturbation[Met[-T4α, -T4β]]
```
Out[]= $\tau_{\alpha\beta} + \tau_{\beta\alpha}$
```
In[]:= Perturbation[TorsionFD[T4α, -T4β, -T4γ]]
```
Out[]= $\dot{\nabla}_{\beta}\tau^{\alpha}{}_{\gamma} - \dot{\nabla}_{\gamma}\tau^{\alpha}{}_{\beta}$

3. Perturbation of field equations defined from mixed form:

```
In[]:= Perturbation[GravField[-T4α, -T4β]]
```
Out[]= $\mathfrak{E}_{\alpha\beta} + E_{\alpha}{}^{\gamma}\tau_{\beta\gamma} + E^{\gamma}{}_{\beta}\tau_{\gamma\alpha} + E_{\alpha}{}^{\gamma}\tau_{\gamma\beta}$

# Work in progress: irreducible decomposition

1. Spatial part of tetrad perturbation:

```
In[]:= ExpandTau[CT[Tau][-T3a, -T3b]]
```

Out[]= $\hat{\psi}\gamma_{ab} + \mathrm{d}_a \mathrm{d}_b \hat{\sigma} + \mathrm{d}_b \hat{c}_a + \upsilon_{abc}(\mathrm{d}^c \hat{\xi} + \hat{w}^c) + \frac{1}{2}\hat{q}_{ab}$

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2. Properties of irreducible components:

```
In[]:= {BD[T3a][CT[TauSSt][-T3a, -T3b]], CT[TauSSt][T3a, -T3a],
    CT[TauSSt][-T3a, -T3b] - CT[TauSSt][-T3b, -T3a]}
```

Out[]= $\{\mathrm{d}^a\hat{q}_{ab}, \hat{q}^a{}_a, \hat{q}_{ab} - \hat{q}_{ba}\}$

```
In[]:= IrrDecomp /@ %
```

Out[]= $\{0, 0, 0\}$

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```
In[]:= IrrDecomp /@ %
```

Out[]= $\{0, 0, 0\}$

3. Similar expansions for gravitational field and energy-momentum:

```
In[]:= ExpandGrav[CT[GravPert][-T3a, -LI[0]]]
```

Out[]= $\mathrm{d}_a\hat{Y} + \hat{V}_a$

```
In[]:= ExpandEnMom[CT[EnMomPert][-LI[0], -LI[0]]]
```

Out[]= $\hat{\mathfrak{e}} + \rho\hat{\phi}$

# Work in progress: gauge-invariant quantities

1. Gauge-invariant tetrad perturbation:

```
In[]:= ConvFromGaugeInvTau[CT[GinvTauSSva][T3a]]
```

Out[]= $\hat{w}^a + \mathscr{A}\hat{c}^a$

```
In[]:= ConvToGaugeInvTau[%]
```

Out[]= $\hat{\mathbf{w}}^a$

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In[]:= ConvToGaugeInvTau[%]
```
Out[]= $\hat{\mathbf{w}}^a$

2. Gauge-invariant gravitational field perturbation:

```
In[]:= ConvFromGaugeInvGrav[CT[GinvGravPertSSsa][]]
```
Out[]= $\hat{\Xi} + \mathscr{A}\mathfrak{H}\hat{\sigma}$
```
In[]:= ConvToGaugeInvGrav[%]
```
Out[]= $\hat{\boldsymbol{\Xi}}$

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```
In[]:= ConvToGaugeInvTau[%]
```

Out[]= $\hat{\mathbf{w}}^a$

2. Gauge-invariant gravitational field perturbation:

```
In[]:= ConvFromGaugeInvGrav[CT[GinvGravPertSSsa][]]
```

Out[]= $\hat{\Xi} + \mathscr{A}\mathfrak{H}\hat{\sigma}$

```
In[]:= ConvToGaugeInvGrav[%]
```

Out[]= $\hat{\boldsymbol{\Xi}}$

3. Gauge-invariant time-time component of field equations:

```
In[]:= CT[GinvGravPert][-LI[0], -LI[0]] -
    CT[GinvEnMomPert][-LI[0], -LI[0]];
In[]:= % // ExpandGrav // ExpandEnMom
```

Out[]= $\hat{\boldsymbol{\Phi}} - \hat{\mathcal{E}} - \rho\hat{\phi}$

# Work in progress: choice of time coordinate

1. Derivatives with respect to cosmological and conformal time:

```
In[]:= {DCosmTime[ScaleF[]], DConfTime[ScaleF[]]}
```

Out[]= $\left\{\frac{\partial_0 A}{N}, \frac{A\partial_0 A}{N}\right\}$

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2. Hubble parameter:

```
In[]:= Hubble[]
```

Out[]= $H$

```
In[]:= HubbleToDScale[%]
```

Out[]= $\frac{\partial_0 A}{NA}$

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```
In[]:= Hubble[]
```
Out[]= $H$
```
In[]:= HubbleToDScale[%]
```
Out[]= $\frac{\partial_0 A}{NA}$

3. Conformal Hubble parameter:

```
In[]:= CHubble[]
```
Out[]= $\mathcal{H}$
```
In[]:= CHubbleToDScale[%]
```
Out[]= $\frac{\partial_0 A}{N}$

# Outline

# Summary

- Metric-affine and teleparallel geometries and their perturbations:
  - Geometric description using Lorentzian metric and affine connection.
  - Alternative description in terms of tetrad and spin connection.
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- Computational tools applicable to perturbation theory:
  - Geometric nature of gravity theories suggest using tensor algebra.
  - Fixed schemes in perturbation theory suitable for algorithmic approach.
  - Example: *xPPN* package for *xAct* / Mathematica allows calculating PPN parameters.
  - Work in progress: further package for cosmological perturbations.

# Outlook

- Further extensions of *xPPN*:
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