xPPN: a tool for calculating the parametrized post-Newtonian limit

http://geomgrav.fi.ut.ee/software/xPPN.html

Manuel Hohmann

Laboratory of Theoretical Physics - Institute of Physics - University of Tartu
Center of Excellence “The Dark Side of the Universe”

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Motivation

- Parametrized post-Newtonian formalism:
  - Weak-field approximation of metric gravity theories.
  - Assumes particular coordinate system (“universe rest frame”).
  - Characterizes gravity theories by 10 (constant) parameters.
  - Parameters closely related to solar system observations.

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⇝ Implement generic PPN formalism using computer tensor algebra.

- Implementation as package using *xAct* for Mathematica:
  - Mathematica offers powerful routines for symbolic calculations.
  - *xAct* implements numerous functions for tensor algebra.
  - *xAct* can easily be extended with new functionality.

Key features

1. Pre-defined geometric objects:
   - Metric and tetrad based geometries.
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2. Variables specific to PPN formalism:
   - Energy-momentum variables: density, pressure, specific internal energy, velocity.
   - Post-Newtonian potentials: $\chi, U, U_{ab}, V_a, W_a, \Phi_1\Phi_2, \Phi_3, \Phi_4, \Phi_W, \mathcal{A}, \mathcal{B}$.
   - Post-Newtonian parameters: $\gamma, \beta, \alpha_1, \alpha_2, \alpha_3, \zeta_1, \zeta_2, \zeta_3, \zeta_4, \xi$.

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   - Post-Newtonian parameters: $\gamma, \beta, \alpha_1, \alpha_2, \alpha_3, \zeta_1, \zeta_2, \zeta_3, \zeta_4, \xi$.
3. Algorithms typically used in PPN formalism:
   - $3 + 1$ decomposition of tensors and connection coefficients into time and space.
   - Perturbative expansion and decomposition into velocity orders.
   - Correct assignment of velocity order $+1$ to time derivative.
   - Both built-in rules and user-defined rules for perturbative expansion.
   - Known transformation rules for transforming between PPN potentials.
   - Transformation of derivatives on PPN potentials to matter source terms.
   - Application of Euler equations of motion to fluid variables.

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4. Open some example from `Examples` folder and run all code:
   - `GeneralRelativity.wl` - General Relativity (GR).
   - `BransDicke.wl` - Brans-Dicke type scalar-tensor gravity with dynamical coupling.
   - `NewGeneralRelativity.wl` - New GR class of teleparallel gravity.
   - `ScalarTorsion.wl` - General scalar-torsion class of teleparallel gravity.
   - `NewerGeneralRelativity.wl` - Newer GR class of symmetric teleparallel gravity.

   NB! For some examples, calculations are time consuming!

Some basic usage

1. Several types of indices are pre-defined (examples):
   - Greek indices $\alpha, \ldots, \omega$, entered as `T4α`, . . . , `T4ω`, on spacetime:

```
In[]:= Met[-T4α, -T4β]
Out[]= g_αβ
```

   - Latin indices $a, \ldots, z$, entered as `T3a`, . . . , `T3z`, on space:

```
In[]:= Velocity[T3a]
Out[]= v^a
```

   - Time components use inert index `LI[0]`.

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In[]:= ParamD[TimePar][Density[]]
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Out[]= $\partial_0 \rho$

3. Selecting single terms in perturbative expansion:

```
In[]:= PPN[Met, 3][-LI[0], -T3a]
```
Out[]= $\overset{3}{g}_{0a}$

Example: PPN expansion of the Ricci tensor

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Out[]= $\{\{R_{00}, R_{0b}\}, \{R_{a0}, R_{ab}\}\}$

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3. Extract second velocity order $\overset{2}{R}_{00}$:

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In[]:= VelocityOrder[%[[1, 1]], 2]
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Out[]= $-\frac{1}{2}\partial_a\partial^a\overset{2}{g}_{00}$

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4. Extract third velocity order $\overset{3}{R}_{a0}$:

```
In[]:= Factor[SortPDs[ToCanonical[VelocityOrder[%%[[2, 1]], 3]]]]
```

Out[]= $\frac{1}{2}\left(-\partial_0\partial_a\overset{2}{g}{}^b{}_b + \partial_0\partial_b\overset{2}{g}{}_a{}^b + \partial_b\partial_a\overset{3}{g}{}_0{}^b - \partial_b\partial^b\overset{3}{g}{}_{0a}\right)$

Example: energy-momentum conservation and Euler equations

1. Consider energy-momentum conservation equation:

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In[]:= InvMet[T4β, T4γ] CD[-T4γ][EnergyMomentum[-T4β, -T4α]]
```

Out[]= $g^{\beta\gamma}\mathring{\nabla}_{\gamma}\Theta_{\beta\alpha}$

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2. Extract third order time component:

```
In[]:= ChangeCovD[%, CD, PD];
In[]:= SpaceTimeSplit[%, {-T4α → -LI[0]}];
In[]:= VelocityOrder[%, 3];
In[]:= ContractMetric[%];
In[]:= ToCanonical[%]
```

Out[]= $-\partial_0\rho - v^a\partial_a\rho - \rho\partial_a v^a$

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Out[]= $-\partial_0\rho - v^a\partial_a\rho - \rho\partial_a v^a$

3. Apply Euler equation of perfect fluid:

```
In[]:= TimeRhoToEuler[%]
```

Out[]= 0

Example: third order metric and vector PPN potentials

1. Standard PPN expansion of third-order metric perturbation:

```
In[]:= MetricToStandard[PPN[Met, 3][-LI[0], -T3a]];
In[]:= Collect[%, {PotentialV[-T3a], PotentialW[-T3a]}, Factor]
```

Out[]= $\frac{1}{2}(-3-\alpha_1+\alpha_2-4\gamma+2\xi-\zeta_1)V_a+\frac{1}{2}(-1-\alpha_2-2\xi+\zeta_1)W_a$

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2. Well-known relations satisfied by vector potentials:
   - Sum of vector potentials is divergence-free vector:

```
In[]:= PD[-T3a][PotentialV[T3a] + PotentialW[T3a]]
```

Out[]= $\partial_a V^a + \partial_a W^a$

```
In[]:= PotentialVToW[%]
```

Out[]= 0

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```
In[]:= PotentialVToW[%]
```

Out[]= 0

   - Difference of vector potentials is pure divergence:

```
In[]:= PotentialV[-T3a] - PotentialW[-T3a]
```

Out[]= V_a-W_a

```
In[]:= PotentialVToChiW[%]
```

Out[]= $\partial_0\partial_a\chi$

Example: defining a new scalar field and its expansion

1. Define scalar field ψ and its constant background value Ψ:

```
In[]:= DefTensor[psi[], {MfSpacetime}, PrintAs → "ψ"]
In[]:= DefConstantSymbol[psi0, PrintAs → "Ψ"]
```

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2. Define rules $\overset{0}{\psi} = \Psi$, $\overset{1}{\psi} = \overset{3}{\psi} = 0$ for PPN expansion:

```
In[]:= OrderSet[PPN[psi, 0][], psi0];
In[]:= OrderSet[PPN[psi, 1][], 0];
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3. Rules are now used automatically, e.g., second-order space component of $\partial^\beta(\psi g_{\beta\alpha})$:

```
In[]:= PD[T4β][Met[-T4β, -T4α] psi[]]
```

Out[]= $\psi\partial^\beta g_{\beta\alpha} + g_{\beta\alpha}\partial^\beta\psi$

```
In[]:= SpaceTimeSplit[%, {-T4α → -T3a}];
In[]:= VelocityOrder[%, 2];
In[]:= ToCanonical[ContractMetric[%]]
```

Out[]= $\partial_a\overset{2}{\psi} + \Psi\partial_b\overset{2}{g}_a{}^b$

Example: general relativity - a screenshot of *xPPN*

PPN metric and parameters

PPN metric

To read off the PPN parameters, we use the following metric components.

In[•]:=
```
metcomp = {PPN[Met,2][-LI[0],-LI[0]], PPN[Met,2][-T3a,-T3b], PPN[Met,3][-LI[0],-T3a], PPN[Met,4][-LI[0],-LI[0]]}
```

Out[•]= $\{ {}^{2}g_{00}, {}^{2}g_{ab}, {}^{3}g_{0a}, {}^{4}g_{00} \}$

Insert the solution we obtained into the metric components.

In[•]:=
```
metcomp /. sol2ru /. sol3ru /. sol4ru;
ToCanonical[%];
Expand[%];
ppnmet = Simplify[%];
metdef = MapThread[Equal, {metcomp, %}, 1]
```

Out[•]= $\left\{ {}^{2}g_{00} == \frac{\kappa^2 U}{4\pi}, {}^{2}g_{ab} == \frac{\kappa^2\, \delta_{ab}\, U}{4\pi}, {}^{3}g_{0a} == -\frac{\kappa^2 (7\, V_a + W_a)}{16\pi}, {}^{4}g_{00} == \frac{8\kappa^2 \pi (2\Phi_1 + \Phi_3 + 3\Phi_4) + \kappa^4 (2\Phi_2 - U^2)}{32\pi^2} \right\}$

PPN parameters

Finally, solve the equations and determine the PPN parameters.

In[•]:=
```
parsol = FullSimplify[Solve[# == 0& /@ eqns, pars][[1]]]
```

Out[•]= $\{\beta \to 1, \gamma \to 1, \xi \to 0, \alpha_1 \to 0, \alpha_2 \to 0, \alpha_3 \to 0, \zeta_1 \to 0, \zeta_2 \to 0, \zeta_3 \to 0, \zeta_4 \to 0\}$

Closer look at one example: general relativity

1. Starting point is trace-reversed Einstein equation:

$$\mathcal{E}_{\alpha\beta} == -\kappa^2\left(\Theta_{\alpha\beta} - \frac{1}{2}\,\Theta_{\gamma\delta}\;\mathrm{ig}^{\gamma\delta}\;\;g_{\alpha\beta}\right) + \mathrm{R}[\overset{\circ}{\nabla}]_{\alpha\beta}$$

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2. Example: second velocity order.
   (1) Extract second-order field equations:

$$\left\{\frac{1}{2}\left(-\kappa^2\,\rho - \partial_a\partial^a \overset{2}{\mathrm{g}}_{00}\right) == 0,\ \frac{1}{2}\left(-\kappa^2\,\delta_{ba}\,\rho + \partial_b\partial_a \overset{2}{\mathrm{g}}_{00} - \partial_b\partial_a \overset{2}{\mathrm{g}}{}^{c}{}_{c} + \partial_c\partial_a \overset{2}{\mathrm{g}}_{b}{}^{c} + \partial_c\partial_b \overset{2}{\mathrm{g}}_{a}{}^{c} - \partial_c\partial^c \overset{2}{\mathrm{g}}_{ab}\right) == 0\right\}$$

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$$\left\{\overset{2}{g}_{00} == a_1\,U,\ \overset{2}{g}_{ab} == a_2\,\delta_{ab}\ U + a_3\ U_{ab}\right\}$$

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$$\left\{\overset{2}{g}_{00} == \frac{\kappa^2\,U}{4\,\pi},\; \overset{2}{g}_{ab} == \frac{\kappa^2\;\delta_{ab}\;U}{4\,\pi},\; \overset{3}{g}_{0a} == -\frac{\kappa^2\left(7\;V_a + W_a\right)}{16\,\pi},\; \overset{4}{g}_{00} == \frac{8\,\kappa^2\,\pi\,(2\,\Phi_1 + \Phi_3 + 3\,\Phi_4) + \kappa^4\,(2\,\Phi_2 - U^2)}{32\,\pi^2}\right\}$$

4. Obtain PPN parameters by comparing with standard PPN metric:

$$\{\beta == 1,\; \gamma == 1,\; \xi == 0,\; \alpha_1 == 0,\; \alpha_2 == 0,\; \alpha_3 == 0,\; \zeta_1 == 0,\; \zeta_2 == 0,\; \zeta_3 == 0,\; \zeta_4 == 0\}$$

$$
\begin{aligned}
\Big\{ & \overset{0}{\mathcal{E}}_{00} == 0,\ \overset{1}{\mathcal{E}}_{00} == 0,\ \overset{2}{\mathcal{E}}_{00} == \frac{1}{2}\left(-\kappa^2 \rho - \partial_a\partial^a \overset{2}{g}_{00}\right),\ \overset{3}{\mathcal{E}}_{00} == 0, \\
& \overset{4}{\mathcal{E}}_{00} == \frac{1}{4}\Big(-2\kappa^2 \rho \Pi - 6\kappa^2 p - 4\kappa^2 \rho\, v_a\, v^a + 4\,\partial_0\partial_a \overset{3}{g}_0{}^a - 2\,\partial_0\partial_0 \overset{2}{g}{}^a{}_a - 2\,\partial_a\partial^a \overset{4}{g}_{00} - \partial_a \overset{2}{g}_{00}\, \partial^a \overset{2}{g}_{00} - \\
& \quad \partial_a \overset{2}{g}{}^b{}_b\, \partial^a \overset{2}{g}_{00} + 2\,\partial^a \overset{2}{g}_{00}\, \partial_b \overset{2}{g}_a{}^b + 2\kappa^2 \rho\, \overset{2}{g}_{00} + 2\,\partial^b\partial^a \overset{2}{g}_{00}\, \overset{2}{g}_{ab}\Big),\ \overset{0}{\mathcal{E}}_{0a} == 0,\ \overset{0}{\mathcal{E}}_{0b} == 0,\ \overset{1}{\mathcal{E}}_{0a} == 0, \\
& \overset{1}{\mathcal{E}}_{0b} == 0,\ \overset{2}{\mathcal{E}}_{0a} == 0,\ \overset{2}{\mathcal{E}}_{0b} == 0,\ \overset{3}{\mathcal{E}}_{0a} == \frac{1}{2}\left(2\kappa^2 \rho\, v_a - \partial_0\partial_a \overset{2}{g}{}^b{}_b + \partial_0\partial_b \overset{2}{g}_a{}^b + \partial_b\partial_a \overset{3}{g}_0{}^b - \partial_b\partial^b \overset{3}{g}_{0a}\right), \\
& \overset{3}{\mathcal{E}}_{0b} == \frac{1}{2}\left(2\kappa^2 \rho\, v_b + \partial_0\partial_a \overset{2}{g}_b{}^a - \partial_0\partial_b \overset{2}{g}{}^a{}_a - \partial_a\partial^a \overset{3}{g}_{0b} + \partial_a\partial_b \overset{3}{g}_0{}^a\right),\ \overset{4}{\mathcal{E}}_{0a} == 0,\ \overset{4}{\mathcal{E}}_{0b} == 0,\ \overset{0}{\mathcal{E}}_{ab} == 0, \\
& \overset{1}{\mathcal{E}}_{ab} == 0,\ \overset{2}{\mathcal{E}}_{ab} == \frac{1}{2}\left(-\kappa^2\, \delta_{ab}\, \rho + \partial_b\partial_a \overset{2}{g}_{00} - \partial_b\partial_a \overset{2}{g}{}^c{}_c + \partial_c\partial_a \overset{2}{g}_b{}^c + \partial_c\partial_b \overset{2}{g}_a{}^c - \partial_c\partial^c \overset{2}{g}_{ab}\right),\ \overset{3}{\mathcal{E}}_{ab} == 0, \\
& \overset{4}{\mathcal{E}}_{ab} == \frac{1}{4}\Big(2\kappa^2\, \delta_{ab}\, (-\rho\Pi + p) - 4\kappa^2 \rho\, v_a\, v_b - 2\,\partial_0\partial_a \overset{3}{g}_{0b} - 2\,\partial_0\partial_b \overset{3}{g}_{0a} + 2\,\partial_0\partial_0 \overset{2}{g}_{ab} + 2\,\partial_b\partial_a \overset{4}{g}_{00} - \\
& \quad 2\,\partial_b\partial_a \overset{4}{g}{}^c{}_c + \partial_a \overset{2}{g}_{00}\, \partial_b \overset{2}{g}_{00} + \partial_a \overset{2}{g}{}^{cd}\, \partial_b \overset{2}{g}_{cd} + 2\,\partial_c\partial_a \overset{4}{g}_b{}^c + 2\,\partial_c\partial_b \overset{4}{g}_a{}^c - 2\,\partial_c\partial^c \overset{4}{g}_{ab} + \partial_a \overset{2}{g}_b{}^c\, \partial_c \overset{2}{g}{}^d{}_d + \\
& \quad \partial_b \overset{2}{g}_a{}^c\, \partial_c \overset{2}{g}{}^d{}_d - \partial_a \overset{2}{g}_{bc}\, \partial^c \overset{2}{g}_{00} - \partial_b \overset{2}{g}_{ac}\, \partial^c \overset{2}{g}_{00} + \partial_c \overset{2}{g}_{ab}\, \partial^c \overset{2}{g}_{00} - \partial_c \overset{2}{g}{}^d{}_d\, \partial^c \overset{2}{g}_{ab} - 2\,\partial_a \overset{2}{g}_b{}^c\, \partial_d \overset{2}{g}_c{}^d - \\
& \quad 2\,\partial_b \overset{2}{g}_a{}^c\, \partial_d \overset{2}{g}_c{}^d + 2\,\partial^c \overset{2}{g}_{ab}\, \partial_d \overset{2}{g}_c{}^d - 2\,\partial_c \overset{2}{g}_{bd}\, \partial^d \overset{2}{g}_a{}^c + 2\,\partial_d \overset{2}{g}_{bc}\, \partial^d \overset{2}{g}_a{}^c + 2\,\partial_b\partial_a \overset{2}{g}_{00}\, \overset{2}{g}_{00} - \\
& \quad 2\kappa^2 \rho\, \overset{2}{g}_{ab} + 2\,\partial_b\partial_a \overset{2}{g}_{cd}\, \overset{2}{g}{}^{cd} - 2\,\partial_d\partial_a \overset{2}{g}_{bc}\, \overset{2}{g}{}^{cd} - 2\,\partial_d\partial_b \overset{2}{g}_{ac}\, \overset{2}{g}{}^{cd} + 2\,\partial_d\partial_c \overset{2}{g}_{ab}\, \overset{2}{g}{}^{cd}\Big)\Big\}
\end{aligned}
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- More general geometric frameworks:
  - Other connections: general teleparallel, Riemann-Cartan, general affine.
  - Multiple metric tensors.

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