Perturbative methods in modified gravity theories

Manuel Hohmann

Laboratory of Theoretical Physics - Institute of Physics - University of Tartu
Center of Excellence "The Dark Side of the Universe"

European Union
European Regional
Development Fund

Investing
in your future

740. WE-Heraeus-Seminar, 5. 2. 2021

Outline

Outline

Field equations in gravity theory

- Partial differential equation (PDE) perspective:
  - Independent variables x^μ and dependent variables $y^A(x)$.
  - Consider partial derivatives $y^A_\mu = \partial_\mu y^A$, $y^A_{\mu\nu} = \partial_\mu\partial_\nu y^A$, $y^A_{\mu\nu\rho} = \partial_\mu\partial_\nu\partial_\rho y^A \ldots$
  - General form of the field equations given by PDE system:

$$\mathcal{E}_A(x^\mu, y^A, y^A_\mu, y^A_{\mu\nu}, \ldots) = 0\,. \tag{1}$$

  - Physical fields given as solutions $(x^\mu) \mapsto y^A(x)$ of PDE system.

Field equations in gravity theory

- Partial differential equation (PDE) perspective:
  - Independent variables x^μ and dependent variables $y^A(x)$.
  - Consider partial derivatives $y^A_\mu = \partial_\mu y^A$, $y^A_{\mu\nu} = \partial_\mu\partial_\nu y^A$, $y^A_{\mu\nu\rho} = \partial_\mu\partial_\nu\partial_\rho y^A \ldots$
  - General form of the field equations given by PDE system:

$$\mathcal{E}_A(x^\mu, y^A, y^A_\mu, y^A_{\mu\nu}, \ldots) = 0\,. \tag{1}$$

  - Physical fields given as solutions $(x^\mu) \mapsto y^A(x)$ of PDE system.

- Differential geometry perspective:
  - Independent variables x^μ are (local) coordinates on a base manifold X.
  - Dependent variables y^A are (local) fiber coordinates on a fiber bundle $\pi : Y \to X$.
  - Variables $x^\mu, y^A, y^A_\mu, y^A_{\mu\nu}, \ldots$ are coordinates on a jet bundle $J^r(\pi)$.
  - Field equations are components of a differential form on $J^r(\pi)$:

$$\mathcal{E} = \mathcal{E}_A \,\mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^n \wedge \mathrm{d}y^A\,. \tag{2}$$

  - Physical geometries given as sections $\sigma : X \to Y$ whose jet prolongation make $\mathcal{E}$ vanish.

Field equations in gravity theory

- Partial differential equation (PDE) perspective:
  - Independent variables x^μ and dependent variables $y^A(x)$.
  - Consider partial derivatives $y^A_\mu = \partial_\mu y^A$, $y^A_{\mu\nu} = \partial_\mu\partial_\nu y^A$, $y^A_{\mu\nu\rho} = \partial_\mu\partial_\nu\partial_\rho y^A \ldots$
  - General form of the field equations given by PDE system:

$$\mathcal{E}_A(x^\mu, y^A, y^A_\mu, y^A_{\mu\nu}, \ldots) = 0\,. \tag{1}$$

  - Physical fields given as solutions $(x^\mu) \mapsto y^A(x)$ of PDE system.

  ⇒ Useful for practical calculations, finding solutions etc.
- Differential geometry perspective:
  - Independent variables x^μ are (local) coordinates on a base manifold X.
  - Dependent variables y^A are (local) fiber coordinates on a fiber bundle $\pi : Y \to X$.
  - Variables $x^\mu, y^A, y^A_\mu, y^A_{\mu\nu}, \ldots$ are coordinates on a jet bundle $J^r(\pi)$.
  - Field equations are components of a differential form on $J^r(\pi)$:

$$\mathcal{E} = \mathcal{E}_A \, \mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^n \wedge \mathrm{d}y^A\,. \tag{2}$$

  - Physical geometries given as sections $\sigma : X \to Y$ whose jet prolongation make $\mathcal{E}$ vanish.

  ⇒ Useful for understanding the structure behind the equations.

Geometries used in the description of gravity

base manifold	X	total space	Y	field	σ
spacetime	M	metric bundle	$\mathrm{LorMet}(M) \subset T^0_2M$	metric	g
		frame bundle	$\mathrm{GL}(M)$	tetrad	θ
		connection bundle	$\mathrm{Aff}(M)$	connection	Γ
		trivial bundle	$M \times Z$	scalar fields	ϕ^A
		tensor bundle	T^r_sM	tensor field	A
tangent bundle	TM	trivial line bundle	$TM \times \mathbb{R}$	Lagrangian	L
cotangent bundle	T^*M	trivial line bundle	$T^*M \times \mathbb{R}$	Hamiltonian	H
projective bundle	PTM^+	associated bundle	$\mathring{T}M \times_{PTM^+} \mathbb{R}^*_+$	Finsler function	F

Perturbations of geometry and field equations

- Consider family of field equations instead of single equation:
  - Additional dependence of $\mathcal{E}_A$ on perturbation parameter ϵ.
  - Background equations given by limit $\epsilon \to 0$.

Perturbations of geometry and field equations

- Consider family of field equations instead of single equation:
  - Additional dependence of $\mathcal{E}_A$ on perturbation parameter ϵ.
  - Background equations given by limit $\epsilon \to 0$.

$\Rightarrow$ Consider family σ_ϵ of fields instead of single field σ:
  - For every value of ϵ, σ_ϵ solves corresponding field equation.
  - $\bar{\sigma} = \sigma_0$ is solution of background equations.
  - ? "Taylor expansion" in ϵ of solution σ_ϵ around $\epsilon = 0$?

Perturbations of geometry and field equations

- Consider family of field equations instead of single equation:
  - Additional dependence of $\mathcal{E}_A$ on perturbation parameter ϵ.
  - Background equations given by limit $\epsilon \to 0$.

$\Rightarrow$ Consider family σ_ϵ of fields instead of single field σ:
  - For every value of ϵ, σ_ϵ solves corresponding field equation.
  - $\bar{\sigma} = \sigma_0$ is solution of background equations.
  - ? "Taylor expansion" in ϵ of solution σ_ϵ around $\epsilon = 0$?

↯ Difficulties of applying a Taylor expansion to fields σ:
  - Values might not form a linear space (e.g., frame bundle) - no well-defined sum of terms.
  - Perturbative expansion defined only locally around background solution.
  - Expansion of coordinate expressions $x \mapsto y^A(x, \epsilon)$ depends on coordinate choice.

Perturbations of geometry and field equations

- Consider family of field equations instead of single equation:
  - Additional dependence of $\mathcal{E}_A$ on perturbation parameter ϵ.
  - Background equations given by limit $\epsilon \to 0$.

$\Rightarrow$ Consider family σ_ϵ of fields instead of single field σ:
  - For every value of ϵ, σ_ϵ solves corresponding field equation.
  - $\bar{\sigma} = \sigma_0$ is solution of background equations.
  - ? "Taylor expansion" in ϵ of solution σ_ϵ around $\epsilon = 0$?

$\lightning$ Difficulties of applying a Taylor expansion to fields σ:
  - Values might not form a linear space (e.g., frame bundle) - no well-defined sum of terms.
  - Perturbative expansion defined only locally around background solution.
  - Expansion of coordinate expressions $x \mapsto y^A(x, \epsilon)$ depends on coordinate choice.

$\Rightarrow$ Proper geometric treatment of perturbation theory:
  - Linear change of solution σ_ϵ given by vertical vector field on Y.
  - Higher order expansion uses theory of jet bundles.
  - ✓ Well-defined expressions for perturbations at arbitrary perturbation order.
  - $\Rightarrow$ Perturbations of coordinate expressions derived from well-defined procedure.

Dependence of field equations on perturbation parameter ϵ

- Approximation of exact field equations around a given solution.
  - Example: linearized gravitational waves in vacuum general relativity.
  - Assume vacuum Einstein equations $G_{\mu\nu} = 0$: no parameter dependence.
  - Consider metric $g_{\mu\nu} = \eta_{\mu\nu} + \epsilon h_{\mu\nu}$ as perturbation of Minkowski metric.

  $\Rightarrow$ Taylor expansion in ϵ yields linear equation for perturbation $h_{\mu\nu}$.

Dependence of field equations on perturbation parameter ϵ

- Approximation of exact field equations around a given solution.
  - Example: linearized gravitational waves in vacuum general relativity.
  - Assume vacuum Einstein equations $G_{\mu\nu} = 0$: no parameter dependence.
  - Consider metric $g_{\mu\nu} = \eta_{\mu\nu} + \epsilon h_{\mu\nu}$ as perturbation of Minkowski metric.

  $\Rightarrow$ Taylor expansion in ϵ yields linear equation for perturbation $h_{\mu\nu}$.
- Approximation of a physical system around a simpler system.
  - Example: weak-field approximations of general relativity.
  - Assume Einstein equations $G_{\mu\nu} = 8\pi G T_{\mu\nu}$ and expand in $\epsilon \sim G$.

  $\Rightarrow$ Background given by vacuum Einstein equations.

Dependence of field equations on perturbation parameter ϵ

- Approximation of exact field equations around a given solution.
  - Example: linearized gravitational waves in vacuum general relativity.
  - Assume vacuum Einstein equations $G_{\mu\nu} = 0$: no parameter dependence.
  - Consider metric $g_{\mu\nu} = \eta_{\mu\nu} + \epsilon h_{\mu\nu}$ as perturbation of Minkowski metric.

  $\Rightarrow$ Taylor expansion in ϵ yields linear equation for perturbation $h_{\mu\nu}$.
- Approximation of a physical system around a simpler system.
  - Example: weak-field approximations of general relativity.
  - Assume Einstein equations $G_{\mu\nu} = 8\pi G T_{\mu\nu}$ and expand in $\epsilon \sim G$.

  $\Rightarrow$ Background given by vacuum Einstein equations.
- Approximation of a modified gravity theory around a well-known theory.
  - Example: modification of Einstein-Hilbert action with higher-order terms:

$$S = \frac{1}{16\pi G}\int_M d^4x\sqrt{-g}\left(R + \alpha R^2 + \beta R^{\mu\nu}R_{\mu\nu} + \ldots\right) . \tag{3}$$

  $\Rightarrow$ Background $\alpha = \beta = 0$ given by Einstein equations.
  - Metric as perturbation $g_{\mu\nu} = \overset{0}{g}_{\mu\nu} + \alpha h_{\mu\nu} + \beta j_{\mu\nu} + \ldots$.

Perturbations around symmetric backgrounds

- Background solution $\bar{\sigma}$ conventionally assumed invariant under group action.

Perturbations around symmetric backgrounds

- Background solution $\bar{\sigma}$ conventionally assumed invariant under group action.
- Common examples in gravity theory:
  - Maximal symmetry (in particular Poincaré symmetry) of Riemannian geometry.
  - Cosmological symmetry (spatial homogeneity and either full or partial isotropy).
  - Spherical or axial symmetry.

Perturbations around symmetric backgrounds

- Background solution $\bar{\sigma}$ conventionally assumed invariant under group action.
- Common examples in gravity theory:
  - Maximal symmetry (in particular Poincaré symmetry) of Riemannian geometry.
    - Propagation of gravitational waves: Newman-Penrose formalism, polarization.
    - Weak-field approximation: Newtonian, post-Newtonian.
  - Cosmological symmetry (spatial homogeneity and either full or partial isotropy).
    - Early universe: inflation, cosmic microwave background.
    - Density fluctuations and growth of structure.
    - Propagation of gravitational waves from distant sources, primordial waves.
  - Spherical or axial symmetry.
    - Quasinormal modes of gravitational waves from compact objects.
    - Extreme mass ratio inspirals: gravitational waves from small orbiting mass.
- Wide applicability to physical systems.

Gauge transformations

- Concept of gauge transformations:
  - Family of maps $\Phi_\epsilon : Y \to Y$ on the values of physical fields.
  - Maps must be fiber preserving: $\pi \circ \Phi_\epsilon = \varphi_\epsilon \circ \pi$ for some $\varphi_\epsilon : X \to X$.
  - $\Rightarrow$ Transformation of fields $\sigma'_\epsilon = \Phi_\epsilon^{-1} \circ \sigma_\epsilon \circ \varphi_\epsilon$.
  - Must preserve background solution: $\sigma'_0 = \sigma_0$.
  - Must preserve field equations: σ'_ϵ is solution if and only if σ_ϵ is.

Gauge transformations

- Concept of gauge transformations:
  - Family of maps $\Phi_\epsilon : Y \to Y$ on the values of physical fields.
  - Maps must be fiber preserving: $\pi \circ \Phi_\epsilon = \varphi_\epsilon \circ \pi$ for some $\varphi_\epsilon : X \to X$.
  - ⇒ Transformation of fields $\sigma'_\epsilon = \Phi_\epsilon^{-1} \circ \sigma_\epsilon \circ \varphi_\epsilon$.
  - Must preserve background solution: $\sigma'_0 = \sigma_0$.
  - Must preserve field equations: σ'_ϵ is solution if and only if σ_ϵ is.

⇒ Transformation of perturbative expansion of σ_ϵ:
  - Consider $\sigma = \bar{\sigma} + \delta\sigma$ and $\sigma' = \bar{\sigma} + \delta\sigma'$ as perturbation around same background $\bar{\sigma}$.
  - Relation between $\delta\sigma$ and $\delta\sigma'$ given by vector field Ξ on Y generating Φ.
  - Vector field Ξ projects to vector field ξ on X generating φ.

Gauge transformations

- Concept of gauge transformations:
  - Family of maps $\Phi_\epsilon : Y \to Y$ on the values of physical fields.
  - Maps must be fiber preserving: $\pi \circ \Phi_\epsilon = \varphi_\epsilon \circ \pi$ for some $\varphi_\epsilon : X \to X$.
  - $\Rightarrow$ Transformation of fields $\sigma'_\epsilon = \Phi_\epsilon^{-1} \circ \sigma_\epsilon \circ \varphi_\epsilon$.
  - Must preserve background solution: $\sigma'_0 = \sigma_0$.
  - Must preserve field equations: σ'_ϵ is solution if and only if σ_ϵ is.

$\Rightarrow$ Transformation of perturbative expansion of σ_ϵ:
  - Consider $\sigma = \bar{\sigma} + \delta\sigma$ and $\sigma' = \bar{\sigma} + \delta\sigma'$ as perturbation around same background $\bar{\sigma}$.
  - Relation between $\delta\sigma$ and $\delta\sigma'$ given by vector field Ξ on Y generating Φ.
  - Vector field Ξ projects to vector field ξ on X generating φ.
- Example: diffeomorphism invariance of metric tensor:
  - Transformation given by pullback of covariant tensor field:

$$g'_{\mu\nu}(x) = \frac{\partial x'^\alpha}{\partial x^\mu}\frac{\partial x'^\beta}{\partial x^\nu} g_{\alpha\beta}(x'(x)) . \tag{4}$$

Gauge transformations

- Concept of gauge transformations:
  - Family of maps $\Phi_\epsilon : Y \to Y$ on the values of physical fields.
  - Maps must be fiber preserving: $\pi \circ \Phi_\epsilon = \varphi_\epsilon \circ \pi$ for some $\varphi_\epsilon : X \to X$.
  - ⇒ Transformation of fields $\sigma'_\epsilon = \Phi_\epsilon^{-1} \circ \sigma_\epsilon \circ \varphi_\epsilon$.
  - Must preserve background solution: $\sigma'_0 = \sigma_0$.
  - Must preserve field equations: σ'_ϵ is solution if and only if σ_ϵ is.

⇒ Transformation of perturbative expansion of σ_ϵ:
  - Consider $\sigma = \bar{\sigma} + \delta\sigma$ and $\sigma' = \bar{\sigma} + \delta\sigma'$ as perturbation around same background $\bar{\sigma}$.
  - Relation between $\delta\sigma$ and $\delta\sigma'$ given by vector field Ξ on Y generating Φ.
  - Vector field Ξ projects to vector field ξ on X generating φ.

- Example: diffeomorphism invariance of metric tensor:
  - Transformation given by pullback of covariant tensor field:

$$g'_{\mu\nu}(x) = \frac{\partial x'^\alpha}{\partial x^\mu}\frac{\partial x'^\beta}{\partial x^\nu} g_{\alpha\beta}(x'(x))\,. \tag{4}$$

  - Transformation $\varphi : X \to X$ of base manifold given by coordinate change $x \mapsto x'(x)$.

Gauge transformations

- Concept of gauge transformations:
  - Family of maps $\Phi_\epsilon : Y \to Y$ on the values of physical fields.
  - Maps must be fiber preserving: $\pi \circ \Phi_\epsilon = \varphi_\epsilon \circ \pi$ for some $\varphi_\epsilon : X \to X$.

  $\Rightarrow$ Transformation of fields $\sigma'_\epsilon = \Phi_\epsilon^{-1} \circ \sigma_\epsilon \circ \varphi_\epsilon$.
  - Must preserve background solution: $\sigma'_0 = \sigma_0$.
  - Must preserve field equations: σ'_ϵ is solution if and only if σ_ϵ is.

$\Rightarrow$ Transformation of perturbative expansion of σ_ϵ:
  - Consider $\sigma = \bar{\sigma} + \delta\sigma$ and $\sigma' = \bar{\sigma} + \delta\sigma'$ as perturbation around same background $\bar{\sigma}$.
  - Relation between $\delta\sigma$ and $\delta\sigma'$ given by vector field Ξ on Y generating Φ.
  - Vector field Ξ projects to vector field ξ on X generating φ.

- Example: diffeomorphism invariance of metric tensor:
  - Transformation given by pullback of covariant tensor field:

$$g'_{\mu\nu}(x) = \frac{\partial x'^\alpha}{\partial x^\mu}\frac{\partial x'^\beta}{\partial x^\nu} g_{\alpha\beta}(x'(x)) \,. \tag{4}$$

  - Transformation $\varphi : X \to X$ of base manifold given by coordinate change $x \mapsto x'(x)$.
  - Transformation $\Phi : Y \to Y$ of total space given by pullback.

Gauge transformations

- Concept of gauge transformations:
  - Family of maps $\Phi_\epsilon : Y \to Y$ on the values of physical fields.
  - Maps must be fiber preserving: $\pi \circ \Phi_\epsilon = \varphi_\epsilon \circ \pi$ for some $\varphi_\epsilon : X \to X$.
  - ⇒ Transformation of fields $\sigma'_\epsilon = \Phi_\epsilon^{-1} \circ \sigma_\epsilon \circ \varphi_\epsilon$.
  - Must preserve background solution: $\sigma'_0 = \sigma_0$.
  - Must preserve field equations: σ'_ϵ is solution if and only if σ_ϵ is.

⇒ Transformation of perturbative expansion of σ_ϵ:
  - Consider $\sigma = \bar{\sigma} + \delta\sigma$ and $\sigma' = \bar{\sigma} + \delta\sigma'$ as perturbation around same background $\bar{\sigma}$.
  - Relation between $\delta\sigma$ and $\delta\sigma'$ given by vector field Ξ on Y generating Φ.
  - Vector field Ξ projects to vector field ξ on X generating φ.
- Example: diffeomorphism invariance of metric tensor:
  - Transformation given by pullback of covariant tensor field:

$$g'_{\mu\nu}(x) = \frac{\partial x'^\alpha}{\partial x^\mu}\frac{\partial x'^\beta}{\partial x^\nu} g_{\alpha\beta}(x'(x)) \,. \tag{4}$$

  - Transformation $\varphi : X \to X$ of base manifold given by coordinate change $x \mapsto x'(x)$.
  - Transformation $\Phi : Y \to Y$ of total space given by pullback.
  - Infinitesimal transformation given by Lie derivative $\delta g_{\mu\nu} - \delta g'_{\mu\nu} = \mathcal{L}_\xi \bar{g}_{\mu\nu}$ of background.

Outline

Teleparallel geometries

- Fundamental fields in the Palatini / metric-affine formulation:
  - Metric tensor $g_{\mu\nu}$.
  - Flat affine connection $\Gamma^{\mu}{}_{\nu\rho} = 0$: vanishing curvature

$$R^{\rho}{}_{\sigma\mu\nu} = \partial_{\mu}\Gamma^{\rho}{}_{\sigma\nu} - \partial_{\nu}\Gamma^{\rho}{}_{\sigma\mu} + \Gamma^{\rho}{}_{\lambda\mu}\Gamma^{\lambda}{}_{\sigma\nu} - \Gamma^{\rho}{}_{\lambda\nu}\Gamma^{\lambda}{}_{\sigma\mu} = 0\,. \tag{5}$$

Teleparallel geometries

- Fundamental fields in the Palatini / metric-affine formulation:
  - Metric tensor $g_{\mu\nu}$.
  - Flat affine connection $\Gamma^{\mu}{}_{\nu\rho} = 0$: vanishing curvature

$$R^{\rho}{}_{\sigma\mu\nu} = \partial_{\mu}\Gamma^{\rho}{}_{\sigma\nu} - \partial_{\nu}\Gamma^{\rho}{}_{\sigma\mu} + \Gamma^{\rho}{}_{\lambda\mu}\Gamma^{\lambda}{}_{\sigma\nu} - \Gamma^{\rho}{}_{\lambda\nu}\Gamma^{\lambda}{}_{\sigma\mu} = 0\,. \tag{5}$$

- The flavors of teleparallel geometries: vanishing curvature
  - Metric teleparallel geometry: vanishing nonmetricity

$$Q_{\rho\mu\nu} = \nabla_{\rho} g_{\mu\nu} = 0\,. \tag{6}$$

  - Symmetric teleparallel gravity: vanishing torsion

$$T^{\rho}{}_{\mu\nu} = \Gamma^{\rho}{}_{\nu\mu} - \Gamma^{\rho}{}_{\mu\nu} = 0\,. \tag{7}$$

  - General teleparallel gravity: allow both torsion $T^{\rho}{}_{\mu\nu}$ and nonmetricity $Q_{\rho\mu\nu}$.

Metric teleparallel geometry: tetrad and spin connection

- Metric teleparallelism conventionally formulated using:
  - Tetrad / coframe: $\theta^A = \theta^A{}_\mu \mathrm{d}x^\mu$ with inverse $e_A = e_A{}^\mu \partial_\mu$.
  - Spin connection: $\omega^A{}_B = \omega^A{}_{B\mu} \mathrm{d}x^\mu$.

Metric teleparallel geometry: tetrad and spin connection

- Metric teleparallelism conventionally formulated using:
  - Tetrad / coframe: $\theta^A = \theta^A{}_\mu \mathrm{d}x^\mu$ with inverse $e_A = e_A{}^\mu \partial_\mu$.
  - Spin connection: $\omega^A{}_B = \omega^A{}_{B\mu} \mathrm{d}x^\mu$.
- Induced metric-affine geometry:
  - Metric:

$$g_{\mu\nu} = \eta_{AB} \theta^A{}_\mu \theta^B{}_\nu \,. \tag{8}$$

  - Affine connection:

$$\Gamma^\mu{}_{\nu\rho} = e_A{}^\mu \left(\partial_\rho \theta^A{}_\nu + \omega^A{}_{B\rho} \theta^B{}_\nu \right) \,. \tag{9}$$

Metric teleparallel geometry: tetrad and spin connection

- Metric teleparallelism conventionally formulated using:
  - Tetrad / coframe: $\theta^A = \theta^A{}_\mu \mathrm{d}x^\mu$ with inverse $e_A = e_A{}^\mu \partial_\mu$.
  - Spin connection: $\omega^A{}_B = \omega^A{}_{B\mu} \mathrm{d}x^\mu$.
- Induced metric-affine geometry:
  - Metric:

$$g_{\mu\nu} = \eta_{AB} \theta^A{}_\mu \theta^B{}_\nu \,. \tag{8}$$

  - Affine connection:

$$\Gamma^\mu{}_{\nu\rho} = e_A{}^\mu \left(\partial_\rho \theta^A{}_\nu + \omega^A{}_{B\rho} \theta^B{}_\nu \right) \,. \tag{9}$$

- Conditions on the spin connection:
  - Flatness $R = 0$:

$$\partial_\mu \omega^A{}_{B\nu} - \partial_\nu \omega^A{}_{B\mu} + \omega^A{}_{C\mu} \omega^C{}_{B\nu} - \omega^A{}_{C\nu} \omega^C{}_{B\mu} = 0 \,. \tag{10}$$

  - Metric compatibility $Q = 0$:

$$\eta_{AC} \omega^C{}_{B\mu} + \eta_{BC} \omega^C{}_{A\mu} = 0 \,. \tag{11}$$

Local Lorentz invariance

- Local Lorentz transformation of the tetrad only:

$$\theta^A{}_\mu \mapsto \theta'^A{}_\mu = \Lambda^A{}_B \theta^B{}_\mu \,. \tag{12}$$

  ✓ Metric is invariant: $g'_{\mu\nu} = g_{\mu\nu}$.
  ↯ Connection is not invariant: $\Gamma'^\mu{}_{\nu\rho} \neq \Gamma^\mu{}_{\nu\rho}$.

Local Lorentz invariance

- Local Lorentz transformation of the tetrad only:

$$\theta^A{}_\mu \mapsto \theta'^A{}_\mu = \Lambda^A{}_B \theta^B{}_\mu\,. \tag{12}$$

  - ✓ Metric is invariant: $g'_{\mu\nu} = g_{\mu\nu}$.
  - ↯ Connection is not invariant: $\Gamma'^\mu{}_{\nu\rho} \neq \Gamma^\mu{}_{\nu\rho}$.
- Perform also transformation of the spin connection:

$$\omega^A{}_{B\mu} \mapsto \omega'^A{}_{B\mu} = \Lambda^A{}_C (\Lambda^{-1})^D{}_B \omega^C{}_{D\mu} + \Lambda^A{}_C \partial_\mu (\Lambda^{-1})^C{}_B\,. \tag{13}$$

  - ✓ Metric is invariant: $g'_{\mu\nu} = g_{\mu\nu}$.
  - ✓ Connection is invariant: $\Gamma'^\mu{}_{\nu\rho} = \Gamma^\mu{}_{\nu\rho}$.

Local Lorentz invariance

- Local Lorentz transformation of the tetrad only:

$$\theta^A{}_\mu \mapsto \theta'^A{}_\mu = \Lambda^A{}_B \theta^B{}_\mu \,. \tag{12}$$

  ✓ Metric is invariant: $g'_{\mu\nu} = g_{\mu\nu}$.
  ↯ Connection is not invariant: $\Gamma'^\mu{}_{\nu\rho} \neq \Gamma^\mu{}_{\nu\rho}$.

- Perform also transformation of the spin connection:

$$\omega^A{}_{B\mu} \mapsto \omega'^A{}_{B\mu} = \Lambda^A{}_C (\Lambda^{-1})^D{}_B \omega^C{}_{D\mu} + \Lambda^A{}_C \partial_\mu (\Lambda^{-1})^C{}_B \,. \tag{13}$$

  ✓ Metric is invariant: $g'_{\mu\nu} = g_{\mu\nu}$.
  ✓ Connection is invariant: $\Gamma'^\mu{}_{\nu\rho} = \Gamma^\mu{}_{\nu\rho}$.

⇒ Metric-affine geometry equivalently described by:
  - Metric $g_{\mu\nu}$ and affine connection $\Gamma^\mu{}_{\nu\rho}$.
  - Equivalence class of tetrad $\theta^A{}_\mu$ and spin connection $\omega^A{}_{B\mu}$.
  - Equivalence defined with respect to local Lorentz transformations.

Local Lorentz invariance

- Local Lorentz transformation of the tetrad only:

$$\theta^A{}_\mu \mapsto \theta'^A{}_\mu = \Lambda^A{}_B \theta^B{}_\mu \,. \tag{12}$$

  ✓ Metric is invariant: $g'_{\mu\nu} = g_{\mu\nu}$.
  ↯ Connection is not invariant: $\Gamma'^\mu{}_{\nu\rho} \neq \Gamma^\mu{}_{\nu\rho}$.

- Perform also transformation of the spin connection:

$$\omega^A{}_{B\mu} \mapsto \omega'^A{}_{B\mu} = \Lambda^A{}_C (\Lambda^{-1})^D{}_B \omega^C{}_{D\mu} + \Lambda^A{}_C \partial_\mu (\Lambda^{-1})^C{}_B \,. \tag{13}$$

  ✓ Metric is invariant: $g'_{\mu\nu} = g_{\mu\nu}$.
  ✓ Connection is invariant: $\Gamma'^\mu{}_{\nu\rho} = \Gamma^\mu{}_{\nu\rho}$.

⇒ Metric-affine geometry equivalently described by:
  - Metric $g_{\mu\nu}$ and affine connection $\Gamma^\mu{}_{\nu\rho}$.
  - Equivalence class of tetrad $\theta^A{}_\mu$ and spin connection $\omega^A{}_{B\mu}$.
  - Equivalence defined with respect to local Lorentz transformations.

- Teleparallel geometry admits Weitzenböck gauge: $\omega^A{}_{B\mu} \equiv 0$.

Linear perturbations of affine connections

- General affine connection perturbation: $\Gamma^{\mu}{}_{\nu\rho} = \bar{\Gamma}^{\mu}{}_{\nu\rho} + \delta\Gamma^{\mu}{}_{\nu\rho}$.

  ⇒ Curvature perturbation:

$$\delta R^{\rho}{}_{\sigma\mu\nu} = \bar{\nabla}_{\mu}\delta\Gamma^{\rho}{}_{\sigma\nu} - \bar{\nabla}_{\nu}\delta\Gamma^{\rho}{}_{\sigma\mu} + \bar{T}^{\omega}{}_{\mu\nu}\delta\Gamma^{\rho}{}_{\sigma\omega}\,. \tag{14}$$

  ⇒ Torsion perturbation:

$$\delta T^{\mu}{}_{\nu\rho} = \delta\Gamma^{\mu}{}_{\rho\nu} - \delta\Gamma^{\mu}{}_{\nu\rho}\,. \tag{15}$$

Linear perturbations of affine connections

- General affine connection perturbation: $\Gamma^{\mu}{}_{\nu\rho} = \bar{\Gamma}^{\mu}{}_{\nu\rho} + \delta\Gamma^{\mu}{}_{\nu\rho}$.
  - ⇒ Curvature perturbation:

$$\delta R^{\rho}{}_{\sigma\mu\nu} = \bar{\nabla}_{\mu}\delta\Gamma^{\rho}{}_{\sigma\nu} - \bar{\nabla}_{\nu}\delta\Gamma^{\rho}{}_{\sigma\mu} + \bar{T}^{\omega}{}_{\mu\nu}\delta\Gamma^{\rho}{}_{\sigma\omega}\,. \tag{14}$$

  - ⇒ Torsion perturbation:

$$\delta T^{\mu}{}_{\nu\rho} = \delta\Gamma^{\mu}{}_{\rho\nu} - \delta\Gamma^{\mu}{}_{\nu\rho}\,. \tag{15}$$

- Restriction to particular geometries:
  - Vanishing torsion $T^{\mu}{}_{\nu\rho} \equiv 0$:

$$0 = \delta T^{\mu}{}_{\nu\rho} \quad \Leftrightarrow \quad \delta\Gamma^{\mu}{}_{\nu\rho} = \delta\Gamma^{\mu}{}_{\rho\nu}\,. \tag{16}$$

  - Vanishing curvature $R^{\rho}{}_{\sigma\mu\nu} \equiv 0$:

$$0 = \delta R^{\rho}{}_{\sigma\mu\nu} \quad \Leftrightarrow \quad \delta\Gamma^{\mu}{}_{\nu\rho} = \bar{\nabla}_{\rho}\tau^{\mu}{}_{\nu}\,. \tag{17}$$

  - Vanishing torsion $T^{\mu}{}_{\nu\rho} \equiv 0$ and curvature $R^{\rho}{}_{\sigma\mu\nu} \equiv 0$:

$$0 = \delta T^{\mu}{}_{\nu\rho} \wedge 0 = \delta R^{\rho}{}_{\sigma\mu\nu} \quad \Leftrightarrow \quad \delta\Gamma^{\mu}{}_{\nu\rho} = \bar{\nabla}_{\nu}\bar{\nabla}_{\rho}\xi^{\mu}\,. \tag{18}$$

Linear perturbations of affine connections

- General affine connection perturbation: $\Gamma^{\mu}{}_{\nu\rho} = \bar{\Gamma}^{\mu}{}_{\nu\rho} + \delta\Gamma^{\mu}{}_{\nu\rho}$. 64 components
  - ⇒ Curvature perturbation:

$$\delta R^{\rho}{}_{\sigma\mu\nu} = \bar{\nabla}_{\mu}\delta\Gamma^{\rho}{}_{\sigma\nu} - \bar{\nabla}_{\nu}\delta\Gamma^{\rho}{}_{\sigma\mu} + \bar{T}^{\omega}{}_{\mu\nu}\delta\Gamma^{\rho}{}_{\sigma\omega} \,. \tag{14}$$

  - ⇒ Torsion perturbation:

$$\delta T^{\mu}{}_{\nu\rho} = \delta\Gamma^{\mu}{}_{\rho\nu} - \delta\Gamma^{\mu}{}_{\nu\rho} \,. \tag{15}$$

- Restriction to particular geometries:
  - Vanishing torsion $T^{\mu}{}_{\nu\rho} \equiv 0$: 40 components

$$0 = \delta T^{\mu}{}_{\nu\rho} \quad \Leftrightarrow \quad \delta\Gamma^{\mu}{}_{\nu\rho} = \delta\Gamma^{\mu}{}_{\rho\nu} \,. \tag{16}$$

  - Vanishing curvature $R^{\rho}{}_{\sigma\mu\nu} \equiv 0$: 16 components

$$0 = \delta R^{\rho}{}_{\sigma\mu\nu} \quad \Leftrightarrow \quad \delta\Gamma^{\mu}{}_{\nu\rho} = \bar{\nabla}_{\rho}\tau^{\mu}{}_{\nu} \,. \tag{17}$$

  - Vanishing torsion $T^{\mu}{}_{\nu\rho} \equiv 0$ and curvature $R^{\rho}{}_{\sigma\mu\nu} \equiv 0$: 4 components

$$0 = \delta T^{\mu}{}_{\nu\rho} \wedge 0 = \delta R^{\rho}{}_{\sigma\mu\nu} \quad \Leftrightarrow \quad \delta\Gamma^{\mu}{}_{\nu\rho} = \bar{\nabla}_{\nu}\bar{\nabla}_{\rho}\xi^{\mu} \,. \tag{18}$$

Linear perturbations of metric-affine geometry

- General metric perturbation: $g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu}$.

Linear perturbations of metric-affine geometry

- General metric perturbation: $g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu}$.

$\Rightarrow$ Nonmetricity perturbation:

$$\delta Q_{\rho\mu\nu} = \bar{\nabla}_\rho \delta g_{\mu\nu} - \bar{g}_{\sigma\nu} \delta\Gamma^\sigma{}_{\mu\rho} - \bar{g}_{\mu\sigma} \delta\Gamma^\sigma{}_{\nu\rho} \,. \tag{19}$$

Linear perturbations of metric-affine geometry

- General metric perturbation: $g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu}$.

$\Rightarrow$ Nonmetricity perturbation:

$$\delta Q_{\rho\mu\nu} = \bar{\nabla}_\rho \delta g_{\mu\nu} - \bar{g}_{\sigma\nu} \delta\Gamma^\sigma{}_{\mu\rho} - \bar{g}_{\mu\sigma} \delta\Gamma^\sigma{}_{\nu\rho} . \tag{19}$$

- Restriction to particular geometries:
  - Riemann-Cartan geometry $Q_{\rho\mu\nu} \equiv 0$:

$$0 = \delta Q_{\rho\mu\nu} \quad \Leftrightarrow \quad \bar{g}_{\sigma\nu} \delta\Gamma^\sigma{}_{\mu\rho} + \bar{g}_{\mu\sigma} \delta\Gamma^\sigma{}_{\nu\rho} = \bar{\nabla}_\rho \delta g_{\mu\nu} . \tag{20}$$

  - Riemannian geometry $Q_{\rho\mu\nu} \equiv 0$ and $T^\mu{}_{\nu\rho} \equiv 0$:

$$0 = \delta T^\mu{}_{\nu\rho} \wedge 0 = \delta Q_{\rho\mu\nu} \quad \Leftrightarrow \quad \delta\Gamma^\rho{}_{\mu\nu} = \frac{1}{2} \bar{g}^{\rho\sigma} \left(\bar{\nabla}_\mu \delta g_{\sigma\nu} + \bar{\nabla}_\nu \delta g_{\mu\sigma} - \bar{\nabla}_\sigma \delta g_{\mu\nu} \right) . \tag{21}$$

  - Metric teleparallel geometry $Q_{\rho\mu\nu} \equiv 0$ and $R^\rho{}_{\sigma\mu\nu} \equiv 0$:

$$0 = \delta R^\rho{}_{\sigma\mu\nu} \wedge 0 = \delta Q_{\rho\mu\nu} \quad \Leftrightarrow \quad \delta g_{\mu\nu} = \tau_{\mu\nu} + \tau_{\nu\mu} . \tag{22}$$

Linear perturbations of metric-affine geometry

- General metric perturbation: $g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu}$. 10 additional components

$\Rightarrow$ Nonmetricity perturbation:

$$\delta Q_{\rho\mu\nu} = \bar{\nabla}_\rho \delta g_{\mu\nu} - \bar{g}_{\sigma\nu}\delta\Gamma^\sigma{}_{\mu\rho} - \bar{g}_{\mu\sigma}\delta\Gamma^\sigma{}_{\nu\rho}\,. \tag{19}$$

- Restriction to particular geometries:
  - Riemann-Cartan geometry $Q_{\rho\mu\nu} \equiv 0$: 10 + 24 = 34 components

$$0 = \delta Q_{\rho\mu\nu} \quad \Leftrightarrow \quad \bar{g}_{\sigma\nu}\delta\Gamma^\sigma{}_{\mu\rho} + \bar{g}_{\mu\sigma}\delta\Gamma^\sigma{}_{\nu\rho} = \bar{\nabla}_\rho \delta g_{\mu\nu}\,. \tag{20}$$

  - Riemannian geometry $Q_{\rho\mu\nu} \equiv 0$ and $T^\mu{}_{\nu\rho} \equiv 0$: 10 components

$$0 = \delta T^\mu{}_{\nu\rho} \wedge 0 = \delta Q_{\rho\mu\nu} \quad \Leftrightarrow \quad \delta\Gamma^\rho{}_{\mu\nu} = \frac{1}{2}\bar{g}^{\rho\sigma}\left(\bar{\nabla}_\mu \delta g_{\sigma\nu} + \bar{\nabla}_\nu \delta g_{\mu\sigma} - \bar{\nabla}_\sigma \delta g_{\mu\nu}\right)\,. \tag{21}$$

  - Metric teleparallel geometry $Q_{\rho\mu\nu} \equiv 0$ and $R^\rho{}_{\sigma\mu\nu} \equiv 0$: 16 components

$$0 = \delta R^\rho{}_{\sigma\mu\nu} \wedge 0 = \delta Q_{\rho\mu\nu} \quad \Leftrightarrow \quad \delta g_{\mu\nu} = \tau_{\mu\nu} + \tau_{\nu\mu}\,. \tag{22}$$

Metric teleparallel perturbations and gauge transformations

- Spin connection perturbation $\omega^A{}_{B\mu} = \bar{\omega}^A{}_{B\mu} + \delta\omega^A{}_{B\mu}$:
  - Vanishing curvature $\delta R^A{}_{B\mu\nu} = 0$ requires $\delta\omega^A{}_{B\mu} = \partial_\mu \lambda^A{}_B$.
  - Vanishing nonmetricity requires $\lambda_{AB} + \lambda_{BA} = 0$.

$\Rightarrow$ Only allowed perturbations are infinitesimal Lorentz transformations.

$\Rightarrow$ Impose Weitzenböck gauge $\omega^A{}_{B\mu} = 0$ at all perturbation orders.

Metric teleparallel perturbations and gauge transformations

- Spin connection perturbation $\omega^A{}_{B\mu} = \bar{\omega}^A{}_{B\mu} + \delta\omega^A{}_{B\mu}$:
  - Vanishing curvature $\delta R^A{}_{B\mu\nu} = 0$ requires $\delta\omega^A{}_{B\mu} = \partial_\mu \lambda^A{}_B$.
  - Vanishing nonmetricity requires $\lambda_{AB} + \lambda_{BA} = 0$.
  - ⇒ Only allowed perturbations are infinitesimal Lorentz transformations.
  - ⇒ Impose Weitzenböck gauge $\omega^A{}_{B\mu} = 0$ at all perturbation orders.

⇒ Tetrad perturbation $\theta^A{}_\mu = \bar{\theta}^A{}_\mu + \delta\theta^A{}_\mu$ fully encodes perturbation of geometry:

$$\tau_{\mu\nu} = \eta_{AB}\bar{\theta}^A{}_\mu \delta\theta^B{}_\nu \,. \tag{23}$$

Metric teleparallel perturbations and gauge transformations

- Spin connection perturbation $\omega^A{}_{B\mu} = \bar{\omega}^A{}_{B\mu} + \delta\omega^A{}_{B\mu}$:
  - Vanishing curvature $\delta R^A{}_{B\mu\nu} = 0$ requires $\delta\omega^A{}_{B\mu} = \partial_\mu \lambda^A{}_B$.
  - Vanishing nonmetricity requires $\lambda_{AB} + \lambda_{BA} = 0$.
  - $\Rightarrow$ Only allowed perturbations are infinitesimal Lorentz transformations.
  - $\Rightarrow$ Impose Weitzenböck gauge $\omega^A{}_{B\mu} = 0$ at all perturbation orders.

$\Rightarrow$ Tetrad perturbation $\theta^A{}_\mu = \bar{\theta}^A{}_\mu + \delta\theta^A{}_\mu$ fully encodes perturbation of geometry:

$$\tau_{\mu\nu} = \eta_{AB} \bar{\theta}^A{}_\mu \delta\theta^B{}_\nu \,. \tag{23}$$

- Gauge transformation induced by coordinate transformation $x'^\mu = x^\mu + X^\mu(x)$:
  - Tetrad transforms as one-form:

$$\delta\theta^A{}_\mu - \delta\theta'^A{}_\mu = (\mathcal{L}_X \bar{\theta})^A{}_\mu = X^\nu \partial_\nu \bar{\theta}^A{}_\mu + \partial_\mu X^\nu \bar{\theta}^A{}_\nu \,. \tag{24}$$

  - $\Rightarrow$ Transformation of geometry perturbation:

$$\tau_{\mu\nu} - \tau'_{\mu\nu} = \bar{\nabla}_\nu X_\mu - \bar{T}_{\mu\nu}{}^\rho X_\rho \,. \tag{25}$$

Symmetric teleparallel perturbations and gauge transformations

- Independent perturbations of metric and connection:
  - General metric perturbation:
$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu}\,. \tag{26}$$
  - Connection perturbation preserving symmetry and flatness:
$$\Gamma^{\mu}{}_{\nu\rho} = \bar{\Gamma}^{\mu}{}_{\nu\rho} + \delta\Gamma^{\mu}{}_{\nu\rho} = \bar{\Gamma}^{\mu}{}_{\nu\rho} + \bar{\nabla}_{\nu}\bar{\nabla}_{\rho}\xi^{\mu}\,. \tag{27}$$

Symmetric teleparallel perturbations and gauge transformations

- Independent perturbations of metric and connection:
  - General metric perturbation:
$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu}\,. \tag{26}$$
  - Connection perturbation preserving symmetry and flatness:
$$\Gamma^{\mu}{}_{\nu\rho} = \bar{\Gamma}^{\mu}{}_{\nu\rho} + \delta\Gamma^{\mu}{}_{\nu\rho} = \bar{\Gamma}^{\mu}{}_{\nu\rho} + \bar{\nabla}_{\nu}\bar{\nabla}_{\rho}\xi^{\mu}\,. \tag{27}$$

- Gauge transformation induced by coordinate transformation $x'^{\mu} = x^{\mu} + X^{\mu}(x)$:
  - Transformation of connection coefficients:
$$\delta\Gamma^{\mu}{}_{\nu\rho} - \delta\Gamma'^{\mu}{}_{\nu\rho} = X^{\sigma}\partial_{\sigma}\bar{\Gamma}^{\mu}{}_{\nu\rho} - \partial_{\sigma}X^{\mu}\bar{\Gamma}^{\sigma}{}_{\nu\rho} + \partial_{\nu}X^{\sigma}\bar{\Gamma}^{\mu}{}_{\sigma\rho} + \partial_{\rho}X^{\sigma}\bar{\Gamma}^{\mu}{}_{\nu\sigma} + \partial_{\nu}\partial_{\rho}X^{\mu} \tag{28}$$

Symmetric teleparallel perturbations and gauge transformations

- Independent perturbations of metric and connection:
  - General metric perturbation:

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu}\,. \tag{26}$$

  - Connection perturbation preserving symmetry and flatness:

$$\Gamma^{\mu}{}_{\nu\rho} = \bar{\Gamma}^{\mu}{}_{\nu\rho} + \delta\Gamma^{\mu}{}_{\nu\rho} = \bar{\Gamma}^{\mu}{}_{\nu\rho} + \bar{\nabla}_{\nu}\bar{\nabla}_{\rho}\xi^{\mu}\,. \tag{27}$$

- Gauge transformation induced by coordinate transformation $x'^{\mu} = x^{\mu} + X^{\mu}(x)$:
  - Transformation of connection coefficients:

$$\begin{aligned}\delta\Gamma^{\mu}{}_{\nu\rho} - \delta\Gamma'^{\mu}{}_{\nu\rho} &= X^{\sigma}\partial_{\sigma}\bar{\Gamma}^{\mu}{}_{\nu\rho} - \partial_{\sigma}X^{\mu}\bar{\Gamma}^{\sigma}{}_{\nu\rho} + \partial_{\nu}X^{\sigma}\bar{\Gamma}^{\mu}{}_{\sigma\rho} + \partial_{\rho}X^{\sigma}\bar{\Gamma}^{\mu}{}_{\nu\sigma} + \partial_{\nu}\partial_{\rho}X^{\mu}\\ &= \bar{\nabla}_{\rho}\bar{\nabla}_{\nu}X^{\mu} - X^{\sigma}\bar{R}^{\mu}{}_{\nu\rho\sigma} - \bar{\nabla}_{\rho}(X^{\sigma}\bar{T}^{\mu}{}_{\nu\sigma})\end{aligned} \tag{28}$$

  - Difference of connection coefficients constitutes tensor field.

Symmetric teleparallel perturbations and gauge transformations

- Independent perturbations of metric and connection:
  - General metric perturbation:

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu}\,. \tag{26}$$

  - Connection perturbation preserving symmetry and flatness:

$$\Gamma^{\mu}{}_{\nu\rho} = \bar{\Gamma}^{\mu}{}_{\nu\rho} + \delta\Gamma^{\mu}{}_{\nu\rho} = \bar{\Gamma}^{\mu}{}_{\nu\rho} + \bar{\nabla}_{\nu}\bar{\nabla}_{\rho}\xi^{\mu}\,. \tag{27}$$

- Gauge transformation induced by coordinate transformation $x'^{\mu} = x^{\mu} + X^{\mu}(x)$:
  - Transformation of connection coefficients:

$$\begin{aligned}\delta\Gamma^{\mu}{}_{\nu\rho} - \delta\Gamma'^{\mu}{}_{\nu\rho} &= X^{\sigma}\partial_{\sigma}\bar{\Gamma}^{\mu}{}_{\nu\rho} - \partial_{\sigma}X^{\mu}\bar{\Gamma}^{\sigma}{}_{\nu\rho} + \partial_{\nu}X^{\sigma}\bar{\Gamma}^{\mu}{}_{\sigma\rho} + \partial_{\rho}X^{\sigma}\bar{\Gamma}^{\mu}{}_{\nu\sigma} + \partial_{\nu}\partial_{\rho}X^{\mu}\\ &= \bar{\nabla}_{\rho}\bar{\nabla}_{\nu}X^{\mu} - X^{\sigma}\bar{R}^{\mu}{}_{\nu\rho\sigma} - \bar{\nabla}_{\rho}(X^{\sigma}\bar{T}^{\mu}{}_{\nu\sigma})\\ &= \bar{\nabla}_{\rho}\bar{\nabla}_{\nu}X^{\mu}\,.\end{aligned} \tag{28}$$

  - Difference of connection coefficients constitutes tensor field.
  - Expression simplifies in symmetric teleparallel geometry.

Symmetric teleparallel perturbations and gauge transformations

- Independent perturbations of metric and connection:
  - General metric perturbation:

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu}\,. \tag{26}$$

  - Connection perturbation preserving symmetry and flatness:

$$\Gamma^{\mu}{}_{\nu\rho} = \bar{\Gamma}^{\mu}{}_{\nu\rho} + \delta\Gamma^{\mu}{}_{\nu\rho} = \bar{\Gamma}^{\mu}{}_{\nu\rho} + \bar{\nabla}_{\nu}\bar{\nabla}_{\rho}\xi^{\mu}\,. \tag{27}$$

- Gauge transformation induced by coordinate transformation $x'^{\mu} = x^{\mu} + X^{\mu}(x)$:
  - Transformation of connection coefficients:

$$\begin{aligned}\delta\Gamma^{\mu}{}_{\nu\rho} - \delta\Gamma'^{\mu}{}_{\nu\rho} &= X^{\sigma}\partial_{\sigma}\bar{\Gamma}^{\mu}{}_{\nu\rho} - \partial_{\sigma}X^{\mu}\bar{\Gamma}^{\sigma}{}_{\nu\rho} + \partial_{\nu}X^{\sigma}\bar{\Gamma}^{\mu}{}_{\sigma\rho} + \partial_{\rho}X^{\sigma}\bar{\Gamma}^{\mu}{}_{\nu\sigma} + \partial_{\nu}\partial_{\rho}X^{\mu}\\ &= \bar{\nabla}_{\rho}\bar{\nabla}_{\nu}X^{\mu} - X^{\sigma}\bar{R}^{\mu}{}_{\nu\rho\sigma} - \bar{\nabla}_{\rho}(X^{\sigma}\bar{T}^{\mu}{}_{\nu\sigma})\\ &= \bar{\nabla}_{\rho}\bar{\nabla}_{\nu}X^{\mu}\,.\end{aligned} \tag{28}$$

  - Difference of connection coefficients constitutes tensor field.
  - Expression simplifies in symmetric teleparallel geometry.

⇒ Transformation of generator ξ^{μ} of connection perturbation:

$$\xi^{\mu} - \xi'^{\mu} = X^{\mu}\,. \tag{29}$$

Outline

Outline

PPN for metric teleparallel gravity

- Background tetrad given by diagonal tetrad $\bar{\theta}^A{}_\mu = \Delta^A{}_\mu = \mathrm{diag}(1, 1, 1, 1)$.

PPN for metric teleparallel gravity

- Background tetrad given by diagonal tetrad $\bar{\theta}^A{}_\mu = \Delta^A{}_\mu = \mathrm{diag}(1, 1, 1, 1)$.

$\Rightarrow$ Relation between metric and tetrad perturbations $\overset{k}{\theta}{}^A{}_\mu = \Delta^A{}_\nu \eta^{\nu\rho} \overset{k}{\tau}_{\rho\mu}$:

$$\overset{n}{g}_{\mu\nu} = \eta^{\rho\sigma} \sum_{k=0}^{n} \overset{k}{\tau}_{\rho\mu} \overset{n-k}{\tau}_{\sigma\nu} \,. \tag{30}$$

PPN for metric teleparallel gravity

- Background tetrad given by diagonal tetrad $\bar{\theta}^A{}_\mu = \Delta^A{}_\mu = \mathrm{diag}(1,1,1,1)$.

$\Rightarrow$ Relation between metric and tetrad perturbations $\overset{k}{\theta}{}^A{}_\mu = \Delta^A{}_\nu \eta^{\nu\rho} \overset{k}{\tau}_{\rho\mu}$:

$$\overset{n}{g}_{\mu\nu} = \eta^{\rho\sigma} \sum_{k=0}^{n} \overset{k}{\tau}_{\rho\mu} \overset{n-k}{\tau}_{\sigma\nu} \,. \tag{30}$$

$\Rightarrow$ Solve for symmetric part of highest order tetrad perturbation:

$$\overset{n}{\tau}_{\mu\nu} + \overset{n}{\tau}_{\nu\mu} = \overset{n}{g}_{\mu\nu} - \eta^{\rho\sigma} \sum_{k=1}^{n-1} \overset{k}{\tau}_{\rho\mu} \overset{n-k}{\tau}_{\sigma\nu} \,. \tag{31}$$

- Background tetrad given by diagonal tetrad $\bar{\theta}^A{}_\mu = \Delta^A{}_\mu = \mathrm{diag}(1, 1, 1, 1)$.

$\Rightarrow$ Relation between metric and tetrad perturbations $\overset{k}{\theta}{}^A{}_\mu = \Delta^A{}_\nu \eta^{\nu\rho} \overset{k}{\tau}{}_{\rho\mu}$:

$$\overset{n}{g}{}_{\mu\nu} = \eta^{\rho\sigma} \sum_{k=0}^{n} \overset{k}{\tau}{}_{\rho\mu} \overset{n-k}{\tau}{}_{\sigma\nu} \,. \tag{30}$$

$\Rightarrow$ Solve for symmetric part of highest order tetrad perturbation:

$$\overset{n}{\tau}{}_{\mu\nu} + \overset{n}{\tau}{}_{\nu\mu} = \overset{n}{g}{}_{\mu\nu} - \eta^{\rho\sigma} \sum_{k=1}^{n-1} \overset{k}{\tau}{}_{\rho\mu} \overset{n-k}{\tau}{}_{\sigma\nu} \,. \tag{31}$$

- Define antisymmetric part $\overset{n}{a}{}_{\mu\nu} = \overset{n}{\tau}{}_{\mu\nu} - \overset{n}{\tau}{}_{\nu\mu}$.

PPN for metric teleparallel gravity

- Background tetrad given by diagonal tetrad $\bar{\theta}^A{}_\mu = \Delta^A{}_\mu = \mathrm{diag}(1, 1, 1, 1)$.

$\Rightarrow$ Relation between metric and tetrad perturbations $\overset{k}{\theta}{}^A{}_\mu = \Delta^A{}_\nu \eta^{\nu\rho} \overset{k}{\tau}{}_{\rho\mu}$:

$$\overset{n}{g}{}_{\mu\nu} = \eta^{\rho\sigma} \sum_{k=0}^{n} \overset{k}{\tau}{}_{\rho\mu} \overset{n-k}{\tau}{}_{\sigma\nu} \,. \tag{30}$$

$\Rightarrow$ Solve for symmetric part of highest order tetrad perturbation:

$$\overset{n}{\tau}{}_{\mu\nu} + \overset{n}{\tau}{}_{\nu\mu} = \overset{n}{g}{}_{\mu\nu} - \eta^{\rho\sigma} \sum_{k=1}^{n-1} \overset{k}{\tau}{}_{\rho\mu} \overset{n-k}{\tau}{}_{\sigma\nu} \,. \tag{31}$$

- Define antisymmetric part $\overset{n}{a}{}_{\mu\nu} = \overset{n}{\tau}{}_{\mu\nu} - \overset{n}{\tau}{}_{\nu\mu}$.

$\Rightarrow$ Full expansion of tetrad perturbation defined recursively:

$$\overset{n}{\tau}{}_{\mu\nu} = \frac{1}{2} \left(\overset{n}{g}{}_{\mu\nu} + \overset{n}{a}{}_{\mu\nu} - \eta^{\rho\sigma} \sum_{k=1}^{n-1} \overset{k}{\tau}{}_{\rho\mu} \overset{n-k}{\tau}{}_{\sigma\nu} \right) \tag{32}$$

PPN for metric teleparallel gravity

- Background tetrad given by diagonal tetrad $\bar{\theta}^A{}_\mu = \Delta^A{}_\mu = \mathrm{diag}(1,1,1,1)$.

⇒ Relation between metric and tetrad perturbations $\overset{k}{\theta}{}^A{}_\mu = \Delta^A{}_\nu \eta^{\nu\rho} \overset{k}{\tau}_{\rho\mu}$:

$$\overset{n}{g}_{\mu\nu} = \eta^{\rho\sigma} \sum_{k=0}^{n} \overset{k}{\tau}_{\rho\mu} \overset{n-k}{\tau}_{\sigma\nu} \,. \tag{30}$$

⇒ Solve for symmetric part of highest order tetrad perturbation:

$$\overset{n}{\tau}_{\mu\nu} + \overset{n}{\tau}_{\nu\mu} = \overset{n}{g}_{\mu\nu} - \eta^{\rho\sigma} \sum_{k=1}^{n-1} \overset{k}{\tau}_{\rho\mu} \overset{n-k}{\tau}_{\sigma\nu} \,. \tag{31}$$

- Define antisymmetric part $\overset{n}{a}_{\mu\nu} = \overset{n}{\tau}_{\mu\nu} - \overset{n}{\tau}_{\nu\mu}$.

⇒ Full expansion of tetrad perturbation defined recursively:

$$\overset{n}{\tau}_{\mu\nu} = \frac{1}{2}\left(\overset{n}{g}_{\mu\nu} + \overset{n}{a}_{\mu\nu} - \eta^{\rho\sigma} \sum_{k=1}^{n-1} \overset{k}{\tau}_{\rho\mu} \overset{n-k}{\tau}_{\sigma\nu}\right) \tag{32}$$

- Relevant and non-vanishing perturbation components:

$$\overset{2}{g}_{00} \,, \quad \overset{2}{g}_{ij} \,, \quad \overset{3}{g}_{i0} \,, \quad \overset{4}{g}_{00} \,, \quad \overset{4}{g}_{ij} \,, \quad \overset{2}{a}_{ij} \,, \quad \overset{3}{a}_{i0} \,, \quad \overset{4}{a}_{ij} \,. \tag{33}$$

Post-Newtonian limit of teleparallel Horndeski (BDLS) gravity

- Action depends on functions $\mathcal{F}$ and Horndeski's $\mathcal{G}_2, \ldots, \mathcal{G}_5$: [Bahamonde, Dialektopoulos, Levi Said '19]

$$S_g[\theta, \omega, \phi] = S_{\text{Horndeski}}[g = \eta(\theta, \theta), \phi] + \int_M \mathcal{F}(\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3, X, Y, \phi, \mathbb{J})\, \theta \, \mathrm{d}^4 x \,. \tag{34}$$

Post-Newtonian limit of teleparallel Horndeski (BDLS) gravity

- Action depends on functions $\mathcal{F}$ and Horndeski's $\mathcal{G}_2, \ldots, \mathcal{G}_5$: [Bahamonde, Dialektopoulos, Levi Said '19]

$$S_g[\theta, \omega, \phi] = S_{\text{Horndeski}}[g = \eta(\theta, \theta), \phi] + \int_M \mathcal{F}(\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3, X, Y, \phi, \mathbb{J})\, \theta\, \mathrm{d}^4x\,. \quad (34)$$

- Free function $\mathcal{F}$ depends on different scalar invariants:
    - Terms quadratic in the torsion tensor:

$$\mathcal{T}_1 = T^{\mu\nu\rho}T_{\mu\nu\rho}\,, \quad \mathcal{T}_2 = T^{\mu\nu\rho}T_{\rho\nu\mu}\,, \quad \mathcal{T}_3 = T^{\mu}{}_{\mu\rho}T_{\nu}{}^{\nu\rho}\,. \quad (35)$$

    - Terms involving the scalar field ϕ:

$$X = -\frac{1}{2}\partial^{\mu}\phi\partial_{\mu}\phi\,, \quad Y = g^{\mu\nu}T^{\rho}{}_{\rho\mu}\phi_{,\nu}\,. \quad (36)$$

    - Higher order torsion / scalar coupling terms $\mathbb{J}$ do not contribute to PPN limit.

Post-Newtonian limit of teleparallel Horndeski (BDLS) gravity

- Action depends on functions $\mathcal{F}$ and Horndeski's $\mathcal{G}_2, \ldots, \mathcal{G}_5$: [Bahamonde, Dialektopoulos, Levi Said '19]

$$S_g[\theta, \omega, \phi] = S_{\text{Horndeski}}[g = \eta(\theta, \theta), \phi] + \int_M \mathcal{F}(\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3, X, Y, \phi, \mathbb{J})\, \theta\, \mathrm{d}^4x\,. \tag{34}$$

- Free function $\mathcal{F}$ depends on different scalar invariants:
  - Terms quadratic in the torsion tensor:

$$\mathcal{T}_1 = T^{\mu\nu\rho} T_{\mu\nu\rho}\,, \quad \mathcal{T}_2 = T^{\mu\nu\rho} T_{\rho\nu\mu}\,, \quad \mathcal{T}_3 = T^{\mu}{}_{\mu\rho} T_{\nu}{}^{\nu\rho}\,. \tag{35}$$

  - Terms involving the scalar field ϕ:

$$X = -\frac{1}{2}\partial^\mu\phi\partial_\mu\phi\,, \quad Y = g^{\mu\nu} T^{\rho}{}_{\rho\mu}\phi_{,\nu}\,. \tag{36}$$

  - Higher order torsion / scalar coupling terms $\mathbb{J}$ do not contribute to PPN limit.
- Taylor expansion of free functions $\mathcal{F}$ and Horndeski's $\mathcal{G}_2, \ldots, \mathcal{G}_5$:

$$\mathcal{F} = F + \sum_{k=1}^{3} F_{,k}\mathcal{T}_k + F_{,X}X + F_{,Y}Y + F_{,\phi}\phi + \ldots\,. \tag{37}$$

PPN parameters of teleparallel Horndeski (BDLS) gravity

- General formula for PPN parameters: [Bahamonde, Dialektopoulos, MH, Levi Said '20]
  - Formula for γ in terms of Taylor coefficients of $\mathcal{F}, \mathcal{G}_2, \ldots, \mathcal{G}_5$:

$$\gamma = 1 - \frac{(F_{,Y} - 2G_{4,\phi})^2 + 2(2F_{,1} + F_{,2} + F_{,3})(F_{,X} + G_{2,X} - 2G_{3,\phi})}{2(F_{,Y} - 2G_{4,\phi})^2 + 2(2F_{,1} + F_{,2} + 2F_{,3} + G_4)(F_{,X} + G_{2,X} - 2G_{3,\phi})}, \tag{38}$$

  - Formula for β is rather lengthy.
  - Fully conservative theory - all other parameters vanish:

$$\xi = \alpha_1 = \alpha_2 = \alpha_3 = \zeta_1 = \zeta_2 = \zeta_3 = \zeta_4 = 0\,, \tag{39}$$

PPN parameters of teleparallel Horndeski (BDLS) gravity

- General formula for PPN parameters: [Bahamonde, Dialektopoulos, MH, Levi Said '20]
  - Formula for γ in terms of Taylor coefficients of $\mathcal{F}, \mathcal{G}_2, \ldots, \mathcal{G}_5$:

$$\gamma = 1 - \frac{(F_{,Y} - 2G_{4,\phi})^2 + 2(2F_{,1} + F_{,2} + F_{,3})(F_{,X} + G_{2,X} - 2G_{3,\phi})}{2(F_{,Y} - 2G_{4,\phi})^2 + 2(2F_{,1} + F_{,2} + 2F_{,3} + G_4)(F_{,X} + G_{2,X} - 2G_{3,\phi})}\,, \tag{38}$$

  - Formula for β is rather lengthy.
  - Fully conservative theory - all other parameters vanish:

$$\xi = \alpha_1 = \alpha_2 = \alpha_3 = \zeta_1 = \zeta_2 = \zeta_3 = \zeta_4 = 0\,, \tag{39}$$

- Reproduce previously found results for numerous special cases:
  - Pure Horndeski gravity: $\mathcal{F} \equiv 0$. [MH '15]
  - Scalar-torsion gravity: $\mathcal{G}_2 \equiv \ldots \equiv \mathcal{G}_5 \equiv 0$, $4F_{,1} = 2F_{,2} = -F_{,3}$. [Emtsova, MH '19] [Flathmann, MH '19]
  - Pure torsion gravity: $\mathcal{G}_2 \equiv \ldots \equiv \mathcal{G}_5 \equiv 0$, $F_{,X} = F_{,Y} = F_{,\phi} = 0$. [Ualikhanova, MH '19]

PPN parameters of teleparallel Horndeski (BDLS) gravity

- General formula for PPN parameters: [Bahamonde, Dialektopoulos, MH, Levi Said '20]
  - Formula for γ in terms of Taylor coefficients of $\mathcal{F}, \mathcal{G}_2, \ldots, \mathcal{G}_5$:

$$\gamma = 1 - \frac{(F_{,Y} - 2G_{4,\phi})^2 + 2(2F_{,1} + F_{,2} + F_{,3})(F_{,X} + G_{2,X} - 2G_{3,\phi})}{2(F_{,Y} - 2G_{4,\phi})^2 + 2(2F_{,1} + F_{,2} + 2F_{,3} + G_4)(F_{,X} + G_{2,X} - 2G_{3,\phi})}\,, \tag{38}$$

  - Formula for β is rather lengthy.
  - Fully conservative theory - all other parameters vanish:

$$\xi = \alpha_1 = \alpha_2 = \alpha_3 = \zeta_1 = \zeta_2 = \zeta_3 = \zeta_4 = 0\,, \tag{39}$$

- Reproduce previously found results for numerous special cases:
  - Pure Horndeski gravity: $\mathcal{F} \equiv 0$. [MH '15]
  - Scalar-torsion gravity: $\mathcal{G}_2 \equiv \ldots \equiv \mathcal{G}_5 \equiv 0$, $4F_{,1} = 2F_{,2} = -F_{,3}$. [Emtsova, MH '19] [Flathmann, MH '19]
  - Pure torsion gravity: $\mathcal{G}_2 \equiv \ldots \equiv \mathcal{G}_5 \equiv 0$, $F_{,X} = F_{,Y} = F_{,\phi} = 0$. [Ualikhanova, MH '19]
- Two branches with minimally / non-minimally coupled scalar field yield $\beta = \gamma = 1$.

PPN for symmetric teleparallel gravity

- Background satisfies *coincident gauge* condition $\bar{\Gamma}^{\rho}{}_{\mu\nu} = 0$.

PPN for symmetric teleparallel gravity

- Background satisfies *coincident gauge* condition $\bar{\Gamma}^{\rho}{}_{\mu\nu} = 0$.

$\Rightarrow$ Perturbed connection given by infinitesimal coordinate transformation:

$$\Lambda^{\alpha}{}_{\beta} = \frac{\partial x'^{\alpha}}{\partial x^{\beta}} \quad \Rightarrow \quad \Gamma^{\rho}{}_{\mu\nu} = \frac{\partial x^{\rho}}{\partial x'^{\gamma}} \frac{\partial x'^{\gamma}}{\partial x^{\mu} \partial x^{\nu}} = (\Lambda^{-1})^{\rho}{}_{\gamma} \partial_{\nu} \Lambda^{\gamma}{}_{\mu} \,. \tag{40}$$

PPN for symmetric teleparallel gravity

- Background satisfies *coincident gauge* condition $\bar{\Gamma}^{\rho}{}_{\mu\nu} = 0$.

$\Rightarrow$ Perturbed connection given by infinitesimal coordinate transformation:

$$\Lambda^{\alpha}{}_{\beta} = \frac{\partial x'^{\alpha}}{\partial x^{\beta}} \quad \Rightarrow \quad \Gamma^{\rho}{}_{\mu\nu} = \frac{\partial x^{\rho}}{\partial x'^{\gamma}} \frac{\partial x'^{\gamma}}{\partial x^{\mu} \partial x^{\nu}} = (\Lambda^{-1})^{\rho}{}_{\gamma} \partial_{\nu} \Lambda^{\gamma}{}_{\mu} \,. \tag{40}$$

- Coordinate transformation generated by flow of a vector field ξ^{μ}:

$$x'^{\mu} = x^{\mu} + \xi^{\mu} + \frac{1}{2}\xi^{\nu}\partial_{\nu}\xi^{\mu} + \ldots \quad \Rightarrow \quad \Lambda^{\alpha}{}_{\beta} = \delta^{\alpha}_{\beta} + \partial_{\beta}\xi^{\alpha} + \frac{1}{2}\partial_{\beta}(\xi^{\gamma}\partial_{\gamma}\xi^{\alpha}) + \ldots \tag{41}$$

PPN for symmetric teleparallel gravity

- Background satisfies *coincident gauge* condition $\bar{\Gamma}^{\rho}{}_{\mu\nu} = 0$.

$\Rightarrow$ Perturbed connection given by infinitesimal coordinate transformation:

$$\Lambda^{\alpha}{}_{\beta} = \frac{\partial x'^{\alpha}}{\partial x^{\beta}} \quad \Rightarrow \quad \Gamma^{\rho}{}_{\mu\nu} = \frac{\partial x^{\rho}}{\partial x'^{\gamma}} \frac{\partial x'^{\gamma}}{\partial x^{\mu} \partial x^{\nu}} = (\Lambda^{-1})^{\rho}{}_{\gamma} \partial_{\nu} \Lambda^{\gamma}{}_{\mu} \,. \tag{40}$$

- Coordinate transformation generated by flow of a vector field ξ^{μ}:

$$x'^{\mu} = x^{\mu} + \xi^{\mu} + \frac{1}{2} \xi^{\nu} \partial_{\nu} \xi^{\mu} + \ldots \quad \Rightarrow \quad \Lambda^{\alpha}{}_{\beta} = \delta^{\alpha}_{\beta} + \partial_{\beta} \xi^{\alpha} + \frac{1}{2} \partial_{\beta}(\xi^{\gamma} \partial_{\gamma} \xi^{\alpha}) + \ldots \tag{41}$$

$\Rightarrow$ Perturbative expansion of symmetric teleparallel connection:

$$\Gamma^{\rho}{}_{\mu\nu} = \partial_{\mu} \partial_{\nu} \xi^{\rho} + \frac{1}{2} \left(\xi^{\sigma} \partial_{\mu} \partial_{\nu} \partial_{\sigma} \xi^{\rho} + 2 \partial_{(\mu} \xi^{\sigma} \partial_{\nu)} \partial_{\sigma} \xi^{\rho} - \partial_{\mu} \partial_{\nu} \xi^{\sigma} \partial_{\sigma} \xi^{\rho} \right) + \ldots \tag{42}$$

PPN for symmetric teleparallel gravity

- Background satisfies *coincident gauge* condition $\bar{\Gamma}^{\rho}{}_{\mu\nu} = 0$.

⇒ Perturbed connection given by infinitesimal coordinate transformation:

$$\Lambda^{\alpha}{}_{\beta} = \frac{\partial x'^{\alpha}}{\partial x^{\beta}} \quad \Rightarrow \quad \Gamma^{\rho}{}_{\mu\nu} = \frac{\partial x^{\rho}}{\partial x'^{\gamma}} \frac{\partial x'^{\gamma}}{\partial x^{\mu} \partial x^{\nu}} = (\Lambda^{-1})^{\rho}{}_{\gamma} \partial_{\nu} \Lambda^{\gamma}{}_{\mu} \,. \tag{40}$$

- Coordinate transformation generated by flow of a vector field ξ^{μ}:

$$x'^{\mu} = x^{\mu} + \xi^{\mu} + \frac{1}{2}\xi^{\nu}\partial_{\nu}\xi^{\mu} + \ldots \quad \Rightarrow \quad \Lambda^{\alpha}{}_{\beta} = \delta^{\alpha}_{\beta} + \partial_{\beta}\xi^{\alpha} + \frac{1}{2}\partial_{\beta}(\xi^{\gamma}\partial_{\gamma}\xi^{\alpha}) + \ldots \tag{41}$$

⇒ Perturbative expansion of symmetric teleparallel connection:

$$\Gamma^{\rho}{}_{\mu\nu} = \partial_{\mu}\partial_{\nu}\xi^{\rho} + \frac{1}{2}\left(\xi^{\sigma}\partial_{\mu}\partial_{\nu}\partial_{\sigma}\xi^{\rho} + 2\partial_{(\mu}\xi^{\sigma}\partial_{\nu)}\partial_{\sigma}\xi^{\rho} - \partial_{\mu}\partial_{\nu}\xi^{\sigma}\partial_{\sigma}\xi^{\rho}\right) + \ldots \tag{42}$$

- Expansion of generators ξ^{μ} in post-Newtonian velocity orders:

$$\xi^{\mu} = \overset{1}{\xi}{}^{\mu} + \overset{2}{\xi}{}^{\mu} + \overset{3}{\xi}{}^{\mu} + \overset{4}{\xi}{}^{\mu} + \mathcal{O}(5) \,. \tag{43}$$

PPN for symmetric teleparallel gravity

- Background satisfies *coincident gauge* condition $\bar{\Gamma}^{\rho}{}_{\mu\nu} = 0$.

$\Rightarrow$ Perturbed connection given by infinitesimal coordinate transformation:

$$\Lambda^{\alpha}{}_{\beta} = \frac{\partial x'^{\alpha}}{\partial x^{\beta}} \quad \Rightarrow \quad \Gamma^{\rho}{}_{\mu\nu} = \frac{\partial x^{\rho}}{\partial x'^{\gamma}} \frac{\partial x'^{\gamma}}{\partial x^{\mu} \partial x^{\nu}} = (\Lambda^{-1})^{\rho}{}_{\gamma} \partial_{\nu} \Lambda^{\gamma}{}_{\mu} \,. \tag{40}$$

- Coordinate transformation generated by flow of a vector field ξ^{μ}:

$$x'^{\mu} = x^{\mu} + \xi^{\mu} + \frac{1}{2} \xi^{\nu} \partial_{\nu} \xi^{\mu} + \ldots \quad \Rightarrow \quad \Lambda^{\alpha}{}_{\beta} = \delta^{\alpha}_{\beta} + \partial_{\beta} \xi^{\alpha} + \frac{1}{2} \partial_{\beta} (\xi^{\gamma} \partial_{\gamma} \xi^{\alpha}) + \ldots \tag{41}$$

$\Rightarrow$ Perturbative expansion of symmetric teleparallel connection:

$$\Gamma^{\rho}{}_{\mu\nu} = \partial_{\mu} \partial_{\nu} \xi^{\rho} + \frac{1}{2} \left(\xi^{\sigma} \partial_{\mu} \partial_{\nu} \partial_{\sigma} \xi^{\rho} + 2 \partial_{(\mu} \xi^{\sigma} \partial_{\nu)} \partial_{\sigma} \xi^{\rho} - \partial_{\mu} \partial_{\nu} \xi^{\sigma} \partial_{\sigma} \xi^{\rho} \right) + \ldots \tag{42}$$

- Expansion of generators ξ^{μ} in post-Newtonian velocity orders:

$$\xi^{\mu} = \overset{1}{\xi}{}^{\mu} + \overset{2}{\xi}{}^{\mu} + \overset{3}{\xi}{}^{\mu} + \overset{4}{\xi}{}^{\mu} + \mathcal{O}(5) \,. \tag{43}$$

- Only terms $\overset{2}{\xi}{}^{i}, \overset{3}{\xi}{}^{0}, \overset{4}{\xi}{}^{i}$ are relevant and non-vanishing.

Post-Newtonian limit of $\mathcal{F}(\mathcal{Q}_1, \mathcal{Q}_2, \mathcal{Q}_3, \mathcal{Q}_4, \mathcal{Q}_5)$ theories

- Action functional depends on free function $\mathcal{F}$:

$$S_g[g, \Gamma] = \int_M \mathcal{F}(\mathcal{Q}_1, \mathcal{Q}_2, \mathcal{Q}_3, \mathcal{Q}_4, \mathcal{Q}_5) \sqrt{-g}\, \mathrm{d}^4 x\,. \tag{44}$$

Post-Newtonian limit of $\mathcal{F}(\mathcal{Q}_1, \mathcal{Q}_2, \mathcal{Q}_3, \mathcal{Q}_4, \mathcal{Q}_5)$ theories

- Action functional depends on free function $\mathcal{F}$:

$$S_g[g, \Gamma] = \int_M \mathcal{F}\left(\mathcal{Q}_1, \mathcal{Q}_2, \mathcal{Q}_3, \mathcal{Q}_4, \mathcal{Q}_5\right) \sqrt{-g}\, \mathrm{d}^4 x \,. \tag{44}$$

- Nonmetricity invariants:

$$\mathcal{Q}_1 = Q^{\rho\mu\nu} Q_{\rho\mu\nu} \,, \quad \mathcal{Q}_2 = Q^{\mu\nu\rho} Q_{\rho\mu\nu} \,, \quad \mathcal{Q}_3 = Q^{\rho\mu}{}_{\mu} Q_{\rho\nu}{}^{\nu} \,, \quad \mathcal{Q}_4 = Q^{\mu}{}_{\mu\rho} Q_{\nu}{}^{\nu\rho} \,, \quad \mathcal{Q}_5 = Q^{\mu}{}_{\mu\rho} Q^{\rho\nu}{}_{\nu} \tag{45}$$

Post-Newtonian limit of $\mathcal{F}(\mathcal{Q}_1, \mathcal{Q}_2, \mathcal{Q}_3, \mathcal{Q}_4, \mathcal{Q}_5)$ theories

- Action functional depends on free function $\mathcal{F}$:

$$S_g[g, \Gamma] = \int_M \mathcal{F}\left(\mathcal{Q}_1, \mathcal{Q}_2, \mathcal{Q}_3, \mathcal{Q}_4, \mathcal{Q}_5\right)\sqrt{-g}\,\mathrm{d}^4x\,. \tag{44}$$

- Nonmetricity invariants:

$$\mathcal{Q}_1 = Q^{\rho\mu\nu}Q_{\rho\mu\nu}\,, \quad \mathcal{Q}_2 = Q^{\mu\nu\rho}Q_{\rho\mu\nu}\,, \quad \mathcal{Q}_3 = Q^{\rho\mu}{}_{\mu}Q_{\rho\nu}{}^{\nu}\,, \quad \mathcal{Q}_4 = Q^{\mu}{}_{\mu\rho}Q_{\nu}{}^{\nu\rho}\,, \quad \mathcal{Q}_5 = Q^{\mu}{}_{\mu\rho}Q^{\rho\nu}{}_{\nu} \tag{45}$$

- Taylor expansion up to linear order is sufficient:

$$\mathcal{F} = F_0 + \sum_{k=1}^{5} F_k\mathcal{Q}_k + \mathcal{O}(\mathcal{Q}^2)\,. \tag{46}$$

Post-Newtonian limit of $\mathcal{F}(\mathcal{Q}_1, \mathcal{Q}_2, \mathcal{Q}_3, \mathcal{Q}_4, \mathcal{Q}_5)$ theories

- Action functional depends on free function $\mathcal{F}$:

$$S_g[g, \Gamma] = \int_M \mathcal{F}(\mathcal{Q}_1, \mathcal{Q}_2, \mathcal{Q}_3, \mathcal{Q}_4, \mathcal{Q}_5)\sqrt{-g}\,\mathrm{d}^4x\,. \tag{44}$$

- Nonmetricity invariants:

$$\mathcal{Q}_1 = Q^{\rho\mu\nu}Q_{\rho\mu\nu}\,, \quad \mathcal{Q}_2 = Q^{\mu\nu\rho}Q_{\rho\mu\nu}\,, \quad \mathcal{Q}_3 = Q^{\rho\mu}{}_{\mu}Q_{\rho\nu}{}^{\nu}\,, \quad \mathcal{Q}_4 = Q^{\mu}{}_{\mu\rho}Q_{\nu}{}^{\nu\rho}\,, \quad \mathcal{Q}_5 = Q^{\mu}{}_{\mu\rho}Q^{\rho\nu}{}_{\nu} \tag{45}$$

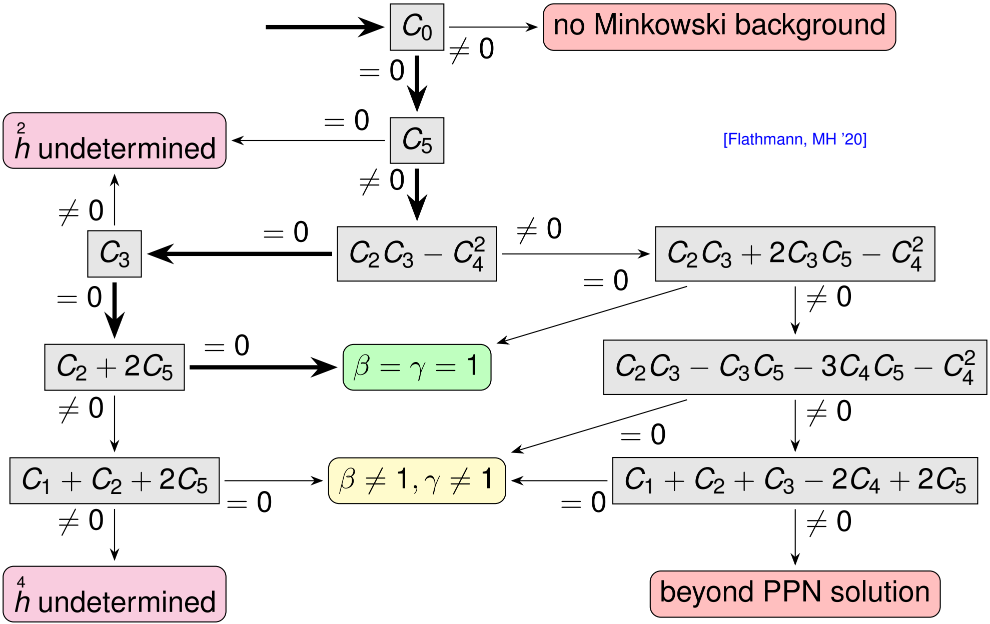
- Taylor expansion up to linear order is sufficient:

$$\mathcal{F} = F_0 + \sum_{k=1}^{5} F_k\mathcal{Q}_k + \mathcal{O}(\mathcal{Q}^2)\,. \tag{46}$$

- Change of parameters:

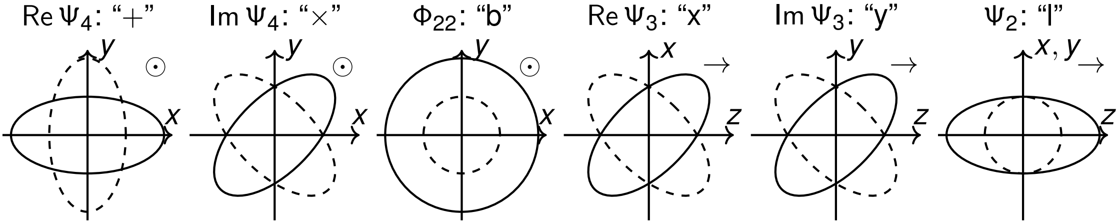
$$F_0 = C_0\,, \quad F_1 = 3C_5\,, \quad F_3 = C_2 - C_5\,, \quad F_5 = 2(-C_2 + C_4 + C_5)\,,$$
$$F_2 = \frac{1}{2}(C_1 + C_2 + C_3 - 2C_4 - 4C_5)\,, \quad F_4 = \frac{1}{2}(-C_1 + C_2 + C_3 - 2C_4 - 4C_5)\,. \tag{47}$$

Post-Newtonian landscape of $\mathcal{F}(\mathcal{Q}_1, \mathcal{Q}_2, \mathcal{Q}_3, \mathcal{Q}_4, \mathcal{Q}_5)$ theories



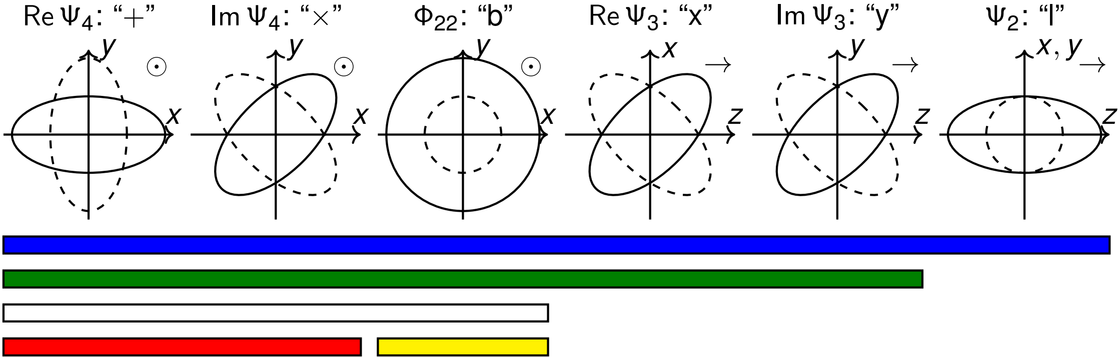
Outline

Gravitational wave polarization and E(2) formalism



Gravitational wave polarization and E(2) formalism

- II_6: 6 polarizations, all modes are allowed.
- III_5: 5 polarizations, $\Psi_2 = 0$, all other modes are allowed.
- N_3: 3 polarizations, $\Psi_2 = \Psi_3 = 0$, tensor and breathing modes are allowed.
- N_2: 2 polarizations, $\Psi_2 = \Psi_3 = \Phi_{22} = 0$, only tensor modes are allowed.
- O_1: 1 polarization, $\Psi_2 = \Psi_3 = \Psi_4 = 0$, only breathing mode is allowed.
- O_0: no gravitational waves.



Polarizations in metric teleparallel gravity

- Action:

$$S_g[\theta] = \int_M \mathcal{F}(\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3)\, \theta\, \mathrm{d}^4x\,. \tag{48}$$

Polarizations in metric teleparallel gravity

- Action:

$$S_g[\theta] = \int_M \mathcal{F}(\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3)\,\theta\,\mathrm{d}^4x\,. \qquad (48)$$

- Polarizations: [MH, Kršššák, Pfeifer, Ualikhanova]
  - N_2 for

$$2F_{,1} + F_{,2} + F_{,3} = 0 \quad \wedge \quad F_{,3} \neq 0\,. \qquad (49)$$

  - N_3 for

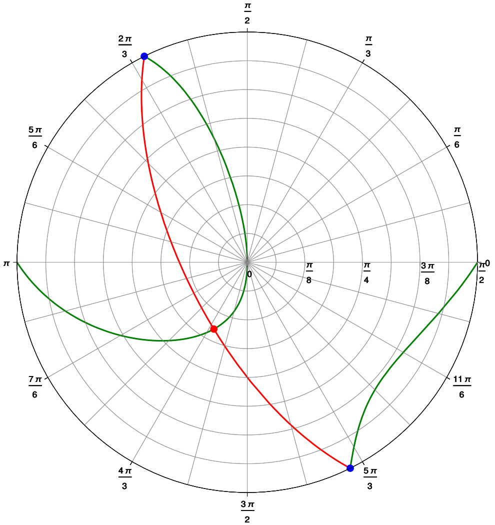
$$2F_{,1}(F_{,2} + F_{,3}) + F_{,2}^2 \neq 0 \quad \wedge \quad 2F_{,1} + F_{,2} + F_{,3} \neq 0\,. \qquad (50)$$

  - III_5 for

$$2F_{,1}(F_{,2} + F_{,3}) + F_{,2}^2 = 0 \quad \wedge \quad 2F_{,1} + F_{,2} + F_{,3} \neq 0\,. \qquad (51)$$

  - II_6 for

$$2F_{,1} + F_{,2} = F_{,3} = 0\,. \qquad (52)$$



Polarizations in symmetric teleparallel gravity

- Action:

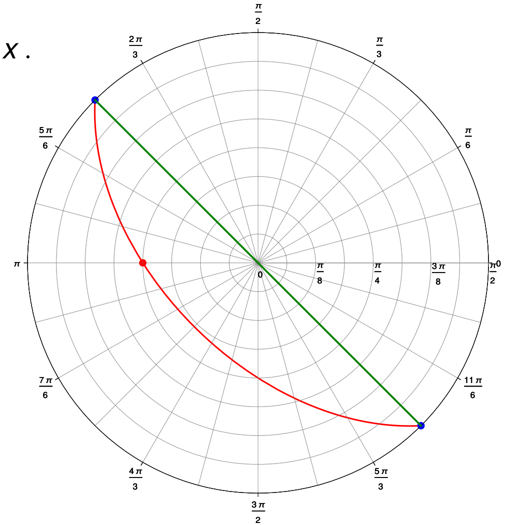
$$S_g[g,\Gamma] = \int_M \mathcal{F}\left(\mathcal{Q}_1, \mathcal{Q}_2, \mathcal{Q}_3, \mathcal{Q}_4, \mathcal{Q}_5\right)\sqrt{-g}\,\mathrm{d}^4x\,. \tag{53}$$

Polarizations in symmetric teleparallel gravity

- Action:

$$S_g[g,\Gamma] = \int_M \mathcal{F}(\mathcal{Q}_1,\mathcal{Q}_2,\mathcal{Q}_3,\mathcal{Q}_4,\mathcal{Q}_5)\sqrt{-g}\,\mathrm{d}^4x\,. \tag{53}$$

- Polarizations: [MH, Levi Said, Pfeifer, Ualikhanova]
  - N_2 for
  $$F_{,2}+F_{,4}+F_{,5}=0 \quad \wedge \quad F_{,5}\neq 0\,. \tag{54}$$
  - N_3 for
  $$F_{,2}+F_{,4}\neq 0 \quad \wedge \quad F_{,2}+F_{,4}+F_{,5}\neq 0\,. \tag{55}$$
  - III_5 for
  $$F_{,2}+F_{,4}=0 \quad \wedge \quad F_{,5}\neq 0\,. \tag{56}$$
  - II_6 for
  $$F_{,2}+F_{,4}=F_{,5}=0\,. \tag{57}$$



Outline

Cosmological metric teleparallel background geometry

- Friedmann-Lemaître-Robertson-Walker metric:

$$g_{\mu\nu}\mathrm{d}x^{\mu} \otimes \mathrm{d}x^{\nu} = -n_{\mu}n_{\nu} + h_{\mu\nu} = -N^2\mathrm{d}t \otimes \mathrm{d}t + A^2\gamma_{ab}\mathrm{d}x^a \otimes \mathrm{d}x^b \,. \tag{58}$$

⇒ Scale factor A, lapse function N, conformal Hubble parameter $\mathcal{H} = \partial_t A/N$.

Cosmological metric teleparallel background geometry

- Friedmann-Lemaître-Robertson-Walker metric:

$$g_{\mu\nu}\mathrm{d}x^{\mu} \otimes \mathrm{d}x^{\nu} = -n_{\mu}n_{\nu} + h_{\mu\nu} = -N^2\mathrm{d}t \otimes \mathrm{d}t + A^2\gamma_{ab}\mathrm{d}x^a \otimes \mathrm{d}x^b\,. \tag{58}$$

⇒ Scale factor A, lapse function N, conformal Hubble parameter $\mathcal{H} = \partial_t A/N$.

- Cosmologically symmetric torsion tensor:

$$\bar{T}_{\mu\nu\rho} = \frac{2\mathscr{V}h_{\mu[\nu}n_{\rho]} + 2\mathscr{A}\varepsilon_{\mu\nu\rho}}{A}\,. \tag{59}$$

Cosmological metric teleparallel background geometry

- Friedmann-Lemaître-Robertson-Walker metric:

$$g_{\mu\nu}\mathrm{d}x^{\mu}\otimes\mathrm{d}x^{\nu} = -n_{\mu}n_{\nu} + h_{\mu\nu} = -N^2\mathrm{d}t\otimes\mathrm{d}t + A^2\gamma_{ab}\mathrm{d}x^a\otimes\mathrm{d}x^b\,. \tag{58}$$

⇒ Scale factor A, lapse function N, conformal Hubble parameter $\mathcal{H} = \partial_t A/N$.

- Cosmologically symmetric torsion tensor:

$$\bar{T}_{\mu\nu\rho} = \frac{2\mathscr{V}h_{\mu[\nu}n_{\rho]} + 2\mathscr{A}\varepsilon_{\mu\nu\rho}}{A}\,. \tag{59}$$

- Two branches of geometries with spatial curvature parameter $k = u^2$: [MH '20]
  1. "Vector" branch:
  $$\mathscr{V} = \mathcal{H} \pm iu\,, \quad \mathscr{A} = 0\,, \tag{60}$$
  2. "Axial" branch:
  $$\mathscr{V} = \mathcal{H}\,, \quad \mathscr{A} = \pm u\,. \tag{61}$$

Spatial geometry and $3+1$ decomposition

- Geometric objects defining spatial geometry:
  - Decomposition of Friedmann-Lemaître-Robertson-Walker metric:
$$g_{\mu\nu} = -n_\mu n_\nu + h_{\mu\nu} \,. \tag{62}$$
  - Unit normal (co-)vector field:
$$n_\mu \mathrm{d}x^\mu = -N\,\mathrm{d}t \,. \tag{63}$$
  - Induced metric $h_{\mu\nu}$ and constant background metric γ_{ab} on spatial hypersurfaces:
$$h_{\mu\nu}\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu = A^2\gamma_{ab}\mathrm{d}x^a \otimes \mathrm{d}x^b \,. \tag{64}$$
  - Totally antisymmetric tensors $\varepsilon_{\mu\nu\rho}$ and υ_{abc} on spatial hypersurfaces:
$$\varepsilon_{\mu\nu\rho} = n^\sigma \epsilon_{\sigma\mu\nu\rho} \,, \quad \varepsilon_{\mu\nu\rho}\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu \otimes \mathrm{d}x^\rho = A^3\upsilon_{abc}\mathrm{d}x^a \otimes \mathrm{d}x^b \otimes \mathrm{d}x^c \,. \tag{65}$$
  - Levi-Civita covariant derivative d_a of background metric γ_{ab}.

Spatial geometry and $3+1$ decomposition

- Geometric objects defining spatial geometry:
  - Decomposition of Friedmann-Lemaître-Robertson-Walker metric:

$$g_{\mu\nu} = -n_\mu n_\nu + h_{\mu\nu} \,. \tag{62}$$

  - Unit normal (co-)vector field:

$$n_\mu \mathrm{d}x^\mu = -N\,\mathrm{d}t \,. \tag{63}$$

  - Induced metric $h_{\mu\nu}$ and constant background metric γ_{ab} on spatial hypersurfaces:

$$h_{\mu\nu}\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu = A^2\gamma_{ab}\mathrm{d}x^a \otimes \mathrm{d}x^b \,. \tag{64}$$

  - Totally antisymmetric tensors $\varepsilon_{\mu\nu\rho}$ and υ_{abc} on spatial hypersurfaces:

$$\varepsilon_{\mu\nu\rho} = n^\sigma \epsilon_{\sigma\mu\nu\rho} \,, \quad \varepsilon_{\mu\nu\rho}\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu \otimes \mathrm{d}x^\rho = A^3\upsilon_{abc}\mathrm{d}x^a \otimes \mathrm{d}x^b \otimes \mathrm{d}x^c \,. \tag{65}$$

  - Levi-Civita covariant derivative d_a of background metric γ_{ab}.
- Perturbation decomposition:

$$\tau_{\mu\nu}\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu = \hat{\tau}_{00}\mathrm{d}t \otimes \mathrm{d}t + \hat{\tau}_{a0}A\,\mathrm{d}x^a \otimes \mathrm{d}t + \hat{\tau}_{0b}A\,\mathrm{d}t \otimes \mathrm{d}x^b + \hat{\tau}_{ab}A^2\,\mathrm{d}x^a \otimes \mathrm{d}x^b \,. \tag{66}$$

Cosmological perturbations in metric teleparallel gravity

- Decomposition of tetrad perturbations $\tau_{\mu\nu}$:

$$\hat{\tau}_{00} = \hat{\phi}\,, \tag{67a}$$

$$\hat{\tau}_{0b} = \mathrm{d}_b\hat{j} + \hat{b}_b\,, \tag{67b}$$

$$\hat{\tau}_{a0} = \mathrm{d}_a\hat{y} + \hat{v}_a\,, \tag{67c}$$

$$\hat{\tau}_{ab} = \hat{\psi}\gamma_{ab} + \mathrm{d}_a\mathrm{d}_b\hat{\sigma} + \mathrm{d}_b\hat{c}_a + \upsilon_{abc}(\mathrm{d}^c\hat{\xi} + \hat{w}^c) + \frac{1}{2}\hat{q}_{ab}\,. \tag{67d}$$

Cosmological perturbations in metric teleparallel gravity

- Decomposition of tetrad perturbations $\tau_{\mu\nu}$:

$$\hat{\tau}_{00} = \hat{\phi}\,, \tag{67a}$$

$$\hat{\tau}_{0b} = \mathrm{d}_b\hat{j} + \hat{b}_b\,, \tag{67b}$$

$$\hat{\tau}_{a0} = \mathrm{d}_a\hat{y} + \hat{v}_a\,, \tag{67c}$$

$$\hat{\tau}_{ab} = \hat{\psi}\gamma_{ab} + \mathrm{d}_a\mathrm{d}_b\hat{\sigma} + \mathrm{d}_b\hat{c}_a + \upsilon_{abc}(\mathrm{d}^c\hat{\xi} + \hat{w}^c) + \frac{1}{2}\hat{q}_{ab}\,. \tag{67d}$$

- Conditions on vector and tensor components:

$$\mathrm{d}_a\hat{b}^a = \mathrm{d}_a\hat{v}^a = \mathrm{d}_a\hat{c}^a = \mathrm{d}_a\hat{w}^a = 0\,, \quad \mathrm{d}_a\hat{q}^{ab} = 0\,, \quad \hat{q}_{[ab]} = 0\,, \quad \hat{q}_a{}^a = 0\,. \tag{68}$$

Cosmological perturbations in metric teleparallel gravity

- Decomposition of tetrad perturbations $\tau_{\mu\nu}$:

$$\hat{\tau}_{00} = \hat{\phi}\,, \tag{67a}$$

$$\hat{\tau}_{0b} = \mathrm{d}_b\hat{j} + \hat{b}_b\,, \tag{67b}$$

$$\hat{\tau}_{a0} = \mathrm{d}_a\hat{y} + \hat{v}_a\,, \tag{67c}$$

$$\hat{\tau}_{ab} = \hat{\psi}\gamma_{ab} + \mathrm{d}_a\mathrm{d}_b\hat{\sigma} + \mathrm{d}_b\hat{c}_a + \upsilon_{abc}(\mathrm{d}^c\hat{\xi} + \hat{w}^c) + \frac{1}{2}\hat{q}_{ab}\,. \tag{67d}$$

- Conditions on vector and tensor components:

$$\mathrm{d}_a\hat{b}^a = \mathrm{d}_a\hat{v}^a = \mathrm{d}_a\hat{c}^a = \mathrm{d}_a\hat{w}^a = 0\,, \quad \mathrm{d}_a\hat{q}^{ab} = 0\,, \quad \hat{q}_{[ab]} = 0\,, \quad \hat{q}_a{}^a = 0\,. \tag{68}$$

- Note that the term $\mathrm{d}_b\hat{c}_a$ is not symmetrized: [Golovnev, Koivisto '18]
  - Antisymmetric part $\mathrm{d}_{[a}\hat{c}_{b]} = \frac{1}{2}\upsilon_{abc}\upsilon^{dec}\mathrm{d}_d\hat{c}_e$ can be absorbed into $\hat{w}^a$.
  - Vanishing divergence follows from Bianchi identity

$$\mathrm{d}_c(\upsilon^{dec}\mathrm{d}_d\hat{c}_e) = \upsilon^{dec}\mathrm{d}_{[c}\mathrm{d}_{d]}\hat{c}_e = \frac{1}{2}\upsilon^{dec}R^f{}_{ecd}\hat{c}_f = 0\,. \tag{69}$$

Cosmological perturbations in metric teleparallel gravity

- Decomposition of tetrad perturbations $\tau_{\mu\nu}$:

$$\hat{\tau}_{00} = \hat{\phi}\,, \tag{67a}$$

$$\hat{\tau}_{0b} = \mathrm{d}_b\hat{j} + \hat{b}_b\,, \tag{67b}$$

$$\hat{\tau}_{a0} = \mathrm{d}_a\hat{y} + \hat{v}_a\,, \tag{67c}$$

$$\hat{\tau}_{ab} = \hat{\psi}\gamma_{ab} + \mathrm{d}_a\mathrm{d}_b\hat{\sigma} + \mathrm{d}_b\hat{c}_a + \upsilon_{abc}(\mathrm{d}^c\hat{\xi} + \hat{w}^c) + \frac{1}{2}\hat{q}_{ab}\,. \tag{67d}$$

- Conditions on vector and tensor components:

$$\mathrm{d}_a\hat{b}^a = \mathrm{d}_a\hat{v}^a = \mathrm{d}_a\hat{c}^a = \mathrm{d}_a\hat{w}^a = 0\,, \quad \mathrm{d}_a\hat{q}^{ab} = 0\,, \quad \hat{q}_{[ab]} = 0\,, \quad \hat{q}_a{}^a = 0\,. \tag{68}$$

- Note that the term $\mathrm{d}_b\hat{c}_a$ is not symmetrized: [Golovnev, Koivisto '18]
  - Antisymmetric part $\mathrm{d}_{[a}\hat{c}_{b]} = \frac{1}{2}\upsilon_{abc}\upsilon^{dec}\mathrm{d}_d\hat{c}_e$ can be absorbed into $\hat{w}^a$.
  - Vanishing divergence follows from Bianchi identity

$$\mathrm{d}_c(\upsilon^{dec}\mathrm{d}_d\hat{c}_e) = \upsilon^{dec}\mathrm{d}_{[c}\mathrm{d}_{d]}\hat{c}_e = \frac{1}{2}\upsilon^{dec}R^f{}_{ecd}\hat{c}_f = 0\,. \tag{69}$$

⇒ Number of components: $6 + 4 \times 2 + 1 \times 2 = 16$.

Gauge transformation of cosmological perturbations

- Split gauge transformation $x'^{\mu} = x^{\mu} + X^{\mu}(x)$:

$$\hat{X}_0 = \hat{X}_{\perp}\,, \quad \hat{X}_a = \mathrm{d}_a\hat{X}_{\|} + \hat{Z}_a \qquad (70)$$

Gauge transformation of cosmological perturbations

- Split gauge transformation $x'^{\mu} = x^{\mu} + X^{\mu}(x)$:

$$\hat{X}_0 = \hat{X}_{\perp}\,, \quad \hat{X}_a = \mathrm{d}_a \hat{X}_{\parallel} + \hat{Z}_a \tag{70}$$

⇒ Transformation of perturbation components:

$$A\delta_X \hat{\tau}_{0b} = \mathrm{d}_b \hat{X}_{\perp} + (\mathrm{d}_b \hat{X}_{\parallel} + \hat{Z}_b)(\mathscr{V} - \mathcal{H})\,, \tag{71a}$$

$$A\delta_X \hat{\tau}_{a0} = \mathrm{d}_a \hat{X}'_{\parallel} + Z'_a - \mathscr{V}(\mathrm{d}_a \hat{X}_{\parallel} + Z_a)\,, \tag{71b}$$

$$A\delta_X \hat{\tau}_{00} = \hat{X}'_{\perp}\,, \tag{71c}$$

$$A\delta_X \hat{\tau}_{ab} = \mathrm{d}_b(\mathrm{d}_a \hat{X}_{\parallel} + \hat{Z}_a) - \mathcal{H}\hat{X}_{\perp}\gamma_{ab} - \mathscr{A}\upsilon_{abc}(\mathrm{d}^c \hat{X}_{\parallel} + \hat{Z}^c)\,. \tag{71d}$$

- Split gauge transformation $x'^{\mu} = x^{\mu} + X^{\mu}(x)$:

$$\hat{X}_0 = \hat{X}_{\perp}\,, \quad \hat{X}_a = \mathrm{d}_a\hat{X}_{\parallel} + \hat{Z}_a \tag{70}$$

⇒ Transformation of perturbation components:

$$A\delta_X\hat{\tau}_{0b} = \mathrm{d}_b\hat{X}_{\perp} + (\mathrm{d}_b\hat{X}_{\parallel} + \hat{Z}_b)(\mathscr{V} - \mathcal{H})\,, \tag{71a}$$

$$A\delta_X\hat{\tau}_{a0} = \mathrm{d}_a\hat{X}'_{\parallel} + Z'_a - \mathscr{V}(\mathrm{d}_a\hat{X}_{\parallel} + Z_a)\,, \tag{71b}$$

$$A\delta_X\hat{\tau}_{00} = \hat{X}'_{\perp}\,, \tag{71c}$$

$$A\delta_X\hat{\tau}_{ab} = \mathrm{d}_b(\mathrm{d}_a\hat{X}_{\parallel} + \hat{Z}_a) - \mathcal{H}\hat{X}_{\perp}\gamma_{ab} - \mathscr{A}\upsilon_{abc}(\mathrm{d}^c\hat{X}_{\parallel} + \hat{Z}^c)\,. \tag{71d}$$

⇒ Irreducible decomposition: [MH '20]

$$A\delta_X\hat{\psi} = -\mathcal{H}\hat{X}_{\perp}\,, \tag{72a}$$

$$A\delta_X\hat{\sigma} = \hat{X}_{\parallel}\,, \tag{72b}$$

$$A\delta_X\hat{y} = \hat{X}'_{\parallel} - \mathscr{V}\hat{X}_{\parallel}\,, \tag{72c}$$

$$A\delta_X\hat{j} = \hat{X}_{\perp} + (\mathscr{V} - \mathcal{H})\hat{X}_{\parallel}\,, \tag{72d}$$

$$A\delta_X\hat{\xi} = -\mathscr{A}\hat{X}_{\parallel}\,, \tag{72e}$$

$$A\delta_X\hat{\phi} = \hat{X}'_{\perp}\,, \tag{72f}$$

$$A\delta_X\hat{c}_a = \hat{Z}_a\,, \tag{72g}$$

$$A\delta_X\hat{v}_a = \hat{Z}'_a - \mathscr{V}\hat{Z}_a\,, \tag{72h}$$

$$A\delta_X\hat{b}_a = (\mathscr{V} - \mathcal{H})\hat{Z}_a\,, \tag{72i}$$

$$A\delta_X\hat{w}_a = -\mathscr{A}\hat{Z}_a\,, \tag{72j}$$

$$A\delta_X\hat{q}_{ab} = 0\,. \tag{72k}$$

Gauge-invariant perturbations

- Gauge-invariant cosmological tetrad perturbations: [MH '20]
  - Scalar perturbations - 3 scalars + 1 pseudo-scalar:

$$\hat{\boldsymbol{\xi}} = \hat{\xi} + \mathscr{A}\hat{\sigma}\,, \tag{73a}$$

$$\hat{\mathbf{y}} = \hat{y} - \hat{\sigma}' - (\mathcal{H} - \mathscr{V})\hat{\sigma}\,, \tag{73b}$$

$$\hat{\boldsymbol{\psi}} = \hat{\psi} + \mathcal{H}[\hat{j} + (\mathcal{H} - \mathscr{V})\hat{\sigma}]\,, \tag{73c}$$

$$\hat{\boldsymbol{\phi}} = \hat{\phi} - \mathcal{H}[\hat{j} + (\mathcal{H} - \mathscr{V})\hat{\sigma}] + [\hat{j} + (\mathcal{H} - \mathscr{V})\hat{\sigma}]'\,. \tag{73d}$$

  - Vector perturbations - 2 divergence-free vectors + 1 pseudo-vector:

$$\hat{\mathbf{v}}_a = \hat{v}_a + (\mathscr{V} - \mathcal{H})\hat{c}_a - \hat{c}'_a\,, \tag{73e}$$

$$\hat{\mathbf{b}}_a = \hat{b}_a + (\mathcal{H} - \mathscr{V})\hat{c}_a\,, \tag{73f}$$

$$\hat{\mathbf{w}}_a = \hat{w}_a + \mathscr{A}\hat{c}_a\,, \tag{73g}$$

  - Tensor perturbation - 1 symmetric, trace-free, divergence-free tensor:

$$\hat{\mathbf{q}}_{ab} = \hat{q}_{ab}\,. \tag{73h}$$

Gauge-invariant perturbations

- Gauge-invariant cosmological tetrad perturbations: [MH '20]
  - Scalar perturbations - 3 scalars + 1 pseudo-scalar:

$$\hat{\boldsymbol{\xi}} = \hat{\xi} + \mathscr{A}\hat{\sigma}\,, \tag{73a}$$
$$\hat{\mathbf{y}} = \hat{y} - \hat{\sigma}' - (\mathcal{H} - \mathscr{V})\hat{\sigma}\,, \tag{73b}$$
$$\hat{\boldsymbol{\psi}} = \hat{\psi} + \mathcal{H}[\hat{j} + (\mathcal{H} - \mathscr{V})\hat{\sigma}]\,, \tag{73c}$$
$$\hat{\boldsymbol{\phi}} = \hat{\phi} - \mathcal{H}[\hat{j} + (\mathcal{H} - \mathscr{V})\hat{\sigma}] + [\hat{j} + (\mathcal{H} - \mathscr{V})\hat{\sigma}]'\,. \tag{73d}$$

  - Vector perturbations - 2 divergence-free vectors + 1 pseudo-vector:

$$\hat{\mathbf{v}}_a = \hat{v}_a + (\mathscr{V} - \mathcal{H})\hat{c}_a - \hat{c}'_a\,, \tag{73e}$$
$$\hat{\mathbf{b}}_a = \hat{b}_a + (\mathcal{H} - \mathscr{V})\hat{c}_a\,, \tag{73f}$$
$$\hat{\mathbf{w}}_a = \hat{w}_a + \mathscr{A}\hat{c}_a\,, \tag{73g}$$

  - Tensor perturbation - 1 symmetric, trace-free, divergence-free tensor:

$$\hat{\mathbf{q}}_{ab} = \hat{q}_{ab}\,. \tag{73h}$$

$\Rightarrow$ Number of components: $4 + 3 \times 2 + 1 \times 2 = 12 = 16 - 4$.

Outline

Challenges and solutions in perturbation theory

- Problem statement and definition:
  - ○ Perturbations of geometry: field equations and solution.
  - ○ Purely analytic approach - no numerical relativity calculations.
  - ↯ Often lengthy and cumbersome tensor equations to be solved.
  - ✓ Successively order-by-order progressing solutions algorithms.

Challenges and solutions in perturbation theory

- Problem statement and definition:
  - ○ Perturbations of geometry: field equations and solution.
  - ○ Purely analytic approach - no numerical relativity calculations.
  - ↯ Often lengthy and cumbersome tensor equations to be solved.
  - ✓ Successively order-by-order progressing solutions algorithms.
- Various software suites include libraries for tensor algebra:
  - ○ Free: SymPy & SciPy for Python; Maxima tensor packages; Cadabra...
  - ○ Commercial: DifferentialGeometry for Maple; Ricci, Tensor & xAct for Mathematica...

Challenges and solutions in perturbation theory

- Problem statement and definition:
  - Perturbations of geometry: field equations and solution.
  - Purely analytic approach - no numerical relativity calculations.
  - ⚡ Often lengthy and cumbersome tensor equations to be solved.
  - ✓ Successively order-by-order progressing solutions algorithms.
- Various software suites include libraries for tensor algebra:
  - Free: SymPy & SciPy for Python; Maxima tensor packages; Cadabra…
  - Commercial: DifferentialGeometry for Maple; Ricci, Tensor & xAct for Mathematica…
- Example: xAct - free tensor algebra package for Mathematica: [Martín-García '02]
  - Versatile package suite for tensor algebra.
  - Build upon powerful Mathematica software, with native binary component.
  - Various packages for component calculus, spinors, field theory…

Challenges and solutions in perturbation theory

- Problem statement and definition:
  - Perturbations of geometry: field equations and solution.
  - Purely analytic approach - no numerical relativity calculations.
  - ↯ Often lengthy and cumbersome tensor equations to be solved.
  - ✓ Successively order-by-order progressing solutions algorithms.
- Various software suites include libraries for tensor algebra:
  - Free: SymPy & SciPy for Python; Maxima tensor packages; Cadabra...
  - Commercial: DifferentialGeometry for Maple; Ricci, Tensor & xAct for Mathematica...
- Example: xAct - free tensor algebra package for Mathematica: [Martín-García '02]
  - Versatile package suite for tensor algebra.
  - Build upon powerful Mathematica software, with native binary component.
  - Various packages for component calculus, spinors, field theory...
- Packages in xAct dedicated to perturbation theory:
  - xPert: Computer algebra for metric perturbation theory [Brizuela, Martín-García, Mena Marugán '08]
  - xPand: Computer algebra for cosmological perturbation theory [Pitrou, Roy, Umeh '13]
  - xPPN: Computer algebra for the PPN formalism [MH '20]

xPPN: Computer algebra for the PPN formalism

- Most common geometric objects pre-defined:
  - Background geometry: Minkowski metric $\eta_{\mu\nu}$, diagonal background tetrad $\Delta^A{}_\mu$…
  - Dynamical geometry: metric $g_{\mu\nu}$, tetrad $\theta^A{}_\mu$, different connections…
  - Matter variables: energy-momentum tensor $\Theta_{\mu\nu}$, density ρ, pressure p…
  - Post-Newtonian approximation: all PPN potentials and PPN parameters

xPPN: Computer algebra for the PPN formalism

- Most common geometric objects pre-defined:
  - Background geometry: Minkowski metric $\eta_{\mu\nu}$, diagonal background tetrad $\Delta^A{}_\mu$...
  - Dynamical geometry: metric $g_{\mu\nu}$, tetrad $\theta^A{}_\mu$, different connections...
  - Matter variables: energy-momentum tensor $\Theta_{\mu\nu}$, density ρ, pressure p...
  - Post-Newtonian approximation: all PPN potentials and PPN parameters
- Algorithms to perform necessary steps for PPN calculation:
  - $3 + 1$ split of tensor components and derivatives into space and time.
  - Perturbative expansion of components into velocity orders.
  - Automatic cancellation of vanishing terms in PPN expansion.
  - Application of relations between PPN potentials and matter variables.

xPPN: Computer algebra for the PPN formalism

- Most common geometric objects pre-defined:
  - Background geometry: Minkowski metric $\eta_{\mu\nu}$, diagonal background tetrad $\Delta^A{}_\mu$...
  - Dynamical geometry: metric $g_{\mu\nu}$, tetrad $\theta^A{}_\mu$, different connections...
  - Matter variables: energy-momentum tensor $\Theta_{\mu\nu}$, density ρ, pressure p...
  - Post-Newtonian approximation: all PPN potentials and PPN parameters
- Algorithms to perform necessary steps for PPN calculation:
  - $3 + 1$ split of tensor components and derivatives into space and time.
  - Perturbative expansion of components into velocity orders.
  - Automatic cancellation of vanishing terms in PPN expansion.
  - Application of relations between PPN potentials and matter variables.
- Applicable to wide range of geometry-based gravity theories:
  - ✓ Riemannian geometry (metric + Levi-Civita connection, curvature only)
  - ✓ Metric teleparallel geometry (tetrad + flat, metric-compatible connection, torsion only)
  - ✓ Symmetric teleparallel geometry (metric + flat, symmetric connection, nonmetricity only)
  - + General teleparallel geometry (metric + flat connection, torsion + nonmetricity)
  - + Riemann-Cartan geometry (metric + metric-compatible connection, curvature + torsion)
  - + Metric-affine geometry (metric + general affine connection)

xPPN: Computer algebra for the PPN formalism

- Most common geometric objects pre-defined:
  - Background geometry: Minkowski metric $\eta_{\mu\nu}$, diagonal background tetrad $\Delta^A{}_\mu$...
  - Dynamical geometry: metric $g_{\mu\nu}$, tetrad $\theta^A{}_\mu$, different connections...
  - Matter variables: energy-momentum tensor $\Theta_{\mu\nu}$, density ρ, pressure p...
  - Post-Newtonian approximation: all PPN potentials and PPN parameters
- Algorithms to perform necessary steps for PPN calculation:
  - $3 + 1$ split of tensor components and derivatives into space and time.
  - Perturbative expansion of components into velocity orders.
  - Automatic cancellation of vanishing terms in PPN expansion.
  - Application of relations between PPN potentials and matter variables.
- Applicable to wide range of geometry-based gravity theories:
  - ✓ Riemannian geometry (metric + Levi-Civita connection, curvature only)
  - ✓ Metric teleparallel geometry (tetrad + flat, metric-compatible connection, torsion only)
  - ✓ Symmetric teleparallel geometry (metric + flat, symmetric connection, nonmetricity only)
  - \+ General teleparallel geometry (metric + flat connection, torsion + nonmetricity)
  - \+ Riemann-Cartan geometry (metric + metric-compatible connection, curvature + torsion)
  - \+ Metric-affine geometry (metric + general affine connection)
- Code and examples: `http://geomgrav.fi.ut.ee/software/xPPN.html`

Outline

Summary

- General treatment of perturbations in gravity theory:
  - Gravitational field and governing equations described by differential geometry.
  - Perturbations and their gauge transformations derived from geometric picture.
  - Practical treatment by differential equations derived from coordinate choice.

Summary

- General treatment of perturbations in gravity theory:
  - Gravitational field and governing equations described by differential geometry.
  - Perturbations and their gauge transformations derived from geometric picture.
  - Practical treatment by differential equations derived from coordinate choice.
- Metric-affine and teleparallel geometries and their perturbations:
  - Suitable class of geometries to describe wide range of gravity theories.
  - Geometric description using Lorentzian metric and affine connection.
  - Alternative connection in terms of tetrad and spin connection.
  - Perturbation can be expressed in terms of tensor fields.

Summary

- General treatment of perturbations in gravity theory:
  - Gravitational field and governing equations described by differential geometry.
  - Perturbations and their gauge transformations derived from geometric picture.
  - Practical treatment by differential equations derived from coordinate choice.
- Metric-affine and teleparallel geometries and their perturbations:
  - Suitable class of geometries to describe wide range of gravity theories.
  - Geometric description using Lorentzian metric and affine connection.
  - Alternative connection in terms of tetrad and spin connection.
  - Perturbation can be expressed in terms of tensor fields.
- Perturbations in teleparallel gravity theories:
  - Gravity theories in which torsion or nonmetricity mediates gravity actively studied.
  - Post-Newtonian limit of torsion theories largely studied; nonmetricity theories classified.
  - Polarization and speed of gravitational waves studied for torsion / nonmetricity.
  - General theory of cosmological perturbations under development.

Summary

- General treatment of perturbations in gravity theory:
  - Gravitational field and governing equations described by differential geometry.
  - Perturbations and their gauge transformations derived from geometric picture.
  - Practical treatment by differential equations derived from coordinate choice.
- Metric-affine and teleparallel geometries and their perturbations:
  - Suitable class of geometries to describe wide range of gravity theories.
  - Geometric description using Lorentzian metric and affine connection.
  - Alternative connection in terms of tetrad and spin connection.
  - Perturbation can be expressed in terms of tensor fields.
- Perturbations in teleparallel gravity theories:
  - Gravity theories in which torsion or nonmetricity mediates gravity actively studied.
  - Post-Newtonian limit of torsion theories largely studied; nonmetricity theories classified.
  - Polarization and speed of gravitational waves studied for torsion / nonmetricity.
  - General theory of cosmological perturbations under development.
- Computational tools applicable to perturbation theory:
  - Geometric nature of gravity theories suggest using tensor algebra.
  - Fixed schemes in perturbation theory suitable for algorithmic approach.
  - Example: xPPN package for xAct / Mathematica allows calculating PPN parameters.

Outlook

- Fundamental questions in perturbation theory:
  - Higher order gauge-invariant perturbations beyond tensor fields.
  - Perturbations of more general geometries (Finsler, Lagrange, Hamilton).
  - Equivalence / inequivalence of perturbations in analogue descriptions of gravity.

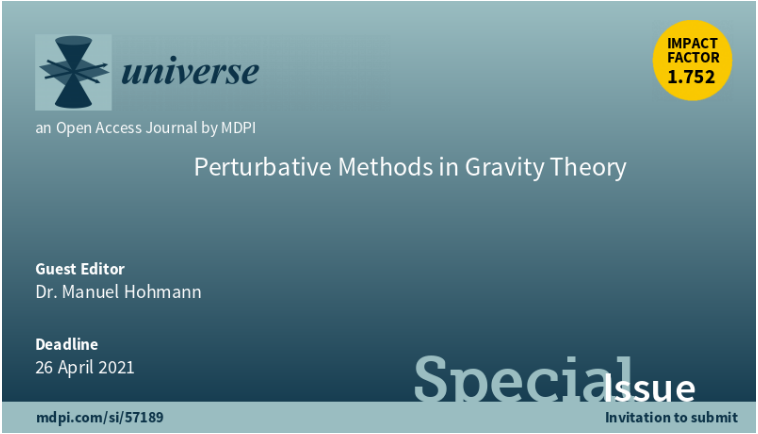
Outlook

- Fundamental questions in perturbation theory:
  - Higher order gauge-invariant perturbations beyond tensor fields.
  - Perturbations of more general geometries (Finsler, Lagrange, Hamilton).
  - Equivalence / inequivalence of perturbations in analogue descriptions of gravity.
- Study of further physical systems using perturbations of teleparallel geometry:
  - Spherical symmetry: quasinormal modes and extreme mass ratio inspirals.
  - Higher-order post-Newtonian approximation and binary dynamics.
  - Asymptotically flat geometries, gravitational wave memory effect and BMS group.

Outlook

- Fundamental questions in perturbation theory:
  - Higher order gauge-invariant perturbations beyond tensor fields.
  - Perturbations of more general geometries (Finsler, Lagrange, Hamilton).
  - Equivalence / inequivalence of perturbations in analogue descriptions of gravity.
- Study of further physical systems using perturbations of teleparallel geometry:
  - Spherical symmetry: quasinormal modes and extreme mass ratio inspirals.
  - Higher-order post-Newtonian approximation and binary dynamics.
  - Asymptotically flat geometries, gravitational wave memory effect and BMS group.
- Development of computational tools:
  - Implementation of more general geometries in tensor algebra software.
  - Software development to address further perturbative algorithms.

Contact & invitation



Questions, suggestions - http://kodu.ut.ee/~manuel/