How to (not) break local Lorentz invariance in gravity theory

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Outline

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- So far unexplained cosmological observations:
  - Accelerating expansion of the universe.
  - Homogeneity of cosmic microwave background.

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- Idea here: modification of the geometric structure of spacetime!
  - Study classical gravity theories based on modified geometry.
  - Consider geometries as effective models of quantum gravity.
  - Derive observable effects to test modified geometry.

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- Questions:
  - How are measurements between detectors at same point related?
  - How does this relation depend on the location of detectors?

The Einstein equivalence principle

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  1. Freely falling test bodies move independent of their composition.
  2. Local non-gravitational experiments independent of velocity.
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- Invariance of physical laws:
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- Consequences for gravitational theory:
  - Spacetime equipped with metric $g_{\mu\nu}$.
  - Freely falling particles follow geodesics of $g_{\mu\nu}$.
  - Local, freely falling laboratories with $g_{\mu\nu} = \eta_{\mu\nu}$.
  - Local, non-gravitational physics respects special relativity.

Orthonormal frames and Lorentz transformations

- Establish orthonormal frame ${e_a}^\mu$ at spacetime point $x \in M$:
  - Four-velocity of observer $\rightsquigarrow$ direction of time component.
  - Clock showing proper time $\rightsquigarrow$ normalization of time component.
  - Light rays / radar experiment $\rightsquigarrow$ direction of spatial components.
  - Light turnaround time $\rightsquigarrow$ normalization of spatial components.
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- Comparing frames established by different observers:
  - Observers with different four-velocities $\dot{\gamma}^\mu, \dot{\gamma}'^\mu$ at same point x.
  - Each observer establishes an orthonormal frame ${e_a}^\mu, {e'_a}^\mu$.
  - LLI: observers' frames are related by Lorentz transformation:

$$ {e'_a}^\mu = {\Lambda_a}^b {e_b}^\mu \,, \quad {\Lambda_a}^c {\Lambda_b}^d \eta_{cd} = \eta_{ab} \,. \tag{1} $$

$\Rightarrow$ Observers find same metric components

$$ g^{\mu\nu} = \eta^{ab} {e_a}^\mu {e_b}^\nu = \eta^{ab} {e'_a}^\mu {e'_b}^\nu \,. \tag{2} $$

  - Frames have same orientation and time-orientation.

Lorentz covariance of observables

- Relating observations made by different observers:
  - Observers measure quantities in their own frames ${e_a}^{\mu}, {e'_a}^{\mu}$.
  - Observers in general obtain different values Q^I, Q'^I.
  - Lorentz covariance: representation ρ of $\mathrm{SO}_0(1,3)$:

$$Q'^I = {\rho^I}_J(\Lambda) Q^J \,. \tag{3}$$

  - Lorentz invariance if $Q'^I = Q^I$.

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- Example: energy-momentum of particles:
  - Observers measure $(p_a) = (E, \vec{p})$ and $(p'_a) = (E', \vec{p}')$.
  - Momentum components form covector: $p'_a = \Lambda_a{}^b p_b$.
  - ⇒ Physical, frame independent quantity p_μ gives observables:

$$p_a = e_a{}^\mu p_\mu \,, \quad p'_a = e'_a{}^\mu p_\mu \,. \tag{4}$$

  - ⇒ Mass m is Lorentz-invariant quantity:

$$\eta^{ab} p_a p_b = \eta^{ab} p'_a p'_b = g^{\mu\nu} p_\mu p_\nu = -m^2 \,. \tag{5}$$

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- Local Lorentz invariance manifest in dispersion relation.

LLI in the PPN formalism

- Perturbative expansion of the metric:

$$g^{(2)}_{00} = 2\alpha U\,, \tag{6a}$$

$$g^{(2)}_{\alpha\beta} = 2\gamma U\delta_{\alpha\beta}\,, \tag{6b}$$

$$\begin{aligned} g^{(3)}_{0\alpha} &= -\frac{1}{2}(3 + 4\gamma + \alpha_1 - \alpha_2 + \zeta_1 - 2\xi)V_\alpha \\ &\quad - \frac{1}{2}(1 + \alpha_2 - \zeta_1 + 2\xi)W_\alpha\,, \end{aligned} \tag{6c}$$

$$\begin{aligned} g^{(4)}_{00} &= -2\beta U^2 - 2\xi\Phi_W + (2 + 2\gamma + \alpha_3 + \zeta_1 - 2\xi)\Phi_1 \\ &\quad + 2(1 + 3\gamma - 2\beta + \zeta_2 + \xi)\Phi_2 + 2(1 + \zeta_3)\Phi_3 \\ &\quad + 2(3\gamma + 3\zeta_4 - 2\xi)\Phi_4 - (\zeta_1 - 2\xi)\mathcal{A}\,. \end{aligned} \tag{6d}$$

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- LLI if $(\alpha_1, \alpha_2, \alpha_3) \neq (0, 0, 0)$.

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Field variables in teleparallel gravity

- Metric teleparallelism conventionally formulated using:
  - Tetrad / coframe: $\theta^a = \theta^a{}_\mu \mathrm{d}x^\mu$ with inverse $e_a = e_a{}^\mu \partial_\mu$.
  - Spin connection: $\omega^a{}_b = \omega^a{}_{b\mu} \mathrm{d}x^\mu$.

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- Induced metric-affine geometry:
  - Metric:

$$g_{\mu\nu} = \eta_{ab}\theta^a{}_\mu \theta^b{}_\nu \,. \tag{7}$$

  - Affine connection:

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- Conditions on the spin connection:
  - Flatness $R = 0$:
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  - Metric compatibility $Q = 0$:
$$\eta_{ac}\omega^c{}_{b\mu} + \eta_{bc}\omega^c{}_{a\mu} = 0 \,. \tag{10}$$

Local Lorentz transformations

- Local Lorentz transformation of the tetrad only:

$$\theta^a{}_\mu \mapsto \theta'^a{}_\mu = \Lambda^a{}_b \theta^b{}_\mu \,. \tag{11}$$

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- Is LLI broken if teleparallel gravity action depends on $\Gamma^\mu{}_{\nu\rho}$?

The Weitzenböck gauge

- Intuitive conclusion: *One can always use the Weitzenböck gauge.*
  - The spin connection is flat:

$$\partial_\mu \omega^a{}_{b\nu} - \partial_\nu \omega^a{}_{b\mu} + \omega^a{}_{c\mu}\omega^c{}_{b\nu} - \omega^a{}_{c\nu}\omega^c{}_{b\mu} \equiv 0\,. \tag{13}$$

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$$\Lambda^a{}_b \mapsto \Lambda'^a{}_b = \Lambda^a{}_c \Omega^c{}_b\,, \quad \overset{w}{\theta}{}^a{}_\mu \mapsto \overset{w}{\theta}{}'^a{}_\mu = (\Omega^{-1})^a{}_b \overset{w}{\theta}{}^b{}_\mu\,. \tag{15}$$

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$$\Lambda^a{}_b \mapsto \Lambda'^a{}_b = \Lambda^a{}_c \Omega^c{}_b\,, \quad \overset{w}{\theta}{}^a{}_\mu \mapsto \overset{w}{\theta}{}'^a{}_\mu = (\Omega^{-1})^a{}_b \overset{w}{\theta}{}^b{}_\mu\,. \tag{15}$$

- Questions posed by the adept of geometry:
  1. How can we determine the transformation $\Lambda^a{}_b$?
  2. Is this even true?
- Remark: this holds also in symmetric and general teleparallelism.

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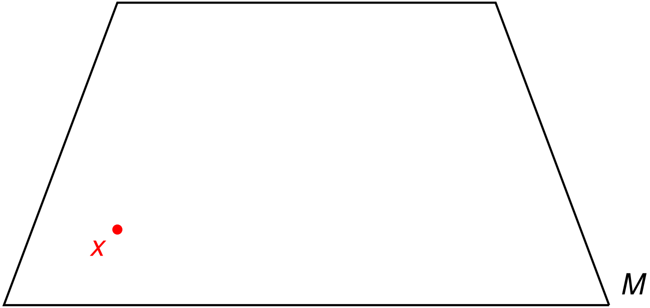

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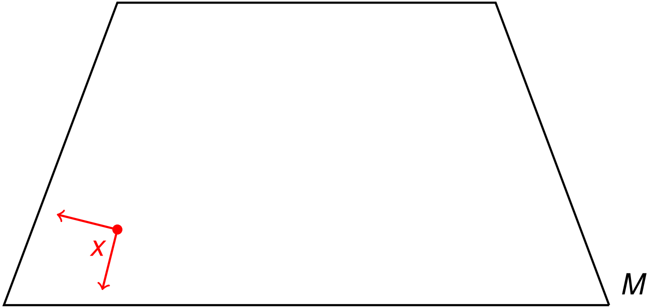

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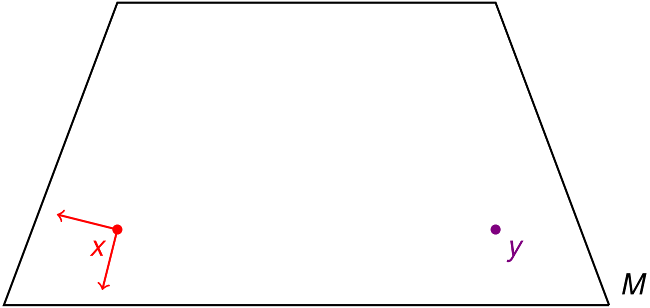

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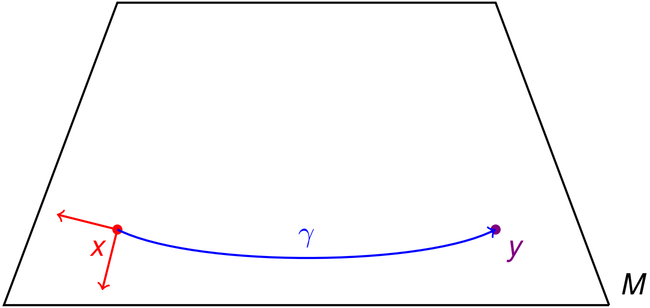

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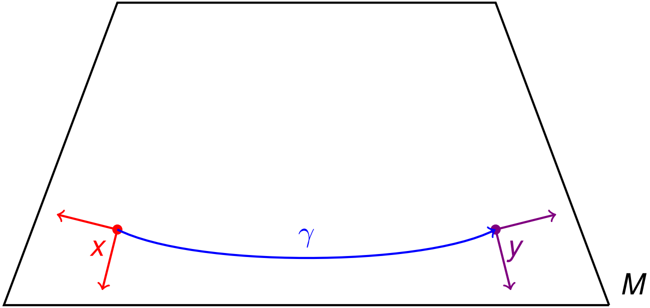

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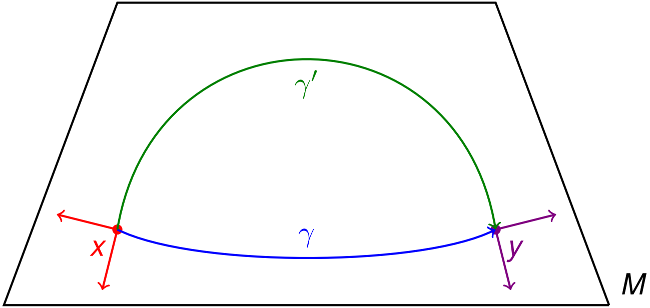

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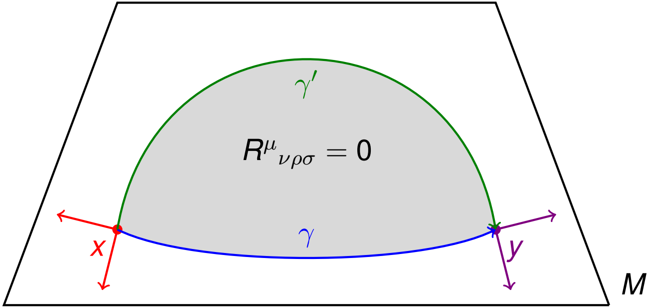

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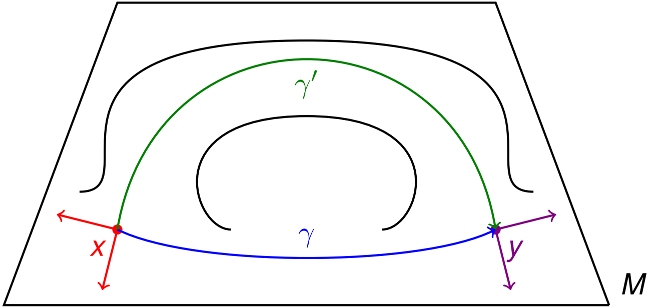

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- Physical geometry: $\mathrm{SO}_0(1,3)$ reduction of the frame bundle & Γ.

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$\Rightarrow$ The “usual rules” for playing with “dark” fields apply:
  - Find out which degrees of freedom couple to physical observables.
  - “Remnant symmetries” may yield gauge degrees of freedom.
  - Make sure physical degrees of freedom obey healthy evolution.
  - ↯ Pay attention to possible pathologies:
    - Is the evolution of physical degrees of freedom determined?
    - Are the physical degrees of freedom stable under perturbations?
    - Does the theory remain healthy under quantization?

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$\Rightarrow$ Most fundamental variables found in geometric picture.

The geometric picture

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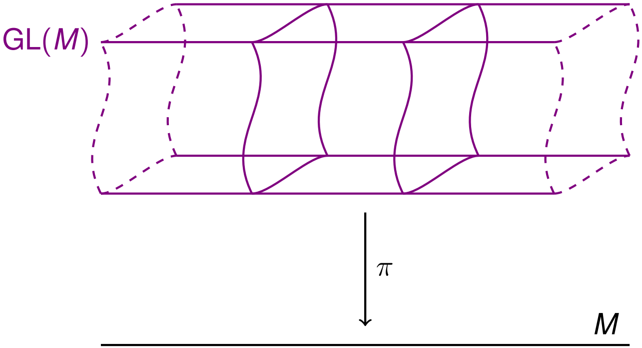

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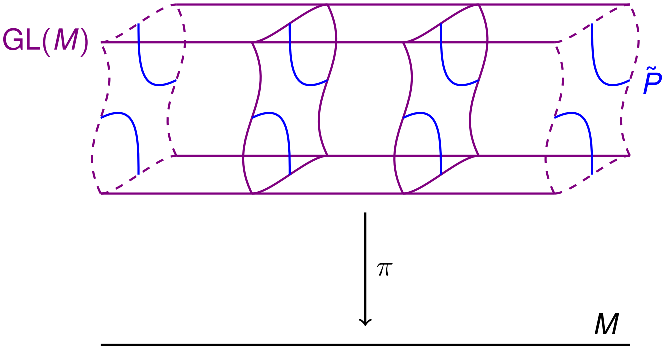

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1. Start with the general linear frame bundle $\pi : \mathrm{GL}(M) \to M$.
2. Metric reduces bundle to orthonormal frame bundle $\tilde{P}$.
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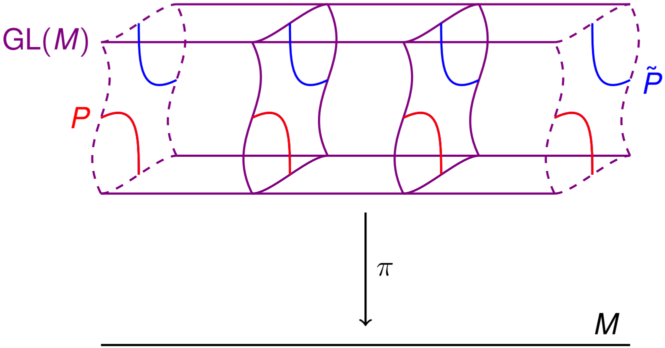

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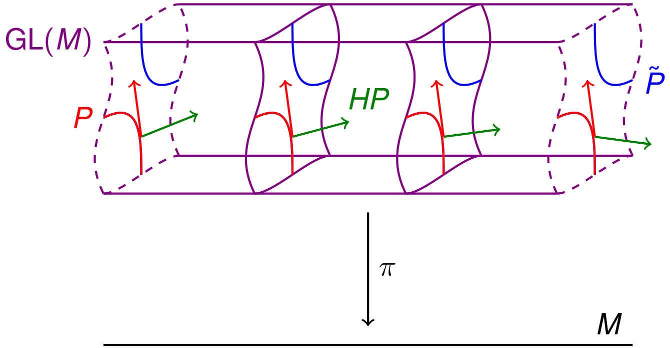
4. Connection specifies horizontal directions $TP = VP \oplus HP$ in P.



Tetrads and spin structure

- How to obtain a spin structure from a tetrad $e : M \to P$?
  1. Spin structure obtained from trivial bundle $Q = M \times \mathrm{SL}(2, \mathbb{C})$.
  2. Use covering map $\sigma : \mathrm{SL}(2, \mathbb{C}) \to \mathrm{SO}_0(1, 3)$.
  3. Define spin structure $\varphi : Q \to P$ as map

$$\varphi(x, z) = e(x) \cdot \sigma(z) . \tag{18}$$

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- Do different tetrads e, e' define the same spin structure?
  - Consider non-simply connected manifold M.
  - Let $\gamma : [0, 1] \to M$ with $\gamma(0) = \gamma(1)$ non-contractible.
  - Let $\Lambda : M \to \mathrm{SO}_0(1, 3)$ such that $\Lambda \circ \gamma$ has odd winding.
  - Tetrads $e = e' \cdot \Lambda$ define spin structures φ, φ'.
  - Assume existence of bundle isomorphism $\mu : Q \to Q$, $\varphi = \varphi' \circ \mu$.

  $\Rightarrow$ Curve connects antipodes: $\mu(\gamma(1), \mathbb{1}) = -\mu(\gamma(0), \mathbb{1})$.

  $\lightning$ Contradicts $\gamma(0) = \gamma(1)$.

  $\Rightarrow$ Spin structures φ, φ' are inequivalent.

Tetrads vs observers

- Clash of two notions of orthonormal frames:
  1. Tetrad $e : M \to P$ solving teleparallel field equation.
  2. Observer frame $\tilde{e} : \mathbb{R} \to \gamma^* P$ along trajectory γ.

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$\Rightarrow$ Observer geometry defined by metric: LLI holds.

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- Split Levi-Civita connection coefficients:

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- Connection appears only as "dark" field coupling to gravity:

$$S = S_{\mathrm{g}}[g, \Gamma] + S_{\mathrm{m}}[g, \chi]\,. \tag{23}$$

LLI violation in post-Newtonian limit?

- Study teleparallel gravity theories:
  1. New General Relativity [Ualikhanova, MH ’19]
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$\Rightarrow$ No violation of LLI.

Outline

Finsler spacetime geometry

- Proper time along a curve in Lorentzian spacetime:

$$\tau = \int_{t_1}^{t_2} \sqrt{-g_{ab}(x(t))\dot{x}^a(t)\dot{x}^b(t)}\mathrm{d}t\,. \tag{24}$$

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$$N^a{}_b = \frac{1}{4}\bar{\partial}_b\left[g^{F\,ac}(y^d\partial_d\bar{\partial}_c F^2 - \partial_c F^2)\right]\,. \tag{27}$$

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- Hamilton equations of motion:

$$\dot{p}_\mu = -\partial_\mu H , \quad \dot{x}^\mu = \bar{\partial}^\mu H . \tag{31}$$

Spherically symmetric MDR

- General spherically symmetric MDR:

$$-m^2 = H(t, r, p_t, p_r, w)\,, \quad w^2 = p_\vartheta^2 + \frac{p_\varphi^2}{\sin^2\vartheta}\,. \tag{32}$$

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Example: κ-Poincarè dispersion relation

- General form of κ-Poincarè dispersion relation:

$$H(x,p) = -\frac{2}{\ell^2}\sinh^2\left(\frac{\ell}{2}Z^\mu p_\mu\right) + \frac{1}{2}e^{\ell Z^\mu p_\mu}[g^{\mu\nu}p_\mu p_\nu + (Z^\mu p_\mu)^2]\,. \tag{36}$$

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  - Lorentzian metric $g_{\mu\nu}$.
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$$H = -\frac{2}{\ell^2}\sinh^2\left[\frac{\ell}{2}(cp_t + dp_r)\right]^2 + \frac{1}{2}e^{\ell(cp_t+dp_r)}\left[(c^2-a)p_t^2 + 2cdp_rp_t + (d^2+b)p_r^2 + \frac{w^2}{r^2}\right]\,. \tag{37}$$

Circular orbits

- Method of calculation:
  - Circular orbit characterized by $\dot{r} = 0$.

  $\Rightarrow$ $\bar{\partial}^r H = 0$ becomes algebraic equation for $p_r = p_r(r, \mathcal{E}, \mathcal{L})$.

  $\Rightarrow$ Determine energy $\mathcal{E} = \mathcal{E}(r, \mathcal{L})$ from dispersion relation $H = -m^2$.

  $\Rightarrow$ Determine radius $r = r(\mathcal{L})$ from $\dot{p}_r = 0 \Rightarrow \partial_r H = 0$.

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  - $\Rightarrow$ Determine radius $r = r(\mathcal{L})$ from $\dot{p}_r = 0 \Rightarrow \partial_r H = 0$.
- Result for κ-Poincarè:

$$r = \frac{3}{2}r_s + \frac{\ell\mathcal{L}}{6} + \mathcal{O}(\ell^2)\,. \tag{38}$$

Shapiro delay

- Method of calculation:
  - Emitter / receiver at r_e, closest encounter at r_c, mirror at r_m.
  - General formula of Shapiro delay:

$$\Delta T = \int_{r_e}^{r_c} \frac{\mathrm{d}t}{\mathrm{d}r}\bigg|_{\text{in}}^{<0} \mathrm{d}r + \int_{r_c}^{r_m} \frac{\mathrm{d}t}{\mathrm{d}r}\bigg|_{\text{out}}^{>0} \mathrm{d}r + \int_{r_m}^{r_c} \frac{\mathrm{d}t}{\mathrm{d}r}\bigg|_{\text{in}}^{<0} \mathrm{d}r + \int_{r_c}^{r_e} \frac{\mathrm{d}t}{\mathrm{d}r}\bigg|_{\text{out}}^{>0} \mathrm{d}r\,. \tag{39}$$

  - At $r = r_c$: $\dot{r} = 0$ relates $\mathcal{E}, \mathcal{L}, r_c, p_{rc}$ by $\bar{\partial}^r H = 0$ and $H = -m^2$.
  - Parametrize trajectory by r and calculate

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- Result for κ-Poincarè:

$$\Delta T(r) \sim r_s e^{-\ell\mathcal{E}} \left[\frac{\ell\mathcal{E}}{2(e^{\ell\mathcal{E}} - 1)}\sqrt{\frac{r - r_c}{r + r_c}} + \frac{(2 - \ell\mathcal{E})}{2} \ln\left(\frac{r + \sqrt{r^2 - r_c^2}}{r_c}\right)\right]\,. \tag{41}$$

Light deflection

- Method of calculation:
  - Emitter / receiver at $r \to \infty$, closest encounter at r_c.
  - Calculate deviation from straight line $\Delta\varphi = \pi$.
  - General formula of deflection angle:

$$\Delta\varphi = \int_{\infty}^{r_c} \left.\frac{\mathrm{d}\varphi}{\mathrm{d}r}\right|_{\mathrm{in}}^{<0} \mathrm{d}r + \int_{r_c}^{\infty} \left.\frac{\mathrm{d}\varphi}{\mathrm{d}r}\right|_{\mathrm{out}}^{>0} \mathrm{d}r - \pi\,. \tag{42}$$

  - At $r = r_c$: $\dot{r} = 0$ relates $\mathcal{E}, \mathcal{L}, r_c, p_{rc}$ by $\bar{\partial}^r H = 0$ and $H = -m^2$.
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$$\Delta\varphi = \frac{r_s}{r_c}\frac{e^{\ell\mathcal{E}} - 1 + \ell\mathcal{E}}{e^{\ell\mathcal{E}} - 1}\,. \tag{44}$$

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- Consider monoenergetic ensemble of gas particles:
  - Fix constant values $\mathcal{E} = \mathcal{E}_0, \mathcal{L} = \mathcal{L}_0, H = H_0$.
  - Particle density $\phi(\mathcal{E}_0, \mathcal{L}_0, H_0) = C \neq 0$ on chosen level set.
  - $\phi = 0$ for all other values of $\mathcal{E}, \mathcal{L}, H$.

Orbiting gas: κ-Poincarè vs. Schwarzschild

- Example: orbiting particles with turnaround radii $r_1 = 3r_s$ and $r_2 = 6r_s$.

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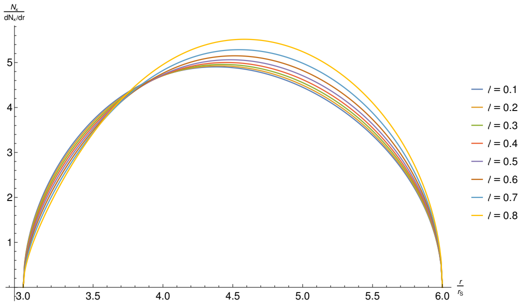

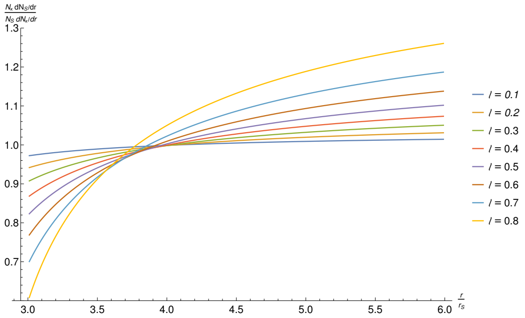

Radially inflowing gas: κ-Poincarè vs. Schwarzschild

- Example: radial inflow ($\mathcal{L}_0 = 0$) of marginally bound particles:

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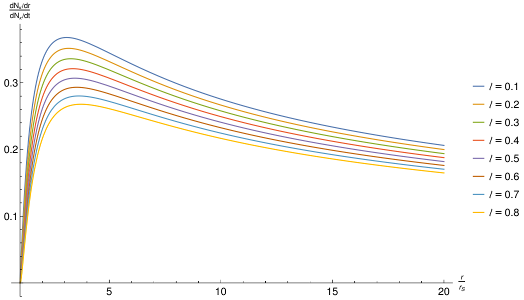

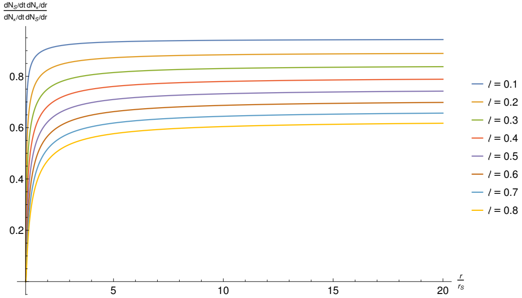

Outline

Conclusion

- Possible signatures of local Lorentz invariance violation:
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  - Post-Newtonian parameters $\alpha_1, \alpha_2, \alpha_3$.

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- Finsler-based gravity theories:
  - Based on generalized length functional.
  - Formulation as modified dispersion relation.
  - Various effects to search for LLI violation.

References

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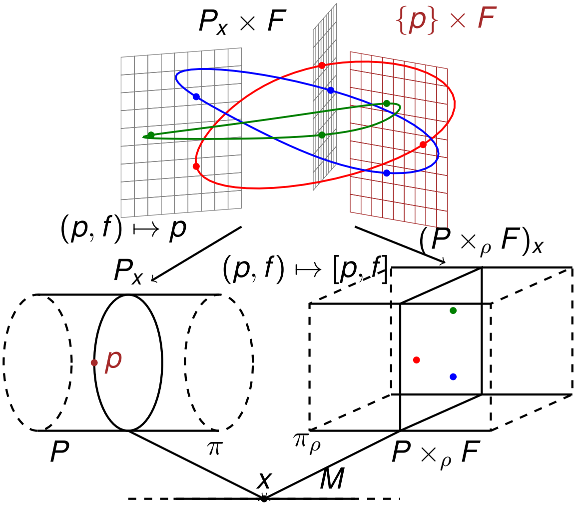
Extra: the associated bundle



Extra: the many faces of connections