The geometric foundation of gravity

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Laboratory of Theoretical Physics - Institute of Physics - University of Tartu
Center of Excellence “The Dark Side of the Universe”

European Union
European Regional
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Outline

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Why should we study gravity?

- Gravity is the dominating force in the universe:
  - Solar system, planetary motion, binary systems. . .
  - Galaxies, galactic clusters and superclusters, structure formation. . .
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$\Rightarrow$ Our description of geometry determines how we describe gravity.

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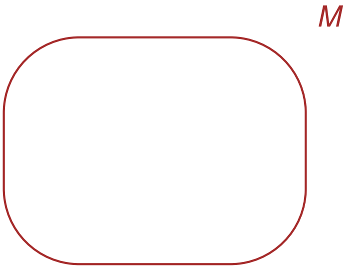

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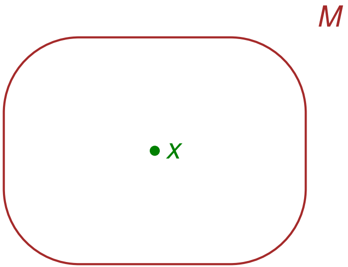

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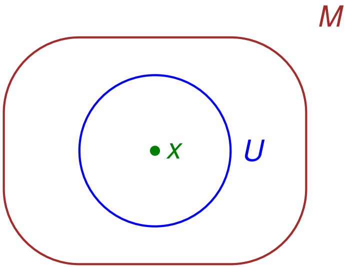

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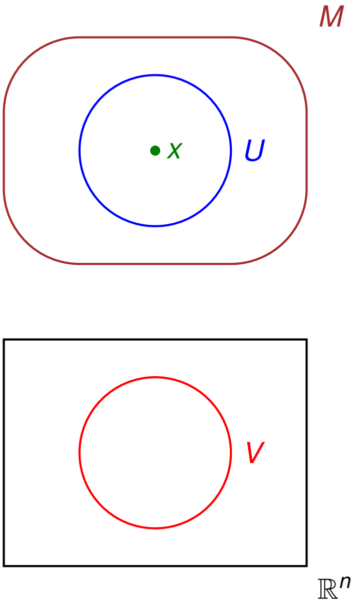

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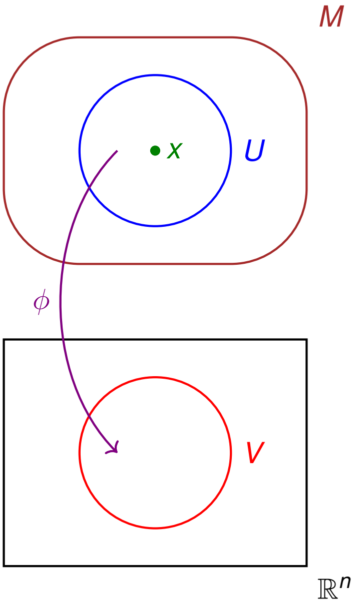

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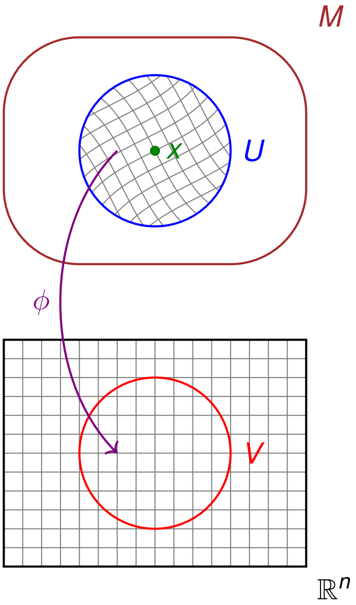

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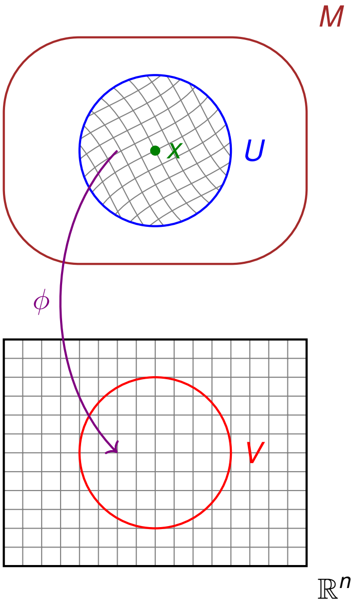
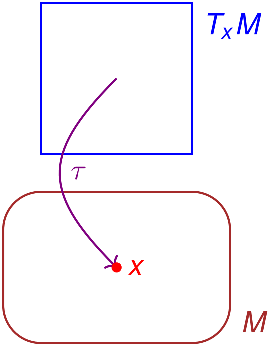


Notions from differential geometry

- $(U, \phi) \leftrightsquigarrow$ chart.
- Collection $\mathcal{A} = \{(U, \phi)\} \leftrightsquigarrow$ atlas.
- $(M, \mathcal{A}) \leftrightsquigarrow$ manifold.

What is a tangent space?

- Every point x of M has a tangent space T_xM.



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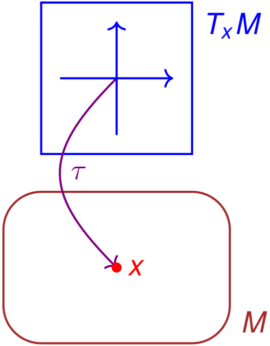

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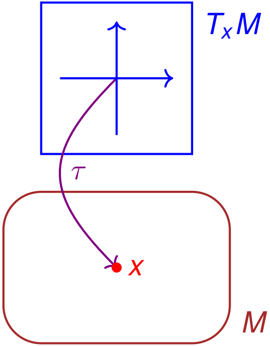

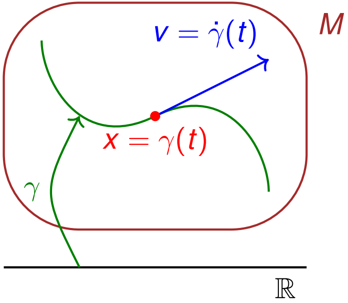

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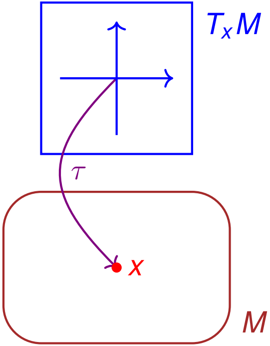

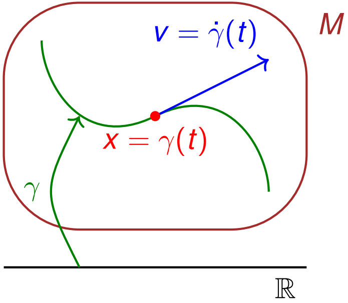

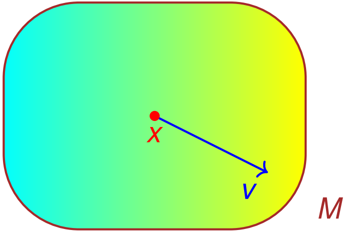

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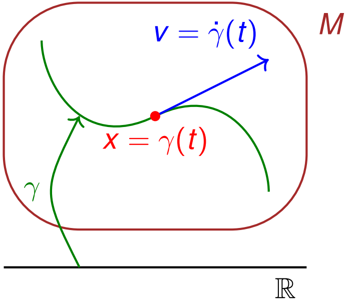
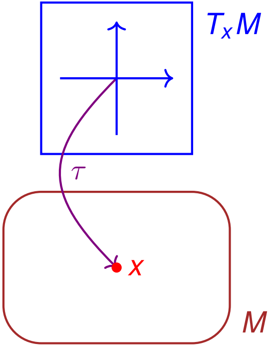
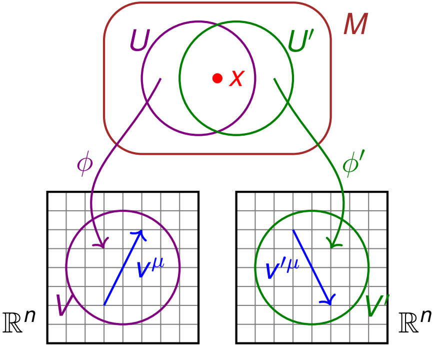
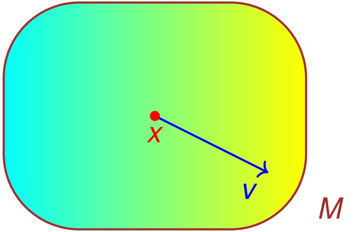




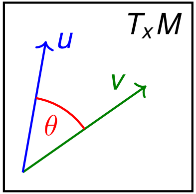




Metric geometry - length, time & causality

- A metric g is a scalar product on every tangent space T_xM:

$$g(u, v) = g_{\mu\nu}(x)u^\mu v^\nu = \|u\|\|v\|\cos\theta\,.$$



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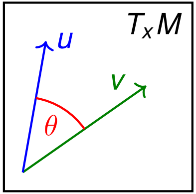

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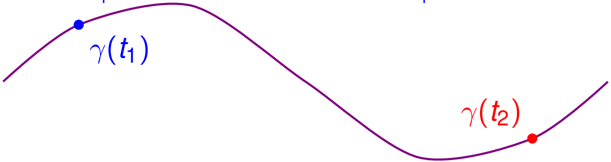

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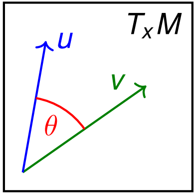

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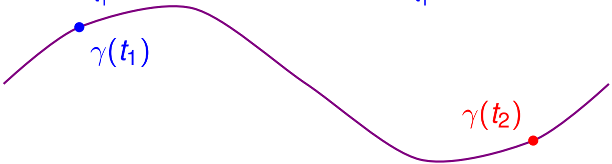

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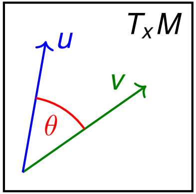

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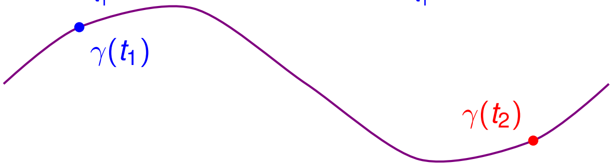
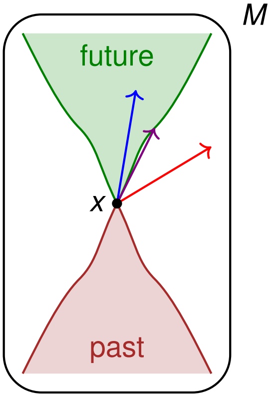


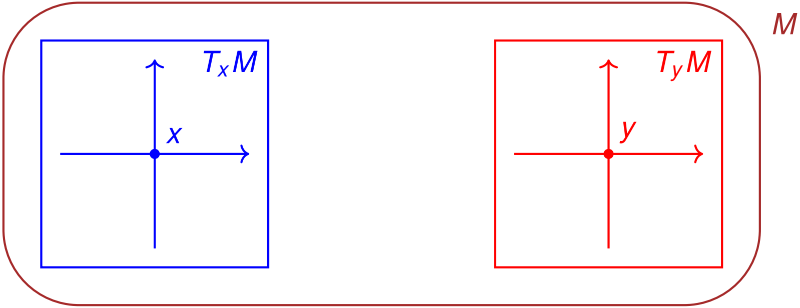


- Length of a trajectory $\ell \leftrightsquigarrow$ time measured by moving clock.
- Metric determines causality and propagation of information:

$g(v, v) > 0$	$g(v, v) = 0$	$g(v, v) < 0$
spacelike	lightlike (null)	timelike

Connections - parallel transport and autoparallel curves

- Tangent spaces $T_x M$ and $T_y M$ are different if $x \neq y$.



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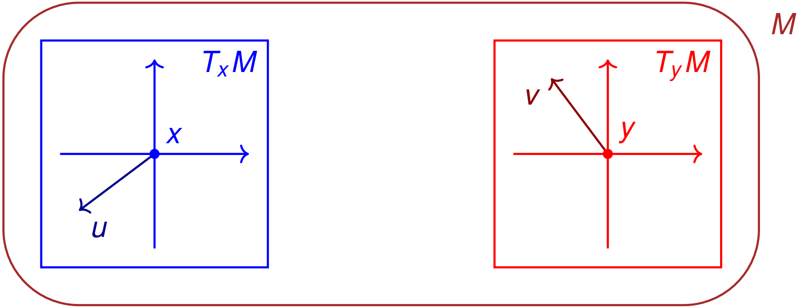

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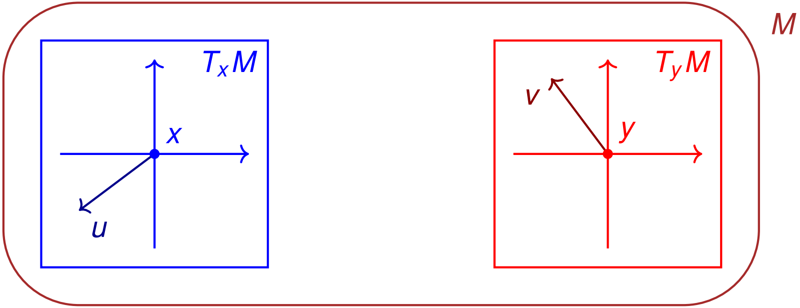

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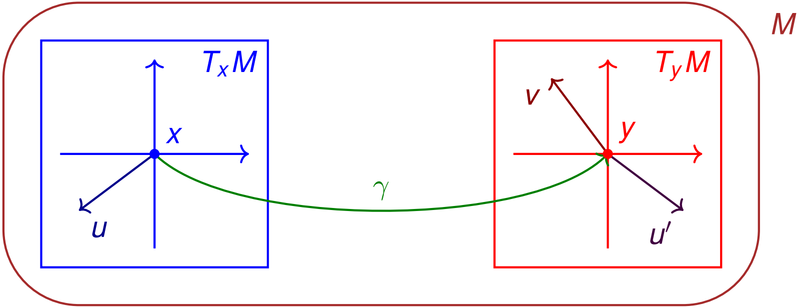

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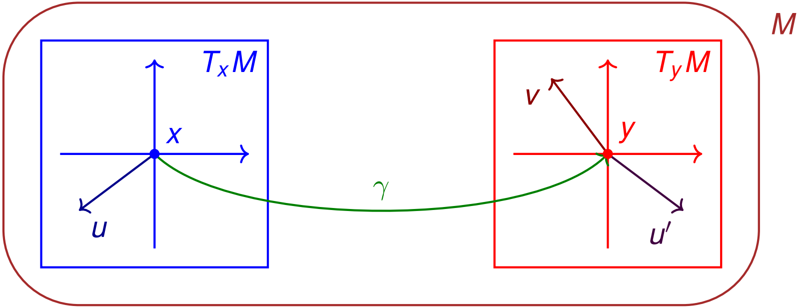

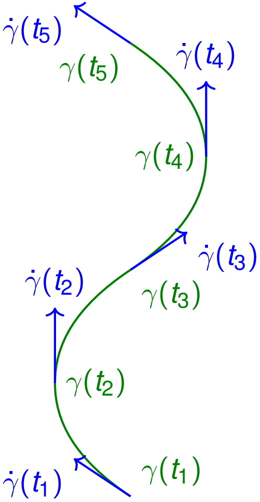

Curvature: the dependence of parallel transport on the path

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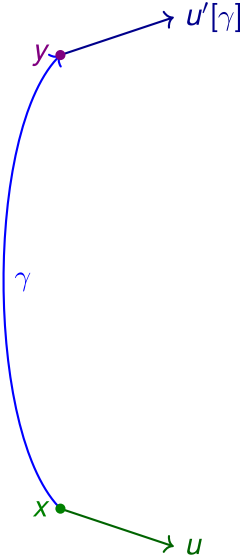

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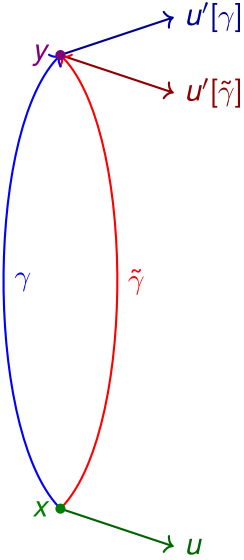

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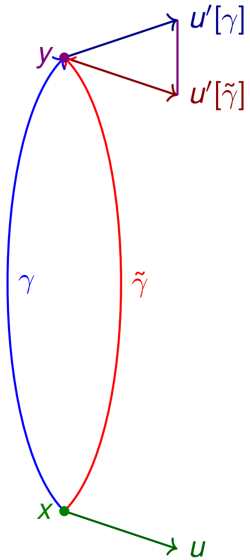

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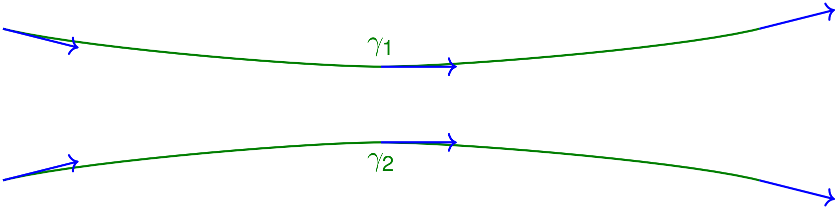

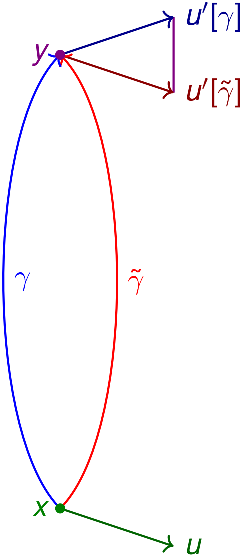

Curvature: the dependence of parallel transport on the path

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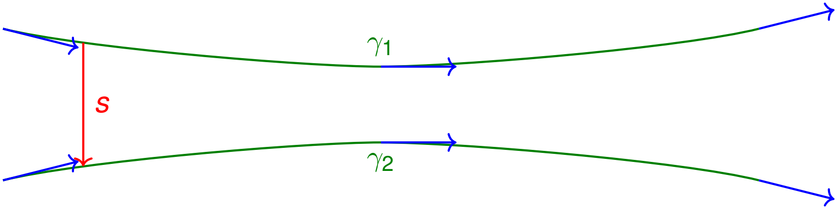

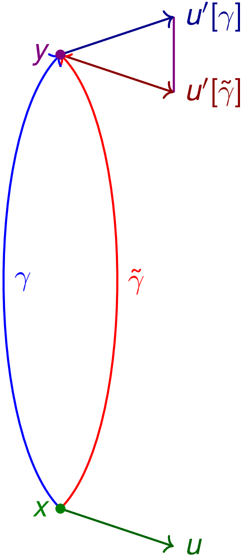

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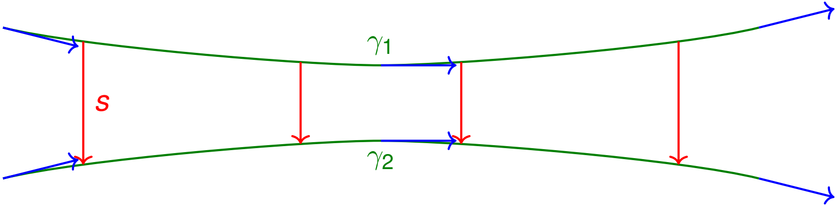

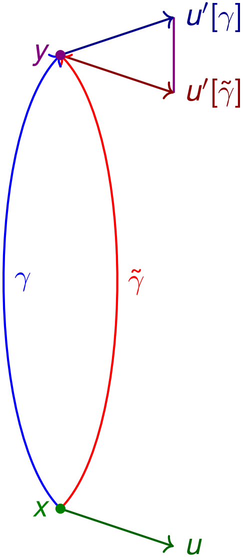

Torsion: the role of a non-symmetric parallel transport

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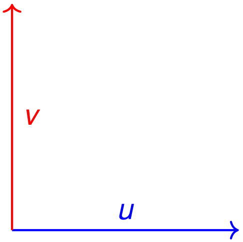

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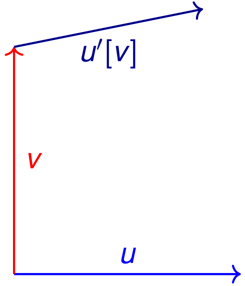

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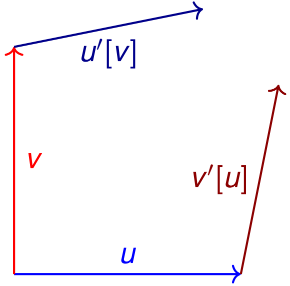

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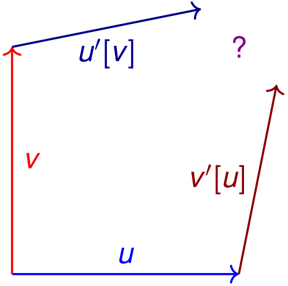

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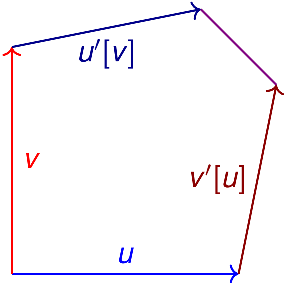

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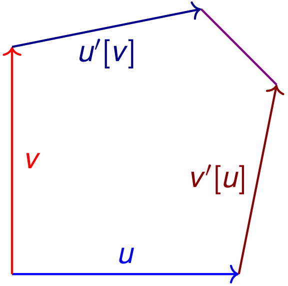
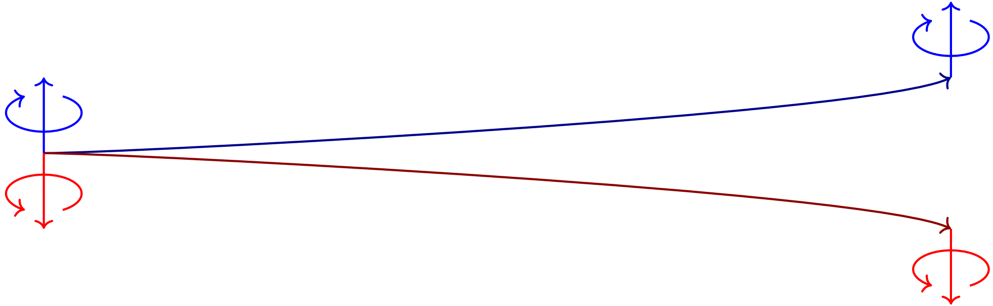
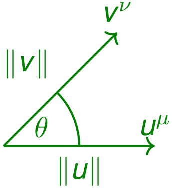
- Torsion may influence particles with spin (fermions).





Nonmetricity: connections which are not metric-compatible

- Notion requires two geometric objects:
  - Metric determines length of tangent vectors.



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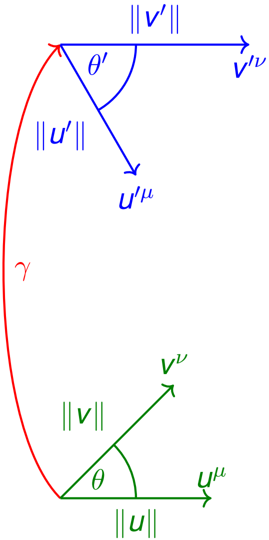

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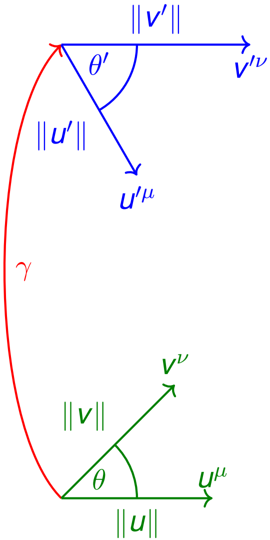

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- Geometric interpretation of $Q \neq 0$?
    - Scalar product of vectors changes along transport:

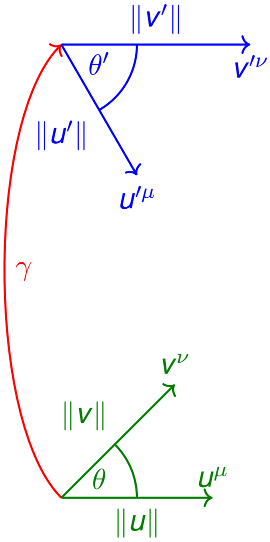
$$g_{\mu\nu} u^\mu v^\nu \neq g'_{\mu\nu} u'^\mu v'^\nu \, .$$

    - Length of vectors changes along transport:

$$\|u\| \neq \|u'\| \, , \quad \|v\| \neq \|v'\| \, .$$

    - Angle between vectors changes along transport:

$$\theta \neq \theta' \, .$$



Connection decomposition: Levi-Civita, contortion and disformation

- In presence of a metric, a connection may be uniquely decomposed:

$$\Gamma^{\mu}{}_{\nu\rho} = \overset{\circ}{\Gamma}{}^{\mu}{}_{\nu\rho} + K^{\mu}{}_{\nu\rho} + L^{\mu}{}_{\nu\rho}\,.$$

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- Components of the connection:
  1. Levi-Civita connection $\leftrightsquigarrow$ metric:

$$\overset{\circ}{\Gamma}{}^{\mu}{}_{\nu\rho} = \frac{1}{2} g^{\mu\sigma} \left(\partial_{\nu} g_{\sigma\rho} + \partial_{\rho} g_{\nu\sigma} - \partial_{\sigma} g_{\nu\rho}\right).$$

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  3. Disformation $\leftrightsquigarrow$ nonmetricity:

$$L^{\mu}{}_{\nu\rho} = \frac{1}{2}\left(Q^{\mu}{}_{\nu\rho} - Q_{\nu}{}^{\mu}{}_{\rho} - Q_{\rho}{}^{\mu}{}_{\nu}\right).$$

Outline

The Nobel Prize in Physics 2020

The Nobel Prize in Physics 2020 was divided, one half awarded to Roger Penrose “for the discovery that black hole formation is a robust prediction of the general theory of relativity”, the other half jointly to Reinhard Genzel and Andrea Ghez “for the discovery of a supermassive compact object at the centre of our galaxy.”

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1. How does general relativity describe gravity?
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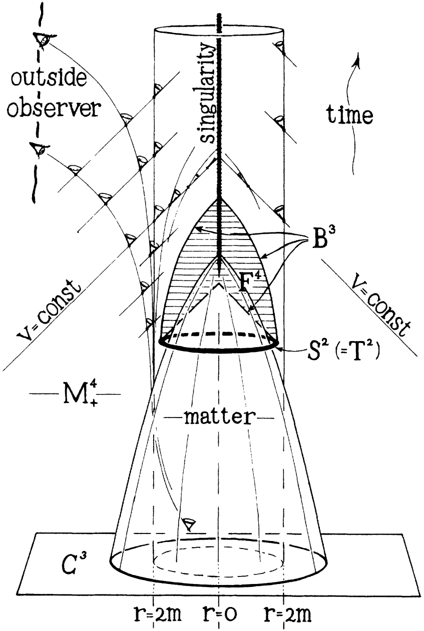
4. How does general relativity predict the formation of black holes?
5. What did Roger Penrose discover?

Penrose's singularity theorem (1965)

Singularity theorem:

The formation of a singularity is unavoidable, if:

1. the null energy condition holds,



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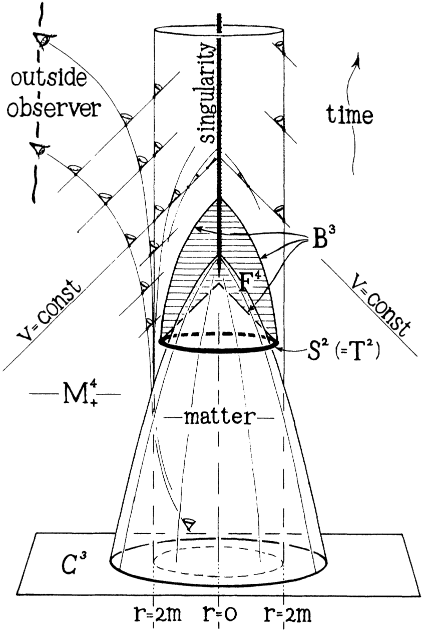

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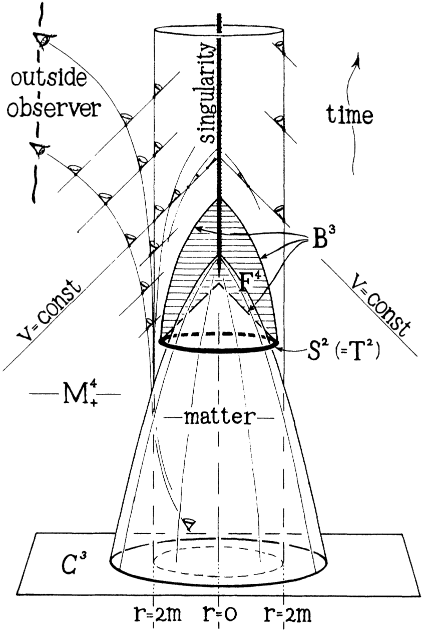

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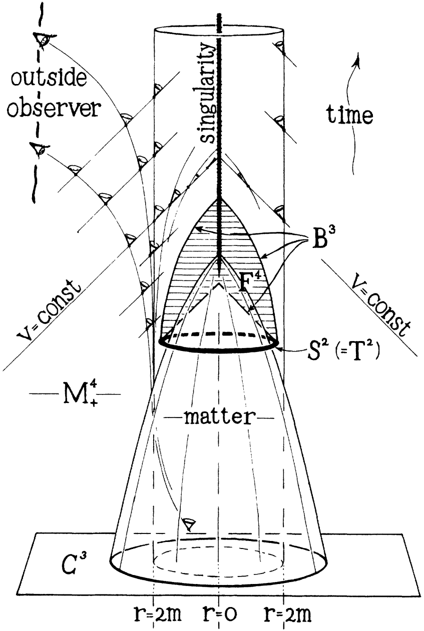

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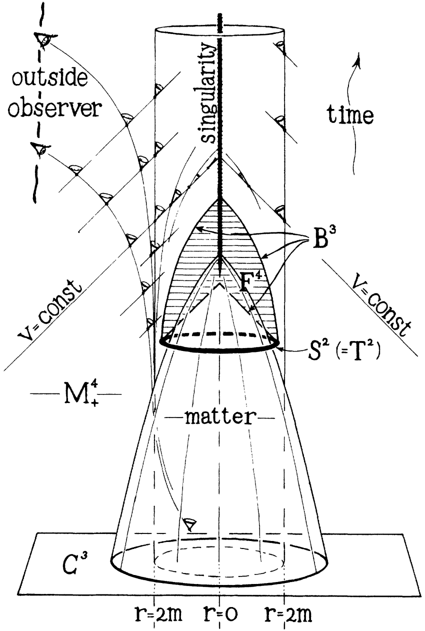

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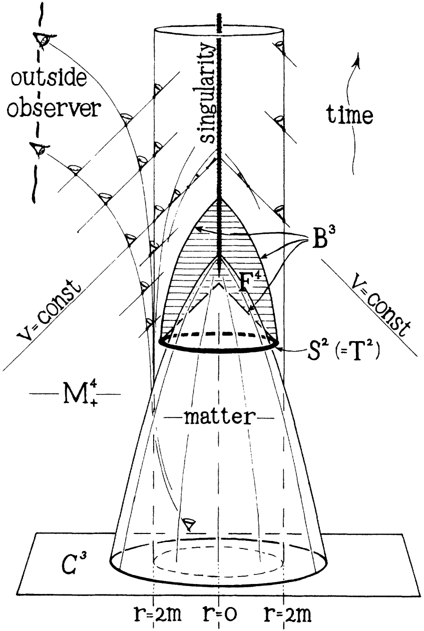

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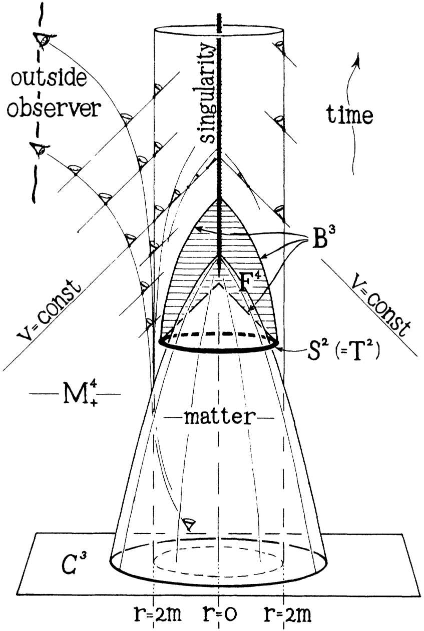

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$$\Theta^{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}.$$

- Classical fluids ($\rho > 0$):
  - Dust: $p = 0$.
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Energy conditions

1. Null energy condition: ✓
   - Projected energy density is non-negative.
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2. Weak energy condition: ✓
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3. Dominant energy condition: ✓
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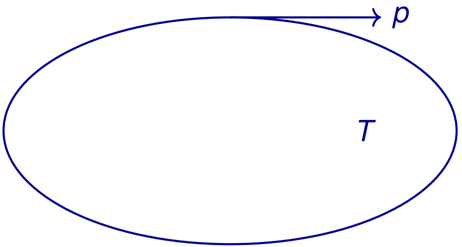
- Quantum field theory:
  - “Quantum inequalities” hold.
  - Averaged energy density.

Trapped surfaces

What is a “trapped surface”?

- Closed, spacelike, two-dimensional surface T.
- Light propagates only inwards from T.

Trapped surfaces



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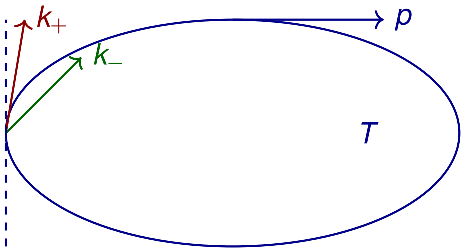

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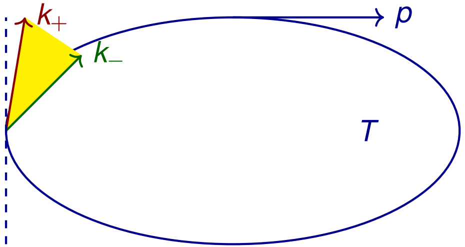

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$\Rightarrow$ The vectors $k_\pm$ describe the propagation of light; all light from T moves inwards.

Singularities and geodesic (in)completeness

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$\lightning$ In presence of singularities, the fate of free-fall observers is not determinable!

Outline

General relativity: gravity described by curvature

- Curvature of the Levi-Civita connection:

$$\overset{\circ}{R}{}^{\mu}{}_{\nu\rho\sigma} = \partial_\rho \overset{\circ}{\Gamma}{}^{\mu}{}_{\nu\sigma} - \partial_\sigma \overset{\circ}{\Gamma}{}^{\mu}{}_{\nu\rho} + \overset{\circ}{\Gamma}{}^{\mu}{}_{\tau\rho} \overset{\circ}{\Gamma}{}^{\tau}{}_{\nu\sigma} - \overset{\circ}{\Gamma}{}^{\mu}{}_{\tau\sigma} \overset{\circ}{\Gamma}{}^{\tau}{}_{\nu\rho} \,.$$

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- Curvature of a general connection:

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The road to teleparallelism

- Curvature of a general connection:

$$R^{\mu}{}_{\nu\rho\sigma} = \partial_\rho \Gamma^{\mu}{}_{\nu\sigma} - \partial_\sigma \Gamma^{\mu}{}_{\nu\rho} + \Gamma^{\mu}{}_{\tau\rho}\Gamma^{\tau}{}_{\nu\sigma} - \Gamma^{\mu}{}_{\tau\sigma}\Gamma^{\tau}{}_{\nu\rho} \,.$$

- Decomposition of the curvature:

$$\begin{aligned} R^{\mu}{}_{\nu\rho\sigma} = \mathring{R}^{\mu}{}_{\nu\rho\sigma} &+ \mathring{\nabla}_\rho (K^{\mu}{}_{\nu\sigma} + L^{\mu}{}_{\nu\sigma}) - \mathring{\nabla}_\sigma (K^{\mu}{}_{\nu\rho} + L^{\mu}{}_{\nu\rho}) \\ &+ (K^{\mu}{}_{\omega\rho} + L^{\mu}{}_{\omega\rho})(K^{\omega}{}_{\nu\sigma} + L^{\omega}{}_{\nu\sigma}) - (K^{\mu}{}_{\omega\sigma} + L^{\mu}{}_{\omega\sigma})(K^{\omega}{}_{\nu\rho} + L^{\omega}{}_{\nu\rho}) \,. \end{aligned}$$

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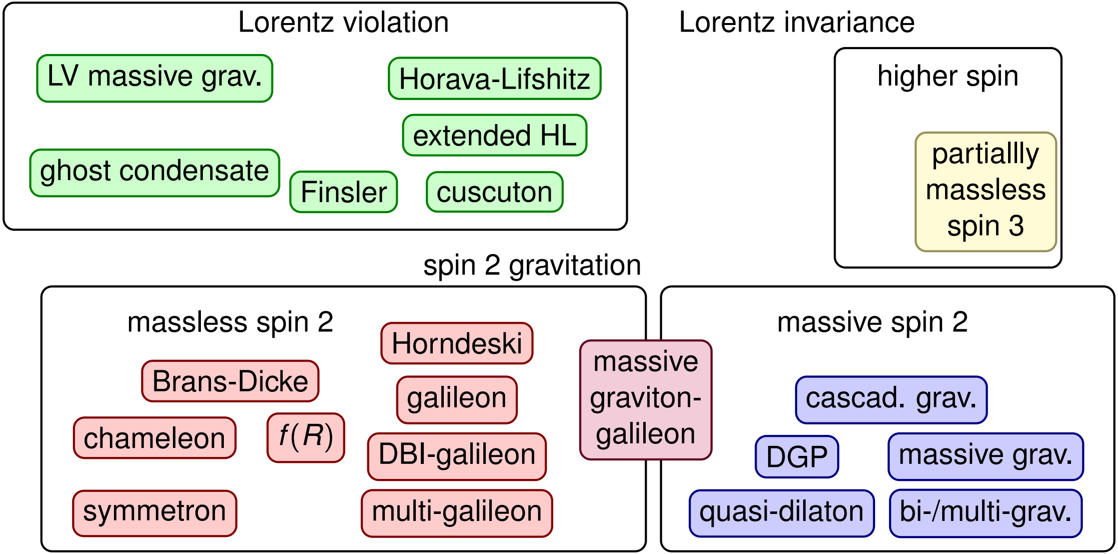
Outline

The “zoo” of modified gravity theories



The $f(\ldots)$ family of gravity theories

- Action with higher order (curvature, torsion, nonmetricity) terms:
  - Possible effects from quantum gravity, Feynman diagrams with loops.
  - New dynamical effects in cosmology modeling inflation and dark energy.
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- Relation between different extensions?
  - Original Lagrangians differ only by boundary terms:

$$-\mathbb{T} + \overset{\circ}{\nabla}_\mu B^\mu_T = \overset{\circ}{R} = -\mathbb{Q} + \overset{\circ}{\nabla}_\mu B^\mu_Q\,.$$

$\Rightarrow$ Corresponding equivalent theories:

$$f(-\mathbb{T} + \overset{\circ}{\nabla}_\mu B^\mu_T) = f(\overset{\circ}{R}) = f(-\mathbb{Q} + \overset{\circ}{\nabla}_\mu B^\mu_Q)\,.$$

  - $f(\mathbb{T})$ and $f(\mathbb{Q})$ Lagrangians lead to essentially different theories.
  - Difference cannot be moved into boundary term $\Rightarrow$ different field equations.

Coupling scalar fields

- Why consider scalar fields Φ non-minimally coupled to gravity?
  - Scalar fields are simplest possibility to add another degree of freedom.
  - Discovery of the Higgs boson showed existence of fundamental scalar fields.
  - Scalar fields appear in effective description of other (e.g., string, quantum) theories.
  - $f(\ldots)$ theories can be expressed as scalar-tensor gravity theories.
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- Scalar field extensions of different formulations of GR:
  - Scalar-curvature gravity:

$$S_{\text{SCG}} = \frac{1}{16\pi G}\int_M d^4x\sqrt{-\det g}\left[\mathcal{A}(\Phi)\mathring{R} - \mathcal{B}(\Phi)g^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi - \mathcal{V}(\Phi)\right].$$

  - Scalar-torsion gravity:

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  - Scalar-nonmetricity gravity:

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New GR, newer GR and even newer theories

- More general theories constructible from torsion terms:
  - "New general relativity":

$$\mathcal{L} = c_1 T^{\mu\nu\rho} T_{\mu\nu\rho} + c_2 T^{\mu\nu\rho} T_{\rho\nu\mu} + c_3 T^{\mu}{}_{\mu\rho} T_{\nu}{}^{\nu\rho} \,.$$

  - Generalized new general relativity:

$$\mathcal{L} = f(T^{\mu\nu\rho} T_{\mu\nu\rho}, T^{\mu\nu\rho} T_{\rho\nu\mu}, T^{\mu}{}_{\mu\rho} T_{\nu}{}^{\nu\rho}) \,.$$

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Non-equivalence of modified gravity theories

- Different geometric formulations of general relativity…

$$\mathcal{L} \sim \overset{\circ}{R}$$

$$\mathcal{L} \sim \mathbb{T}$$

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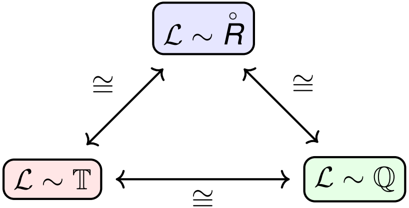
Non-equivalence of modified gravity theories

- Different geometric formulations of general relativity are equivalent.



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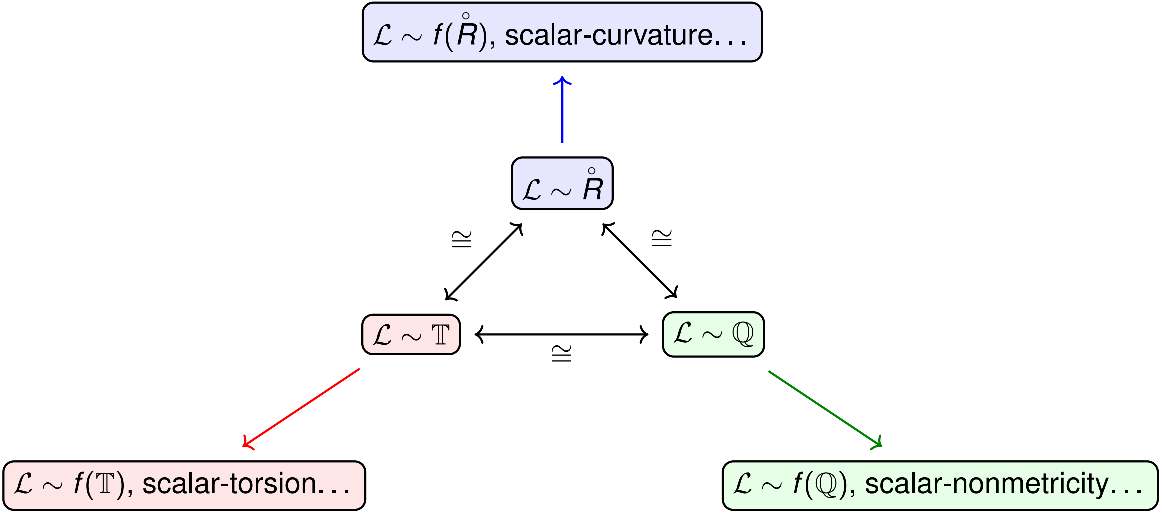

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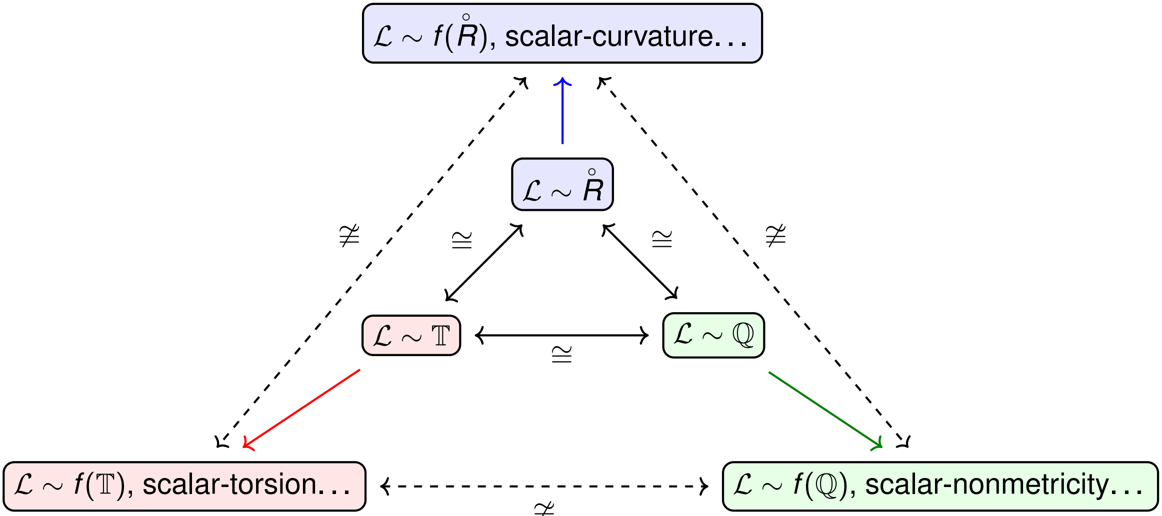

Gravity based on observers and gases: Finsler geometry

- Recall length functional in Riemannian geometry:

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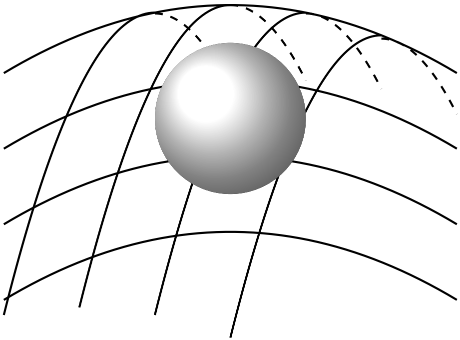
  - Possible to address unsolved questions in gravity and cosmology from new geometry?

A unified picture: Cartan geometry

- Cartan geometry: how a hamster sitting in a ball describes geometry.



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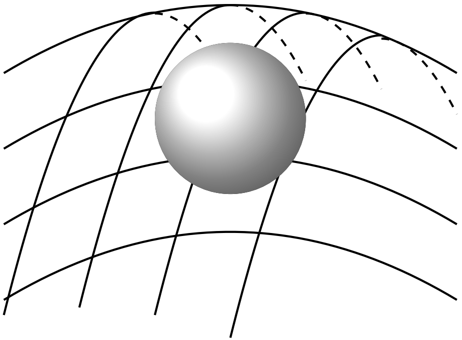

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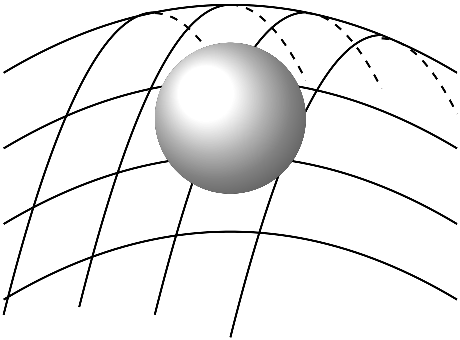

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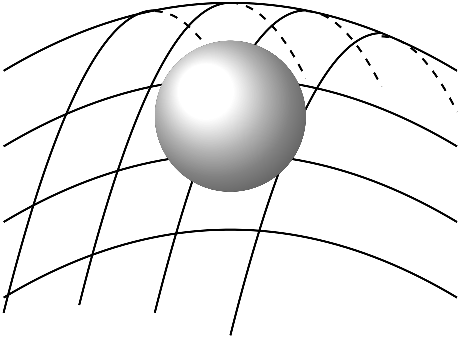

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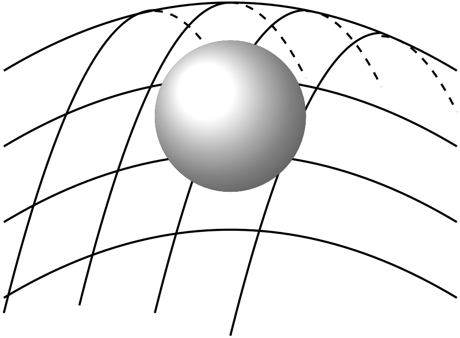

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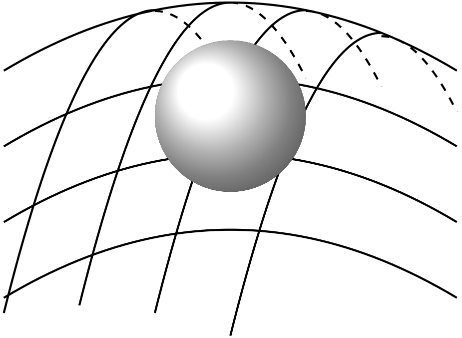

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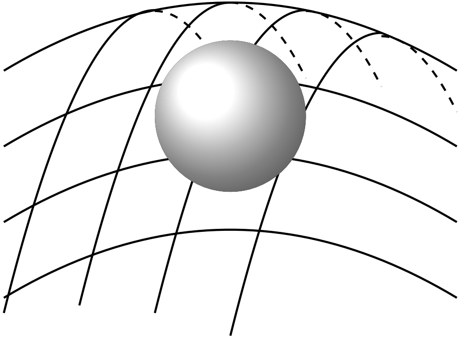
- Possible ways the hamster can move in the ball: Lie group G.
- Motions of the hamster not moving the ball: subgroup $H \subset G$.
- Cartan connection A measures, how motions of hamster move the ball.



- Deviation of surface from ball geometry:

$$F = dA + \frac{1}{2}[A, A]\,.$$

- Action functional in Cartan geometry:

$$S = \int_M \kappa(F_{\mathfrak{h}} \wedge \star F_{\mathfrak{h}})\,.$$

- Unified model of previous geometries.

Outline

Summary

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  - Most observations and dynamical processes in the universe dominated by gravity.
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- Nobel Prize in Physics 2020 for using geometry to prove fundamental physics.

The road ahead: from the cosmos to quantum gravity

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$\Rightarrow$ Understanding gravity as geometry is a crucial part of today's physics.